ABELIAN SURFACES WITH SUPERSINGULAR GOOD REDUCTION AND NON SEMISIMPLE TATE MODULE

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Abstract. We show the existence of abelian surfaces $\mathcal{A}$ over $\mathbb{Q}_p$ having good reduction with supersingular special fibre whose associated p-adic Galois module $V_p(\mathcal{A})$ is not semisimple.

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Contents

Introduction

Fix a prime number p and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Write $G = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for the absolute Galois group of $\mathbb{Q}_p$. For a d-dimensional abelian variety $\mathcal{A}$ over $\mathbb{Q}_p$ let $\mathcal{A}[p^n]$ be the group of p^n-torsion points with values in $\overline{\mathbb{Q}}_p$ and

$$V_p(\mathcal{A}) \underset{\mathrm{def}}{=} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geq 1} \mathcal{A}[p^n].$$

This is a $2d$-dimensional $\mathbb{Q}_p$-vector space on which G acts linearly and continuously. We want to consider the following problem: find abelian varieties $\mathcal{A}$ over $\mathbb{Q}_p$ having good reduction with supersingular special fibre and such that the Galois module $V_p(\mathcal{A})$ is *not* semisimple. In this paper we show the existence of two such varieties with nonisogenous special fibres for the least dimension possible, namely for $d = 2$. In fact our procedure easily generalises to any $d \geq 2$, however we stick to surfaces as they furnish low-dimensional hence simple to describe representations.

The existence of such surfaces follows from the characterisation of p-adic representations of G arising from abelian varieties with (tame) potential good reduction obtained in [Vo], and indeed provides an example of application of this result. In order to explicitely describe our objects we use Fontaine's contravariant functor establishing an equivalence between crystalline p-adic representations of G and admissible filtered φ-modules. In section 1 we briefly review this theory as well as the characterisation in [Vo] (Theorem 1.2), and outline the general strategy. In sections 2 and 3 we construct two filtered φ-modules arising from abelian surfaces over $\mathbb{Q}_p$ with good reduction that enjoy the required properties (Propositions 2.1 and 3.1).

1. The general method

Recall from [Fo2] that the objects D in the category $\mathbf{MF}_{\mathbb{Q}_p}(\varphi)$ of filtered φ-modules are finite dimensional $\mathbb{Q}_p$-vector spaces together with a Frobenius map $\varphi \in \mathrm{Aut}_{\mathbb{Q}_p}(D)$ and a decreasing filtration $\mathrm{Fil} = (\mathrm{Fil}^i D)_{i\in\mathbb{Z}}$ on D by subspaces such that $\mathrm{Fil}^i D = D$ for $i \ll 0$ and $\mathrm{Fil}^i D = 0$ for $i \gg 0$, and the morphisms are $\mathbb{Q}_p$-linear maps commuting with φ and preserving the filtration. The dual of (D,Fil) is the $\mathbb{Q}_p$-linear dual D^* with $\varphi_{D^*} = \varphi^{*\,-1}$ and $\mathrm{Fil}^i D^*$ consists of linear forms on D vanishing on $\mathrm{Fil}^j D$ for all $j > -i$. The Tate twist $D\{-1\}$ of (D,Fil) is D as a $\mathbb{Q}_p$-vector space with $\varphi_{D\{-1\}} = p\varphi$ and $\mathrm{Fil}^i D\{-1\} = \mathrm{Fil}^{i-1} D$. The filtration Fil has Hodge-Tate type $(0,1)$ if $\mathrm{Fil}^i D = D$ for $i \le 0$, $\mathrm{Fil}^i D = 0$ for $i \ge 2$, and $\mathrm{Fil}^1 D$ is a nontrivial subspace. The full subcategory $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$ of $\mathbf{MF}_{\mathbb{Q}_p}(\varphi)$ consists of objects (D,Fil) satisfying a property relating the Frobenius with the filtration, called admissibility and defined as follows. For a φ-stable sub-$\mathbb{Q}_p$-vector space D' of D consider the Hodge and Newton invariants

$$t_H(D') \underset{\mathrm{def}}{=} \sum_{i\in\mathbb{Z}} i \dim_{\mathbb{Q}_p}\big(D'\cap \mathrm{Fil}^i D \big/ D'\cap \mathrm{Fil}^{i+1} D\big) \quad \text{and} \quad t_N(D') \underset{\mathrm{def}}{=} v_p(\det \varphi_{|D'})$$

where v_p is the normalised p-adic valuation on $\mathbb{Q}_p$. Then (D,Fil) is admissible if

(i) $t_H(D) = t_N(D)$
(ii) $t_H(D') \le t_N(D')$ for any sub-$\mathbb{Q}_p[\varphi]$-module D' of D.

A sub-$\mathbb{Q}_p[\varphi]$-module D' endowed with the induced filtration $\mathrm{Fil}^i D' = D' \cap \mathrm{Fil}^i D$ is a subobject of (D,Fil) in $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$ if and only if $t_H(D') = t_N(D')$.

Let B_{cris} be the ring of p-adic periods constructed in [Fo1] and for a p-adic representation V of G put

$$\mathbf{D}^*_{\mathrm{cris}}(V) \underset{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Q}_p[G]}(V, B_{\mathrm{cris}}).$$

We always have $\dim_{\mathbb{Q}_p} \mathbf{D}^*_{\mathrm{cris}}(V) \le \dim_{\mathbb{Q}_p} V$ and V is said to be crystalline when equality holds. The functor $V \mapsto \mathbf{D}^*_{\mathrm{cris}}(V)$ establishes an anti-equivalence between the category of crystalline p-adic representations of G and $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$, a quasi-inverse being $\mathbf{V}^*_{\mathrm{cris}}(D,\mathrm{Fil}) = \mathrm{Hom}_{\varphi,\mathrm{Fil}}(D, B_{\mathrm{cris}})$ ([Co-Fo]). These categories are well-suited to our problem since for an abelian variety $\mathcal{A}$ over $\mathbb{Q}_p$ the G-module $V_p(\mathcal{A})$ is crystalline if and only if $\mathcal{A}$ has good reduction ([Co-Io] Thm.4.7 or [Br] Cor.5.3.4.).

A p-Weil number is an algebraic integer such that all its conjugates have absolute value $\sqrt{p}$ in $\mathbb{C}$. Call a monic polynomial in $\mathbb{Z}[X]$ a p-Weil polynomial if all its roots in $\overline{\mathbb{Q}}$ are p-Weil numbers and its valuation at $X^2 - p$ is even. Consider the following conditions on a filtered φ-module (D,Fil) over $\mathbb{Q}_p$:

(1) φ acts semisimply and $\mathrm{P}_{\mathrm{char}}(\varphi)$ is a p-Weil polynomial
(2) the filtration has Hodge-Tate type $(0,1)$
(3) there exists a nondegenerate skew form on D under which φ is a p-similitude and $\mathrm{Fil}^1 D$ is totally isotropic.

Recall that φ is a p-similitude under a bilinear form β if $\beta(\varphi x, \varphi y) = p\beta(x, y)$ for all $x, y \in D$ and $\mathrm{Fil}^1 D$ is totally isotropic if $\beta(x, y) = 0$ for all $x, y \in \mathrm{Fil}^1 D$. The map sending $\delta \in \mathrm{Isom}_{\mathbb{Q}_p}(D^*, D)$ to $\beta : (x, y) \mapsto \delta^{-1}(x)(y)$ identifies the antisymmetric isomorphisms of filtered φ-modules from $D^*\{-1\}$ to D with the forms satisfying (3). A $\mathbb{Q}_p$-linear map $\delta : D^* \to D$ is an antisymmetric morphism in $\mathbf{MF}_{\mathbb{Q}_p}(\varphi)$ if $\delta^* = -\delta$ (under the canonical isomorphism $D^{**} \simeq D$), $\varphi\delta = p\delta\varphi^{*\,-1}$, and $\delta(\mathrm{Fil}^1 D)^{\perp} \subseteq \mathrm{Fil}^1 D$.

Remark 1.1. Let $\mathrm{Hom}^{\mathrm{a}}_{\varphi}(D^*\{-1\}, D)$ be the $\mathbb{Q}_p$-vector space of antisymmetric φ-module morphisms from $D^*\{-1\}$ to D and pick any $\delta \in \mathrm{Isom}^{\mathrm{a}}_{\varphi}(D^*\{-1\}, D)$. Then $\alpha^{\dagger} = \delta\alpha^*\delta^{-1}$ defines an involution $\dagger$ on $\mathrm{End}_{\varphi}(D)$ and the map $\alpha \mapsto \alpha\delta$ establishes an isomorphim $\mathrm{End}_{\varphi}(D)^{\dagger} \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{a}}_{\varphi}(D^*\{-1\}, D)$ where $\mathrm{End}_{\varphi}(D)^{\dagger}$ is the subspace of elements fixed by $\dagger$.

Theorem 1.2 ([Vo] Corollary 5.9)**.** *Let V be a crystalline p-adic representation of G. The following are equivalent:*

(i) *there is an abelian variety $\mathcal{A}$ over $\mathbb{Q}_p$ such that $V \simeq V_p(\mathcal{A})$*
(ii) *$\mathbf{D}^*_{\mathrm{cris}}(V)$ satisfies conditions* (1), (2) *and* (3)*.*

Note that the restriction $p \neq 2$ in [Vo] Theorem 5.7 and its Corollary 5.9 is unnecessary as Kisin shows that a crystalline representation with Hodge-Tate weights in $\{0, 1\}$ arises from a p-divisible group unrestrictidly on the prime p ([Ki] Thm.0.3).

Let $\mathcal{A}$ be an abelian variety over $\mathbb{Q}_p$ having good reduction and $(D, \mathrm{Fil}) = \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{A}))$. The φ-module D satisfies (1) by the Weil conjectures for abelian varieties over $\mathbb{F}_p$. Tate's theorem on endomorphisms of the latter (see [Wa-Mi] II) shows that the isomorphism class of the φ-module D, given by semisimplicity by $\mathrm{P}_{\mathrm{char}}(\varphi)$, determines the isogeny class of the special fibre of $\mathcal{A}$ over $\mathbb{F}_p$. Any polarisation on $\mathcal{A}$ induces a form on D satisfying (3) and the filtration satisfies (2) by the Hodge decomposition for p-divisible groups and (3).

Conversely let V be a crystalline p-adic representation of G such that $\mathbf{D}^*_{\mathrm{cris}}(V)$ satisfies (1), (2), (3). From (1) the Honda-Tate theory ([Ho-Ta]) furnishes an abelian variety A over $\mathbb{F}_p$ with the right Frobenius. From (2) Kisin's result [Ki] furnishes a p-divisible group over $\mathbb{Z}_p$ lifting $A(p)$. The Serre-Tate theory of liftings then produces a formal abelian scheme $\mathcal{A}$ over $\mathbb{Z}_p$ with special fibre isogenous to A. Finally (3) furnishes a polarisation on $\mathcal{A}$ which ensures by Grothendieck's theorem on algebraisation of formal schemes ([Gr] 5.4.5) that $\mathcal{A}$ is a true abelian scheme. The proof of Theorem 5.7 in [Vo] details this construction.

Thus we want to construct an admissible filtered φ-module (D, Fil) over $\mathbb{Q}_p$ satisfying conditions (1), (2), (3) of theorem 1.2 and such that

(a) $\mathrm{P}_{\mathrm{char}}(\varphi)$ is a supersingular p-Weil polynomial
(b) (D, Fil) is not semisimple.

Recall that a p-Weil polynomial is supersingular if its roots are of the form $\zeta\sqrt{p}$ with $\zeta \in \overline{\mathbb{Q}}$ a root of unity, and that an abelian variety A over $\mathbb{F}_p$ is supersingular if and only if the characteristic polynomial of its Frobenius is supersingular. Regarding (a) in section 2 we take $\mathrm{P}_{\mathrm{char}}(\varphi)(X) = (X^2 + p)^2$ which is the characteristic polynomial of the Frobenius of the product of a supersingular elliptic curve E over $\mathbb{F}_p$ with itself. In section 3 we take $\mathrm{P}_{\mathrm{char}}(\varphi)(X) = X^4 + pX^2 + p^2$ which is the characteristic polynomial of the Frobenius of a simple supersingular abelian surface over $\mathbb{F}_p$.

Regarding (b) we assume $p \equiv 1 \bmod 3\mathbb{Z}$ in section 3. In each (a)-case we find a subobject D_1 of (D, Fil) in $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$ and a quotient object D_2 (endowed with the quotient filtration $\mathrm{Fil}^i D_2 = \mathrm{Fil}^i D \bmod D_1$) such that the sequence

$$\text{(S)} \qquad 1 \longrightarrow D_1 \xrightarrow{\text{incl}} D \xrightarrow{\text{proj}} D_2 \longrightarrow 1$$

is exact and D_2 is *not* a subobject. Thus (S) does not split and therefore (D, Fil) is not semisimple. Of course when $(D, \mathrm{Fil}) \simeq \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{A}))$ this means that there is a nonsplit short exact sequence of G-modules

$$1 \longrightarrow V_2 \longrightarrow V_p(\mathcal{A}) \longrightarrow V_1 \longrightarrow 1$$

with $V_i \simeq \mathbf{V}^*_{\mathrm{cris}}(D_i)$ for $i = 1, 2$, and it follows that $V_p(\mathcal{A})$ is not a semisimple G-module.

2. A lift of the twofold product of a supersingular elliptic curve

Consider the filtered φ-module (D, Fil) over $\mathbb{Q}_p$ defined as follows. There is a $\mathbb{Q}_p$-basis $\mathcal{B} = (x_1, y_1, x_2, y_2)$ for D so that

$$D = \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p y_1 \oplus \mathbb{Q}_p x_2 \oplus \mathbb{Q}_p y_2$$

is a 4-dimensional $\mathbb{Q}_p$-vector space. The matrix of φ over $\mathcal{B}$ is

$$\mathrm{Mat}_{\mathcal{B}}(\varphi) = \begin{pmatrix} 0 & -p & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL_4(\mathbb{Q}_p)$$

and the filtration is given by

$$\mathrm{Fil}^0 D = D, \quad \mathrm{Fil}^1 D = \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p(y_1 + x_2), \quad \mathrm{Fil}^2 D = 0.$$

Proposition 2.1. *There is an abelian surface $\mathcal{A}$ over $\mathbb{Q}_p$ such that $(D, \mathrm{Fil}) \simeq \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{A}))$. Further*

(a) *$\mathcal{A}$ has good reduction with special fibre isogenous to the product of two supersingular elliptic curves over $\mathbb{F}_p$*
(b) *the G-module $V_p(\mathcal{A})$ is not semisimple.*

Proof. The filtration has Hodge-Tate type $(0, 1)$ with $\dim \mathrm{Fil}^1 D = 2$ and $\det \varphi = p^2$ hence $t_H(D) = 2 = t_N(D)$. Since $\mathrm{P}_{\mathrm{char}}(\varphi)(X) = (X^2 + p)^2$ the nontrivial φ-stable subspaces of D are the $D_i = \mathbb{Q}_p x_i \oplus \mathbb{Q}_p y_i$ for $i = 1, 2$ both having Newton invariant $t_N(D_i) = 1$. However $D_1 \cap \mathrm{Fil}^1 D = \mathbb{Q}_p x_1$ whereas $D_2 \cap \mathrm{Fil}^1 D = 0$, so $t_H(D_1) = 1$ and $t_H(D_2) = 0$. Therefore (D, Fil) is admissible, D_1 is a subobject, D_2 is a quotient that is not a subobject, the short exact sequence (S) does not split and (D, Fil) is not semisimple.

The action of φ is semisimple and $\mathrm{P}_{\mathrm{char}}(\varphi) = \mathrm{P}_{\mathrm{char}}(\mathrm{Fr}_E)^2$ where E is a supersingular elliptic curve over $\mathbb{F}_p$ with $\mathrm{P}_{\mathrm{char}}(\mathrm{Fr}_E)(X) = X^2 + p$. Thus (D, Fil) satisfies condition (1) of theorem 1.2 as well as condition (a) of section 1 and it obviously satisfies (2). It remains to check condition (3) that is to find a $\delta \in \mathrm{Isom}_{\mathbb{Q}_p}(D^*, D)$ satisfying $\delta^* = -\delta$, $\varphi\delta = p\delta\varphi^{*-1}$, and $\delta(\mathrm{Fil}^1 D)^{\perp} = \mathrm{Fil}^1 D$. Let $\mathcal{B}^* = (x_1^*, y_1^*, x_2^*, y_2^*)$ be the dual basis of $\mathcal{B}$ for D^* where z^*

is the linear form on D sending $z \in D$ to 1 and vanishing on all vectors noncolinear to z. The matrix of $p\varphi^{*\,-1}$ over $\mathcal{B}^*$ is

$$p\,\mathrm{Mat}_{\mathcal{B}}(\varphi^{-1})^t = \begin{pmatrix} 0 & -1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & p & 0 \end{pmatrix}$$

where M^t is the transpose of M and

$$(\mathrm{Fil}^1 D)^\perp = \mathbb{Q}_p y_2^* \oplus \mathbb{Q}_p(y_1^* - x_2^*).$$

Let $\delta : D^* \to D$ be the $\mathbb{Q}_p$-linear morphism with matrix over the bases $\mathcal{B}^*$ and $\mathcal{B}$

$$\mathrm{Mat}_{\mathcal{B}^*\!,\mathcal{B}}(\delta) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then δ is invertible and satisfies the relations $\delta^* = -\delta$ and $\varphi\delta = p\delta\varphi^{*\,-1}$. Further $\delta(\mathrm{Fil}^1 D)^\perp = \delta\big(\mathbb{Q}_p y_2^* \oplus \mathbb{Q}_p(y_1^* - x_2^*)\big) = \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p(y_1 + x_2) = \mathrm{Fil}^1 D.$ □

Remark 2.2. Any 2-dimensional object satisfying conditions (1) and (2) of theorem 1.2 also satisfies condition (3). Hence theorem 1.2 applied to the admissible filtered φ-modules $(D_1, \mathrm{Fil}^i D \cap D_1)$ and $(D_2, \mathrm{Fil}^i D \bmod D_1)$ shows the existence of elliptic schemes $\mathcal{E}_i$ over $\mathbb{Z}_p$ such that $D_i \simeq \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{E}_i))$ for $i = 1, 2$. The special fibres of the $\mathcal{E}_i$ are $\mathbb{F}_p$-isogenous to E. Thus we obtain a nonsplit exact sequence of G-modules

$$1 \longrightarrow V_p(\mathcal{E}_2) \longrightarrow V_p(\mathcal{A}) \longrightarrow V_p(\mathcal{E}_1) \longrightarrow 1\,.$$

By Tate's full faithfulness theorem [Ta] the G-module $V_p(\mathcal{A})$ determines the p-divisible group $\mathcal{A}(p)$ over $\mathbb{Z}_p$ up to isogeny, therefore $\mathcal{A}(p)$ is not $\mathbb{Z}_p$-isogenous to $\mathcal{E}_1(p) \times \mathcal{E}_2(p)$.

Remark 2.3. The same construction works starting with the square of any supersingular p-Weil polynomial of degree two (when $p \geq 5$ there is only $X^2 + p$ but when $p = 2$ or 3 there are also the $X^2 \pm pX + p$). However it fails when dealing with the product of two distinct such. Indeed let $\alpha_1 \neq \alpha_2 \in p\mathbb{Z}_p$ and D be a semisimple 4-dimensional φ-module with $\mathrm{P}_{\mathrm{char}}(\varphi)(X) = (X^2 + \alpha_1 X + p)(X^2 + \alpha_2 X + p)$. Then $D = D_1 \oplus D_2$ with $D_i = \mathrm{Ker}(\varphi^2 + \alpha_i\varphi + p)$, which are the nontrivial φ-stable subspaces of D, and $t_N(D_i) = 1$. Since $\alpha_1 \neq \alpha_2$ one checks that any $\mathbb{Q}_p$-linear $\delta : D^* \to D$ satisfying $\delta^* = -\delta$ and $\varphi\delta = p\delta\varphi^{*\,-1}$ sends $D_2^\perp$ into D_1 and $D_1^\perp$ into D_2. Endowing D with an admissible Hodge-Tate $(0, 1)$ filtration such that (s) does not split amounts to picking a 2-dimensional subspace $\mathrm{Fil}^1 D$ such that $\dim D_1 \cap \mathrm{Fil}^1 D = 1$ and $\dim D_2 \cap \mathrm{Fil}^1 D = 0$ (or vice versa) ; then $\dim D_1 \cap \delta(\mathrm{Fil}^1 D)^\perp = 0$ and $\dim D_2 \cap \delta(\mathrm{Fil}^1 D)^\perp = 1$, therefore $\delta(\mathrm{Fil}^1 D)^\perp \neq \mathrm{Fil}^1 D$. This shows that the p-adic Tate modules of abelian schemes over $\mathbb{Z}_p$ with special fibre $\mathbb{F}_p$-isogenous to the product of two nonisogenous supersingular elliptic curves are semisimple.

Remark 2.4. One constructs in a similar fashion for each integer $n \geq 2$ a lift of the n-fold product of a supersingular elliptic curve over $\mathbb{F}_p$ with nonsemisimple p-adic Tate module.

3. A lift of a simple supersingular abelian surface

In this section we assume $p \equiv 1 \bmod 3\mathbb{Z}$ which is equivalent to $\zeta_3 \in \mathbb{Q}_p$ where ζ_3 is a primitive 3rd root of unity. Consider the filtered φ-module (D, Fil) defined as follows. There is a $\mathbb{Q}_p$-basis $\mathcal{B} = (x_1, y_1, x_2, y_2)$ for D so that

$$D = \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p y_1 \oplus \mathbb{Q}_p x_2 \oplus \mathbb{Q}_p y_2$$

is a 4-dimensional $\mathbb{Q}_p$-vector space. The matrix of φ over $\mathcal{B}$ is

$$\mathrm{Mat}_{\mathcal{B}}(\varphi) = \begin{pmatrix} 0 & \zeta_3 p & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} p \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL_4(\mathbb{Q}_p)$$

and the filtration is given by

$$\mathrm{Fil}^0 D = D, \quad \mathrm{Fil}^1 D = \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p (y_1 + x_2), \quad \mathrm{Fil}^2 D = 0.$$

Proposition 3.1. *There is an abelian surface $\mathcal{A}$ over $\mathbb{Q}_p$ such that $(D, \mathrm{Fil}) \simeq \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{A}))$. Further*

(a) *$\mathcal{A}$ has good reduction with special fibre isogenous to a supersingular simple abelian surface over $\mathbb{F}_p$*
(b) *the G-module $V_p(\mathcal{A})$ is not semisimple.*

Proof. Just as in the proof of proposition 2.1 we have $t_H(D) = 2 = t_N(D)$. Since

$$\mathrm{P}_{\mathrm{char}}(\varphi)(X) = X^4 + pX^2 + p^2 = (X^2 - \zeta_3 p)(X^2 - \zeta_3^{-1} p)$$

the nontrivial sub-$\mathbb{Q}_p[\varphi]$-modules of D are the $D_i = \mathbb{Q}_p x_i \oplus \mathbb{Q}_p y_i$ for $i = 1, 2$ both having Newton invariant $t_N(D_i) = 1$, and Hodge invariants $t_H(D_1) = 1$, $t_H(D_2) = 0$. Again we obtain a nonsplit exact sequence (S) in $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$ and (D, Fil) is not semisimple.

The action of φ is semisimple and $\mathrm{P}_{\mathrm{char}}(\varphi) = \mathrm{P}_{\mathrm{char}}(\mathrm{Fr}_A)$ where A is a supersingular simple abelian surface over $\mathbb{F}_p$ with $\mathrm{P}_{\mathrm{char}}(\mathrm{Fr}_A)(X) = X^4 + pX^2 + p^2$. Thus (D, Fil) satisfies condition (1) of theorem 1.2 as well as condition (a) of section 1. It obviously satisfies (2) and it remains to check (3). Let $\mathcal{B}^* = (x_1^*, y_1^*, x_2^*, y_2^*)$ be the dual basis of $\mathcal{B}$ for D^*. Again $(\mathrm{Fil}^1 D)^\perp = \mathbb{Q}_p y_2^* \oplus \mathbb{Q}_p (y_1^* - x_2^*)$ and the matrix of $p\varphi^{*\,-1}$ over $\mathcal{B}^*$ is

$$p\,\mathrm{Mat}_{\mathcal{B}}(\varphi^{-1})^t = \begin{pmatrix} 0 & \zeta_3^{-1} & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 \\ 0 & 0 & p & 0 \end{pmatrix}.$$

Let $\delta : D^* \to D$ be the $\mathbb{Q}_p$-linear morphism with matrix over the bases $\mathcal{B}^*$ and $\mathcal{B}$

$$\mathrm{Mat}_{\mathcal{B}^*_,\mathcal{B}}(\delta) = \begin{pmatrix} 0 & 0 & 0 & \zeta_3 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -\zeta_3 & 0 & 0 & 0 \end{pmatrix}.$$

As in the proof of proposition 2.1 one checks that δ is invertible, satisfies $\delta^* = -\delta$, $\varphi\delta = p\delta\varphi^{*\,-1}$, and that $\delta(\mathrm{Fil}^1 D)^\perp = \mathrm{Fil}^1 D$. ∎

Remark 3.2. The objects $(D_1, \mathrm{Fil}^i D \cap D_1)$ and $(D_2, \mathrm{Fil}^i D \bmod D_1)$ in $\mathbf{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$ do not arise from elliptic schemes over $\mathbb{Z}_p$, however [Ki] Thm.0.3 shows the existence of p-divisible groups $\mathcal{G}_i$ over $\mathbb{Z}_p$ such that $D_i \simeq \mathbf{D}^*_{\mathrm{cris}}(V_p(\mathcal{G}_i))$. The special fibre of $\mathcal{A}(p)$ is $\mathbb{F}_p$-isogenous to the product of the special fibres of the $\mathcal{G}_i$, themselves being nonisogenous. Thus we obtain a nonsplit exact sequence of G-modules

$$1 \longrightarrow V_p(\mathcal{G}_2) \longrightarrow V_p(\mathcal{A}) \longrightarrow V_p(\mathcal{G}_1) \longrightarrow 1$$

and Tate's full faithfulness theorem shows that $\mathcal{A}(p)$ is not $\mathbb{Z}_p$-isogenous to $\mathcal{G}_1 \times \mathcal{G}_2$.

Remark 3.3. Starting with $X^4 - pX^2 + p^2$ when $p \equiv 1 \bmod 3\mathbb{Z}$ and $X^4 + p^2$ when $p \equiv 1 \bmod 4\mathbb{Z}$ one obtains alike nonsemisimple 4-dimensional supersingular representations (just replace ζ_3 by ζ_6 or ζ_4). More generally the

$$p^d \Phi_n\Big(\frac{X^2}{p}\Big) = \prod_{i \in (\mathbb{Z}/n\mathbb{Z})^\times} (X^2 - \zeta_n^i p) \qquad \text{with } d = \#\big(\mathbb{Z}/n\mathbb{Z}\big)^\times \geq 2$$

where Φ_n is the nth cyclotomic polynomial are supersingular p-Weil polynomials leading when $p \equiv 1 \bmod n\mathbb{Z}$ to similar higher-dimensional constructions.

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