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On differential topological exponential fields

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Chapter 1

Introduction

Throughout this text we will only consider commutative characteristic 0 rings and fields.
Let R be a ring with multiplicative group of units $(U(R), ., 1)$ and let E be a group morphism from $(R, +, 0)$ to $(U(R), ., 1)$. Then (R, E) is called an exponential ring or E-ring. An E-field is an E-ring and a field. Examples are $(\mathbb{R}, \exp)$, $(\mathbb{C}, \exp)$, where $\exp : x \mapsto \sum_{i=0}^{\infty} \frac{x^i}{i!}$.
Let $D : R \to R$ be an additive group morphism that also satisfy the Leibniz rule (namely $\forall x \forall y D(xy) = xDy + yDx$)). Then D is called a derivation on R, and (R, D) a differential ring. Elements of $C_R := \{x \in R : Dx = 0\}$ are called constants and form a subring of R. A differential field is a differential ring and a field.
A differential E-field is an E-field and a differential field, for which the derivation D is also an E-derivation:

$$\forall x \quad D(E(x)) = E(x)Dx,$$

while a topological E-field is an E-field and a topological field for which E is continuous. A differential topological E-field is simply a topological E-field and a differential E-field, without any interaction between the derivation and the topology.

Recall that a theory T is said to be model-complete if every embedding of models of T is elementary A.0.1, equivalently if all models of T are existentially closed in T.
We consider first-order model-complete theories T of topological E-fields and $T_D := T \cup \{D \,\text{is an}\, E - \text{derivation}\}$ of their differential expansions.

We show that the subclass of existentially closed models of T_D satisfy a geometric scheme, in the spirit of [51], where D.Pierce and A.Pillay, propose alternative geometric axioms to the previous axiomatization by L.Blum [9] of DCF_0–the theory of differentially closed fields of characteristic 0–.

Later in [48], C.Michaux and C.Rivière adapt those geometric axioms to the ordered case, in order to axiomatize $CODF$, the theory of closed ordered differential fields, while in [64], M.Tressl shows there is a theory of differential fields (in several commuting derivatives) of characteristic 0, which is a model-companion for every theory of large differential fields extending a model-complete theory of pure fields. N.Guzy and F.Point extend the geometric axioms to theories of topological fields–with possibly extra structure–that admit a model-completion, to axiomatize the model-completion of their differential expansion ([23]). We adapt these axioms further to a "differential lifting" scheme $(DL)_E$.

A.Wilkie has shown model-completeness of the theory $T_{\mathbb{R},\exp}$ of the ordered exponential field of real numbers $(\mathbb{R}, \exp, <)$ [65]; while N.Mariaule has shown model-completeness of the theory $T_{\mathcal{O}_p,E_p}$ of the valuation ring $\mathcal{O}_p$ of the valued field of complex p-adic numbers $\mathbb{C}_p$ with the p-adic exponential $E_p : x \mapsto \exp(px)$.

In [18], L.van den Dries, D.Marker and A.Macintyre construct $\mathbb{R}((t))^{LE}$, a 'logarithmic-exponential' field of generalised power series. As an ordered E-field, it is an elementary extension of $(\mathbb{R}, \exp)$ ([19]); it also contains the field of Laurent series $\mathbb{R}((t))$. They endow $\mathbb{R}((t))^{LE}$ with an E-derivation ∂ that admits $\mathbb{R}$ as field of constants. As a differential field, $(\mathbb{R}((t))^{LE}, \partial)$ is naturally involved in asymptotic differential algebra (see M.Aschenbrenner, L.van den Dries, J.van der Hoeven's [1]). Obviously it is not a model of $CODF$, as this would imply the density of $\mathbb{R}$ in $\mathbb{R}((t))^{LE}$.

Let E-polynomials be terms of the language $\{+, -, ., E\}$, and E-varieties be zero sets of E-polynomials. These objects have already been well studied, in analytic contexts–see for example A.Wilkie [65], T.Servi [61, 60]– as well as in algebraic contexts–see for example L.van den Dries [15], G.Terzo [63].

The notion of E-algebraicity introduced by A.Macintyre in the 80s is more complex than the notion of algebraicity: a point a in an E-field $(L,E) \supseteq (K,E)$ is E-algebraic over K if it is in a projection of a regular variety defined over K by a squared system of E-polynomials. J.Kirby [28] has shown that it defines a closure operator ecl to which one can associate a good notion of dimension, by relying it to another operator defined by formal E-derivations.
He also extends E-derivations on E-field extensions for which a Schanuel's conjecture is true, as this allows to avoid hidden exponential-algebraic relations.

Schanuels's conjecture for $\mathbb{C}$ states that if $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then $\mathbb{Q}(\lambda_1, \dots, \lambda_n, e^{\lambda_1}, \dots, e^{\lambda_n})$ has (algebraic) transcendence degree at least n over $\mathbb{Q}$. This conjecture is far from being proven and would imply the algebraic independence of e and π. The decidability of $T_{\mathbb{R},exp}$ relies on this conjecture restricted to $\mathbb{R}$ and is actually equivalent to a weaker form of it, see A.Wilkie and A.Macintyre's proof in [41].

In order to state the main results, let (K,E,D) be a differential E-field, $n \in \mathbb{N}\backslash\{0\}$, $\bar{X} := X_1, \cdots, X_n$ and $\frac{\partial}{\partial X_i}$ denote the usual partial derivative with respect to the variable X_i. Similarly to the algebraic case, we define the torsor of an E-variety $A \subseteq K^n$ as the set:

$$\tau(A) := \{(\bar{a}, \bar{v}) \in K^{2n} : \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a})v_i + P^D(\bar{a}) = 0 \text{ for all } P(\bar{X}) \in I(A)\}$$

where $I(A)$ is the ideal of E-polynomials that vanish on all points of A, and $P^D(\bar{X})$ is an E-polynomial defined by induction on the number of iterations of E in P (if $P(\bar{X}) \in K[\bar{X}]$, then $P^D(\bar{X})$ is the polynomial obtained when applying D to the coefficients of P).

Let $\mathcal{V}$ be a base of neighborhoods of 0 on K, such that $(K,E,D,\mathcal{V})$ is a differential topological E-field. Let A^{reg} be the set of regular zeros of the variety A.
Given $A \subseteq K^{2n}$, $B \subseteq K^n$, we say that a point $\bar{a} \in A^{reg}$ that projects on a point in B^{reg} virtually projects generically on B, if, roughly speaking, one can find a point close to $\bar{a}$ in A^{reg} that projects on a generic point in

B^{reg}, in some extension of $(K, E, \mathcal{V})$.
$(DL)_E$ expresses that a differential E-polynomial system has a zero close to a regular zero of an associated E-polynomial system:

Definition 1.0.1 *We say that $(K, E, D, \mathcal{V})$ satisfies the scheme $(DL)_E$ if:*

for any $U \in \mathcal{V}$,
for any E-varieties $A \subseteq K^{2n}$, $B \subseteq K^n$ defined over K as zero-sets of finitely many E-polynomials, such that $A^{reg} \subseteq \tau(B)$, and that there is a tuple $(\bar{a}, \bar{c}) \in K^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$, that virtually projects generically on B then

there is $\bar{b} \in K^n$ such that $(\bar{b}, D\bar{b}) \in A^{reg}$ and

$$(\bar{a}, \bar{c}) - (\bar{b}, D\bar{b}) \in U^{2n}$$

If $(K, E, D, \mathcal{V})$ satisfies the scheme $(DL)_E$, then the subfield of constants C_K is dense in K.

We show that if T is model-complete and its models satisfy either a hypothesis we call $(I)_E$, which imply that the underlying fields are large, or an implicit function theorem, and if the topology is definable, then:

Theorem 1.0.2 *The models of T_D that are existentially closed satisfy $(DL)_E$.*

Our results apply to differential expansions of models of $T_{\mathbb{R},\exp}$ and of $T_{\mathcal{O}_p,E_p}$:

Corollary 1.0.3 *The existentially closed models of T_D, where $T := T_{\mathbb{R},\exp}$ or $T_{\mathcal{O}_p,E_p}$, satisfy $(DL)_E$.*

Finally we extend a result of Q.Brouette [10] to endow $(\mathbb{R}, \exp, <)$, and $(\mathbb{R}((t))^{LE}, \exp, <)$ with an E-derivation D that makes them models of $(DL)_E$.

To construct the tools used in the proof of Theorem 1.0.2, we heavily rely on J.Kirby's characterization of *ecl* by formal E-derivations:

1. Given (K, E, D) and (L, E) such that $(K, E) \subseteq (L, E)$, we show that the E-derivation D extends to (L, E).

2. We adapt algebraic results on torsors to the exponential setting.

3. Using also an adaptation of the notion of large fields (F.Pop [55]), to our exponential context, and Hensel's Lemma in Laurent series, we construct, given (K, E), and V a regular E-variety defined on K, an elementary extension (L, E) of (K, E), and generic points of V in L.
   Alternatively we construct generic points of V assuming an implicit function theorem instead of large fields and Hensel's lemma in Laurent series.

On our way, we also investigate some problems appearing naturally, although the extra results are not involved in Theorem 1.0.2's proof. We show some Nullstellensätze for E-fields in Chapter 5, we construct an E-field containing Laurent series over an unordered E-field in 3.3.3, we notice a few results on E-varieties defined over E-fields satisfying an implicit function theorem in 6.3, we adapt some results on torsors from D.Pierce and A.Pillay [51] to the exponential context in the second part of 6.2, and show some saturation results in Appendix B.

This thesis is organised in several parts. Chapters 4, 5 and 6 are mainly algebraic and do not really involve model theory.

In Chapter 3 we look at classes of topological E-fields in which we have an implicit function theorem, in order to encompass models of $T_{\mathbb{R},\exp}$ and of $T_{\mathcal{O}_p,E_p}$. We then state a Hensel's Lemma for regular systems of E-polynomials in Laurent series $K((\bar{t}))$. As the latter cannot be endowed with the structure of an E-field, starting from a topological unordered E-field $(K, E, \mathcal{V})$, we construct a topological unordered E-field of power series $(K((t))^E, E, \mathcal{W})$ containing $K((t))$ adapting the construction of L.van den Dries, D.Marker and A.Macintyre [18] done in the ordered case.

Chapter 4 is heavily based on [28]. Starting from $(K, E) \subseteq (L, E)$, and D an E-derivation on K, we:

- use the characterization of *ecl* by formal E-derivations to link the *ecl*-dimension over K of a tuple $\bar{a} \subseteq L^n$ to the linear dimension of spaces of E-derivations,

- construct an elementary extension of K in which we have infinitely small elements relatively to K, that are E-algebraic independent, by constructing linearly independent E-derivations in some chosen suitably saturated elementary extension,

- simply adapt a classical algebraic result to the exponential context to show D extends on $ecl^L(K)$,

- adapt J.Kirby's notion of strong extensions of partial E-field–in which a Schanuel property is satisfied–to extend D from $ecl^L(K)$ to L. A problem is that J.Kirby works with fields in which E is partially defined on a subgroup of the additive group of K, and that the latter has to be a $\mathbb{Q}$-vector space. But we want to encompass the cases of $\mathbb{Q}_p$ and $\mathbb{C}_p$, in which E_p is defined on the valuation rings $\mathbb{Z}_p$ and $\mathcal{O}_p$.

Chapter 6 is about E-varieties. We use the results of Chapter 4 to:

- construct generic points of regular E-varieties, using the 2d item in the presentation of Chapter 4 above and then Hensel's Lemma in $K((\bar{t}))$, and

- avoid the difficulty given by the non-noetherianity of $R[\bar{X}]^E$ when we

- extend results on algebraic torsors to E-torsors.

Then we consider E-varieties in topological E-fields (K, E) that satisfy an implicit function theorem. In particular we also construct generic points of regular E-varieties in an elementary extension of (K, E).

Chapter 7 is in three parts. We first state a hypothesis we call $(I)_E$ that will imply, when T is model-complete, that the underlying field K of a model of T is large, in other words existentially closed in $K((\bar{t}))$. Then we construct generic points of a regular E-variety defined on K in an elementary extension of K containing $K((\bar{t}))$, using the first item in the description of Chapter 6 above, and $(I)_E$. We show that $\aleph_1$-saturated

models of $T_{\mathbb{R},\exp}$ satisfy $(I)_E$.
Finally we show that if $(K, E, \mathcal{V})$ satisfies either $(I)_E$ or an implicit function theorem then $(K, E, D, \mathcal{V})$ can be embedded in a differential topological E-field satisfying a pre-scheme $(DL)_E$, using results of Chapters 4 and 6. We conclude with the proof of Theorem 1.0.2, Corollary 1.0.3 and endow $(\mathbb{R}, \exp, <)$, and $(\mathbb{R}((t))^{LE}, \exp, <)$ with an E-derivation D that makes them models of $(DL)_E$.

Chapter 5 is independent of all other chapters except Chapter 2, Section 2.1. We show some versions of Strong and Weak Nullstellensätze for E-fields (K, E), as well as a version of a Real Nullstellensatz for ordered E-fields $(K, E, <)$. A problem is that the E-ring $(R[\bar{X}]^E, E)$ of E-polynomials over a given E-ring (R, E) is not a Hilbert ring. Another question is to contruct a maximal ideal M which is also an E-ideal, that is which satisfies

$$P \in M \to E(P) - 1 \in M$$

We need M to be maximal as an (algebraic non-E-) ideal. We use iterative constructions of E-ideals and augmentation ideals within group rings of E-polynomials.

While we were finishing this thesis, A.Fornasiero and A.Kaplan [21] posted on ArXiv an axiomatization of the existentially closed models of a differential extension of a complete, model complete o-minimal theory extending the theory RCF of real closed ordered fields (for example $T_{\mathbb{R},\exp}$).

Chapter 2

Setting

We will assume that basic notions of first-order logic and model theory are known, namely, given a language $\mathcal{L}$: $\mathcal{L}$-terms, $\mathcal{L}$-formula, $\mathcal{L}$-sentence, $\mathcal{L}$-theory, $\mathcal{L}$-structure, $\mathcal{L}$-definable set, $\mathcal{L}$-embedding, as well as ordinals, cardinals and basic set theory notions.We refer the reader to [46, Section 1, p.7-32 & Appendix A, p.315] for example. The following model theoretic notions: elementary extensions, saturated models, existentially closed models, model complete theories, are recalled in Appendix A.

Recall that throughout this text we will only consider commutative characteristic 0 rings and fields.

Let $\mathcal{L}_{rings} = \{+, -, ., 0, 1\}$, $\mathcal{L}_{fields} = \mathcal{L}_{rings} \cup \{^{-1}\}$.

Definition 2.0.1 Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$. A *topological $\mathcal{L}$-structure* (M, τ) is a first-order $\mathcal{L}$-structure with a Hausdorff topology τ such that every n-ary function symbol of $\mathcal{L}$ is interpreted by a continuous function from M^n to M, and every m-ary relation symbol of $\mathcal{L}$ and its complement is interpreted by the union of an open subset of M^m and a set of zeros of $\mathcal{L}$-terms (M^n and M^m are endowed with the product topology).

As M has the underlying structure of an additive group, a fundamental system $\mathcal{V}$ of neighborhoods of 0 determines the topology: for each $x \in M$, $x + \mathcal{V}$ is a fundamental system of neighborhoods of x.

Fact 2.0.2 [6, III.4 & III.49] *Let R be a ring and a topological space. R is a topological ring if and only if the filter $\mathcal{V}$ of neighborhoods of* 0 *in R satisfies all the following conditions:*

(TG1) For all $U \in \mathcal{V}$, there is $V \in \mathcal{V}$ such that $V + V \subseteq U$.

(TG2) For all $U \in \mathcal{V}$, we have that $-U \in \mathcal{V}$.

(TM1) For all $x_0 \in R$ and $V \in \mathcal{V}$, there is $W \in \mathcal{V}$ such that $x_0 W \subseteq V$ and $W x_0 \subseteq V$.

(TM2) For all $V \in \mathcal{V}$, there is $W \in \mathcal{V}$ such that $WW \subseteq V$.

Definition 2.0.3 Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$. A *topological $\mathcal{L}$-field* (M, τ) is a $\mathcal{L} \cup \{^{-1}\}$-structure and a topological $\mathcal{L}$-structure for which the inverse function $^{-1}$ is continuous on $M^\times := M \setminus \{0\}$.

For example a field K endowed with an absolute value $|.| \to \mathbb{R}^{\geq 0}$ is a topological $\mathcal{L}_{rings}$-field with as basis of neighborhoods of 0 the sets

$$\{a \in K : |a| < b\}$$

with $b \in \mathbb{R}^{>0}$.

Definition 2.0.4 Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$. Let K be an $\mathcal{L}$-structure expanding a field. If there is a $\mathcal{L}$-formula $\phi(x, \overline{y})$ such that the set of subsets of the form

$$\phi(K, \overline{a}) := \{x \in K \ : \ K \vDash \phi(x, \overline{a})\}$$

where $\overline{a} \subseteq K$, can be chosen as a basis $\mathcal{V}$ of neighborhoods of 0 in K in such a way that $(K, \mathcal{V})$ is a topological field, we will say that we have a *definable $\mathcal{L}$-topology* on K, and that $(K, \mathcal{V})$ satisfies *Hypothesis $\mathcal{D}$*.

Set $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\} \cup \{R_i, i \in I\}$, where R_i, $i \in I$ are relations symbols, a first-order language, and $\mathcal{L}_D := \mathcal{L} \cup \{D\}$.

Our framework being characteristic 0 differential partial E-fields (where the exponential can be only defined on a subring), endowed with a definable topology for which E is continuous, we first set our algebraic context of differential partial E-fields before going back to specific differential topological $\mathcal{L}$-structures (which we do not call topological $\mathcal{L}_D$-structures as we do not require continuity of the derivation).

2.1 Exponential and differential rings and fields

2.1.1 Partial E-fields

Recall that (R, E) is an E-ring if R is a ring endowed with a group morphism E from $(R, +, 0)$ to its multiplicative group of units $(U(R), ., 1)$.

Definition 2.1.1 A *partial E-field* is a two-sorted structure

$$\langle K, R; +_K, ._K, +_R, ._R, i, E_R \rangle$$

where $\langle K, +_K, ._K \rangle$ is a field, $\langle R, +_R, ._R, E_R \rangle$ is an E-ring;

$$i : \langle R, +_R, ._R \rangle \to \langle K, +_K, ._K \rangle$$

is an injective homomorphism of rings; R is identified with its image under i; and $+_R$ and $+_K$ are both written $+$; $._R$ and $._K$ are both written $.$, E_R is written E.

We will only consider partial E-fields where R is a subring of K; let us denote them (K, R, E). Recall that (K, D) is a differential field if K is a field endowed with a mapping $D : K \to K$ that is an additive group morphism which satisfies Leibniz rule.

Definition 2.1.2 An *E-derivation* on a partial E-field (K, R, E) is a derivation D on K such that for all $x \in R$,

$$D(E(x)) = E(x)Dx$$

We call a partial E-field equipped with an E-derivation a *differential partial E-field*.

Notice that there are several definitions of partial E-fields in the literature, ours is different than for example, the definition of partial E-domains of J.Kirby in [28], as we assume, contrary to [28], that the domain of definition of E is a ring, but not especially a $\mathbb{Q}$-vector space, and that the image of E is included in its domain.

Definition 2.1.3 [28, Definition 2.2] A *partial E-domain* is a two-sorted structure

$$\langle K, R; +_K, ., +_R, (q.)_{q\in\mathbb{Q}}, \alpha, E_R\rangle$$

where $\langle K, +_K, .\rangle$ is a $\mathbb{Q}$-algebra, $\langle R, +_R, (q.)_{q\in\mathbb{Q}}\rangle$ is a $\mathbb{Q}$-vector space,

$$\alpha : \langle R, +_R\rangle \to \langle K, +_K\rangle$$

is an injective homomorphism of additive groups and

$$E_R : \langle R, +_R\rangle \to \langle K, .\rangle$$

is a homomorphism. R is identified with its image under α, and $+_R$ and $+_K$ are both written $+$.

2.1.2 Exponential and differential polynomials

Let (R, E) be an E-ring.

The structure $R[\bar{X}]^E$ of E-polynomials in n indeterminates over the E-ring (R, E) is constructed as a group ring over the ring of polynomials $R[\bar{X}]$, while the exponential map is extended 'step by step', allowing to 'count' its number of iterations. It has a natural E-ring structure. We recall here the construction that the reader can find in [15] for more details.

- Let
$$R_{-1} := R, \quad R_0 := R[\bar{X}], \quad A_0 := \bar{X}R[\bar{X}]$$
Consequently
$$R_0 = R \oplus A_0 = R_{-1} \oplus A_0$$
E is extended as follows: $E_{-1} : R_{-1} \to R_0$ is defined as the composition of the inclusion $i : R \hookrightarrow R[\bar{X}]$ with the exponential E of the E-ring R.

- Let $k \geq 0$, and suppose that R_{k-1}, R_k, A_k and E_{k-1} have been constructed and verify
$$R_k = R_{k-1} \oplus A_k$$
Let $\exp(A_k)$ be a multiplicative copy of the additive group A_k.

Let R_{k+1} be the group ring of the multiplicative group $\exp(A_k)$ over the ring R_k:

$$R_{k+1} := R_k[\exp(A_k)]$$

(its elements are thus linear combinations of $\exp(a)$ for $a \in A_k$, with coefficients in $R_k = R_{k-1} \oplus A_k$)

Furthermore let A_{k+1} be the R_k-submodule of R_{k+1} freely generated by the $\exp(a)$, $a \in A_k \setminus \{0\}$. $R_{k+1} = R_k \oplus A_{k+1}$ as additive groups.

Now, in order to define E_k, let $r \in R_k$. As $r = p + a$, with $p \in R_{k-1}$ and $a \in A_k$, it is possible to define

$$\begin{array}{llll} E_k : & R_k & \to R_{k+1} & \\ & r & \mapsto E_k(r) & = E_{k-1}(p)\exp(a) \end{array}$$

The underlying ring of $R[\bar{X}]^E$ will be taken as $\bigcup R_k$. E is given by $E(x) = E_k(x)$ if $x \in R_k$. Hence $R[\bar{X}]^E$ is an E-ring extension of R. As a group ring over $R[\bar{X}]$, $R[\bar{X}]^E = \left(R[\bar{X}]\right)[\exp(A_0 \oplus A_1 \oplus \cdots \oplus A_k \oplus \cdots)]$.

Note that the free E-ring on the indeterminates $\bar{X} := X_1, \cdots, X_m$, corresponds to $[\overline{X}]^E = \mathbb{Z}[\overline{X}]^E$.

Fact 2.1.4 [15, Proposition 1.6] *If R is an integral domain of characteristic 0, then $R[\bar{X}]^E$ is an integral domain whose units are of the form $u.E(p)$, u a unit of R, $p \in R[\bar{X}]^E$.*

There is a notion *ord* of *degree of an E-polynomial* (see for example [15, 1.9]):

Definition 2.1.5 Let

$$P = P_0 + \sum_{i=1}^{k} P_i \in R[X]^E = R_0 \oplus A_1 \oplus \cdots \oplus A_k \oplus \cdots$$

with $P_0 \in R[X]$, $P_i \in A_i$, $i > 0$. There is $k \geq 0$, $P \in R_k \setminus R_{k-1}$, let us first define the *height* of P by

$$h(P) := k$$

Then P of height k can be written uniquely as $P = P_0 + \sum_{i=1}^{k} P_i$, hence we let

$$\begin{array}{rccc} ord: & R[X]^E & \to & On \\ & P_0 + \sum_{i=1}^{k} P_i & \mapsto & \sum_{i=1}^{k} \omega^i t(P_i) + t(P_0) \end{array}$$

where

- $t(P) = 0$ if $P = 0$
- $t(P) = deg_X(P) + 1$ if $P \in R[X] \setminus \{0\}$
- $t(P) = d$ if $P = \sum_{i=1}^{d} r_i.E(a_i) \in A_k = R_{k-1}[\exp(A_{k-1} \setminus \{0\})]$, $k > 0$

and On is the class of ordinals.

Fact 2.1.6 [15, Lemma 1.10] *If $P_0 = 0$, there is $Q \in R[X]^E$ s.t.*

$$ord(E(Q).P) < ord(P)$$

Let $\frac{\partial}{\partial X} : r \in R \mapsto 0, X \mapsto 1$; the usual partial differentiation.

Fact 2.1.7 [39, Th.16 p.199] *If $P \in R[X]^E$, there is $Q \in R[X]^E$ s.t. $ord(Q) < ord(P)$ and*

$$ord\left(\frac{\partial(E(Q).P)}{\partial X}\right) < ord(P)$$

Now let (R, E, D) be an E-differential ring.

For $1 \leq i \leq n =: |\bar{X}|$, let $X_i^{(0)} := X_i$ and for $j \in \mathbb{N}$, let $X_i^{(j+1)} := DX_i^{(j)}$. This formally endows the ring of polynomials $R[\bar{X}^{(0)}, \bar{X}^{(1)}, \cdots]$ with a derivation. This differential ring is denoted $R\{\bar{X}\}$ and is called the ring of differential polynomials in variables $\bar{X} := X_1, \cdots, X_n$.

In [15], L.van den Dries extends a given E-derivation of R to an E-derivation on $R[\bar{X}]^E$, with $DX_i = 1$, $1 \leq i \leq n$, uniquely such that the constants are exactly R (see Lemma 3.2 and Proposition 3.4 in [15]). The derivation maps R_k onto itself, $k \geq 1$. Although we do not require that $DX_i = 1$, $1 \leq i \leq n$, we can extend the derivation D of $R[\bar{X}^{(0)}, \bar{X}^{(1)}, \cdots]$, where $D_{|R}$ is an E-derivation, to an E-derivation on $R[\bar{X}^{(0)}, \bar{X}^{(1)}, \cdots]^E$.

For $\forall i, j \in \{1, \cdots, |\bar{X}|\}$, let $\frac{\partial}{\partial X_i} : r \in R \mapsto 0, X_j \mapsto \delta_{ij}$; the usual partial differentiation.

Recall that if $P(\bar{X}) \in R[\bar{X}]$, then $DP(\bar{X}) = P^D(\bar{X}) + \sum_{i=1}^{|\bar{X}|} \frac{\partial P}{\partial X_i}(\bar{X}) DX_i$ in $R\{\bar{X}\}$, where $P^D(\bar{X})$ is the polynomial obtained when applying D to the coefficients of P.

If $P(\bar{X}) \in R[\bar{X}]^E$, let us define $P^D(\bar{X})$ by induction on the number of iterations of E in P, given $D : R \to R$ an E-derivation: suppose if $P(\bar{X}) \in R_k$, then $P^D(\bar{X})$ has been defined, and let $P(\bar{X}) \in R_{k+1}$. Thus

$$P(\bar{X}) = Q_0(\bar{X}) + \sum_{j=1}^{p} Q_{1,j}(\bar{X}) E(Q_{2,j}(\bar{X}))$$

with for all $j = 1, \cdots, p$, Q_0, $Q_{1,j}$ in R_k and $Q_{2,j}$ in $R_k \setminus \{0\}$. Let

$$P^D(\bar{X}) := Q_0^D(\bar{X}) + \sum_{j=1}^{p} Q_{1,j}^D(\bar{X}) E(Q_{2,j}(\bar{X})) + \sum_{j=1}^{p} Q_{1,j}(\bar{X}) Q_{2,j}^D(\bar{X}) E(Q_{2,j}(\bar{X}))$$

Lemma 2.1.8 *Let (R, E) be an E-ring,*

$$P(\bar{X}) \in R[\bar{X}]^E \subseteq R[\bar{X}^{(0)}, \bar{X}^{(1)}, \cdots]^E$$

and D an E-derivation on R. Then D extends on $R\{\bar{X}\}$ and there is exactly one E-derivation extending D on $R[\bar{X}^{(0)}, \bar{X}^{(1)}, \cdots]^E$ such that

$$DP(\bar{X}) = P^D(\bar{X}) + \sum_{i=1}^{|\bar{X}|} \frac{\partial P}{\partial X_i}(\bar{X}) DX_i$$

Proof. By induction on the number of iterations of E in P. True if $P \in R[\bar{X}]$. Suppose if $P(\bar{X}) \in R_k$, then $DP(\bar{X}) = P^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{X}).DX_i$, and let $P(\bar{X}) \in R_{k+1}$. Thus

$$P(\bar{X}) = Q_0(\bar{X}) + \sum_{j=1}^{p} Q_{1,j}(\bar{X}) E(Q_{2,j}(\bar{X}))$$

with for all $j = 1, \cdots, p$, Q_0, $Q_{1,j}$ in R_k and $Q_{2,j}$ in $R_k \setminus \{0\}$. To be an E-derivation, D must satisfy:

$$\begin{aligned}
DP(\bar{X}) &= DQ_0(\bar{X}) + \sum_{j=1}^{p} \left[DQ_{1,j}(\bar{X})E(Q_{2,j}(\bar{X})) + Q_{1,j}(\bar{X})DQ_{2,j}(\bar{X})E(Q_{2,j}(\bar{X}))\right] \\
&= Q_0^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial Q_0}{\partial X_i}(\bar{X}).DX_i \\
&+ \sum_{j=1}^{p} \left[Q_{1,j}^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial Q_{1,j}}{\partial X_i}(\bar{X}).DX_i\right] E(Q_{2,j}(\bar{X})) \\
&+ \sum_{j=1}^{p} Q_{1,j}(\bar{X}) \left[Q_{2,j}^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial Q_{2,j}}{\partial X_i}(\bar{X}).DX_i\right] E(Q_{2,j}(\bar{X})) \\
&= Q_0^D(\bar{X}) + \sum_{j=1}^{p} Q_{1,j}^D(\bar{X})E(Q_{2,j}(\bar{X})) \\
&+ \sum_{j=1}^{p} Q_{1,j}(\bar{X})Q_{2,j}^D(\bar{X})E(Q_{2,j}(\bar{X})) \\
&+ \sum_{i=1}^{n} \left[\frac{\partial Q_0}{\partial X_i}(\bar{X}) + \sum_{j=1}^{p} \left(\frac{\partial Q_{1,j}}{\partial X_i}(\bar{X}) + Q_{1,j}(\bar{X})\frac{\partial Q_{2,j}}{\partial X_i}(\bar{X})\right) E(Q_{2,j}(\bar{X}))\right] .DX_i
\end{aligned}$$

i.e.

$$DP(\bar{X}) = P^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{X}).DX_i$$

Unicity of the derivation is obvious. ■

Let us denote $R\{\bar{X}\}^E$ the E-ring $R[\bar{X}^{(0)}, X^{(1)}, \cdots]^E$ endowed with such a derivation.

2.1.3 E-polynomial functions

Let (R, E) be an E-ring. Recall ([15, (1.5)]) that a set I gives rise to the E-ring R^I of functions $I \to R$ where the operations are defined pointwise. For $I \neq \emptyset$ we identify the constant functions with their values in R, so $R \subseteq R^I$. If $I = R^m$, then the coordinate functions are denoted by $x_1, \cdots, x_m$ where $x_i(r_1, \cdots, r_m) = r_i$. The E-ring morphism $R[X_1, \cdots, X_m]^E \to R^{R^m}$ fixing R and sending each X_i to x_i will be indicated by $P \mapsto \tilde{P}$. The $\tilde{P}$'s are called E-polynomial functions (in m

variables), let us denote by $R[x_1, \cdots, x_m]^E$ these E-polynomial functions $R^m \to R$.

Fact 2.1.9 [15, Proposition 4.1] *Suppose the E-ring R is an integral domain of characteristic 0, that there is a non-zero $r \in R$, and that there are derivations $d_1, \cdots, d_m$ on a ring extension of $R[x_1, \cdots, x_m]^E$ which are trivial on R and satisfy, for $1 \leq i, j \leq m$ and for all $f \in R[x_1, \cdots, x_m]$:*

$$d_i(x_j) = \delta_{ij} \text{ and } d_i(E(f)) = r.d_i(f).E(f)$$

Then the map $P \mapsto \tilde{P}$, $R[X_1, \cdots, X_m]^E \to R[x_1, \cdots, x_m]^E$, is an isomorphism.

Fact 2.1.10 [15, Proposition 4.4] *Suppose R is an ordered E-field and its exponential map E satisfies $E(x) \geq 1 + rx$ for a fixed non-zero $r \in R$ and all $x \in R$. Then the map $P \mapsto \tilde{P}$, $R[X_1, \cdots, X_m]^E \to R[x_1, \cdots, x_m]^E$, is an isomorphism.*

Fact 2.1.11 [15, Lemma 2.3] *Let $Q(\bar{Y}) \in R[\bar{Y}]^E$. Then there is $m \geq 0$ and $\bar{X} = X_1, \cdots, X_m$ such that $Q(\bar{Y}) = P(\bar{r}, \bar{Y})$ for some $P(\bar{X}, \bar{Y}) \in \mathbb{Z}[\bar{X}, \bar{Y}]^E$ and $\bar{r} \in R^m$.*

Fact 2.1.12 [15, 1.8] *Suppose we have $m+n$ indeterminates $X_1, \ldots, X_m$, $X_{m+1}, \ldots, X_{m+n}$. There is a unique $R \cup \{X_1, \ldots, X_{m+n}\}$-fixing E-ring isomorphism*

$$R[X_1, \ldots, X_{m+n}]^E \cong (R[X_1, \ldots, X_m]^E)[X_{m+1}, \ldots, X_{m+n}]^E$$

2.1.4 E-algebraicity

Let (K, R, E) be a partial E-field.

We would like to consider E-polynomials defined over K, supposedly applying the associated exponential polynomial functions only to elements of R, following J.Kirby [28].
We will either consider elements of $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ the tensor product of $K[\bar{X}]$ and $R[\bar{X}]^E$ over $R[\bar{X}]$, or we may, using the facts of Subsection

2.1.3, consider E-polynomials $P(\bar{X}, \bar{Y})$ of $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]^E$, together with parameters $b_1, \cdots, b_m$ lying in R or K, in such a way that the associated exponential polynomial function

$$\bar{X} \mapsto P(\bar{X}, \bar{b})$$

is well-defined. The former option is less general than the latter but it allows to simplify notations.
Moreover, and still in order to avoid unnecessary heavy formalism, for $P(\bar{X}, \bar{b})$ an E-polynomial of $\mathbb{Z}[\bar{X}\bar{Y}]^E$ with parameters $\bar{b}$ such that $b_1, \cdots, b_p \in R$ and $b_{p+1}, \cdots, b_n \in K \setminus R$, we write

$$P(\bar{X}, \bar{b}) \in \mathbb{Z}[\bar{b}]^E[\bar{X}]^E \text{ instead of}$$

$$P(\bar{X}, \bar{b}) \in \mathbb{Z}[b_{p+1}, \cdots, b_n][b_1, \cdots, b_p]^E[\bar{X}]^E$$

A.Macintyre introduced as a definition of E-algebraicity the fact of being in a projection of a certain E-algebraic variety [39, Definition 5 & 6], namely being a coordinate of a tuple solution of a Hovanskii system of E-polynomials:

Definition 2.1.13 Let $B \subseteq K$. An element $a \in K$ is said to be *E-algebraic over* B, or *ecl*-dependant over B in K if there exists $n, m \in \mathbb{N}$, $b_1, \cdots, b_m \in B$, $a_1, \cdots, a_n \in K$ and $h_1, \ldots h_n \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]^E$, where $a = a_1$ and with equations

$$h_i(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0 \text{ for } i = 1, \ldots, n$$

and the inequation

$$\begin{vmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \frac{\partial h_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial X_1} & \cdots & \frac{\partial h_n}{\partial X_n} \end{vmatrix} (a_1, \ldots, a_n, b_1, \ldots, b_m) \neq 0$$

If $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}_p$, this corresponds to being a coordinate of a tuple of an isolated zero of the squared system defined by the h_i's, $1 \leq i \leq n$; the latter, together with the inequation, is called a Hovanskii system.
We will either denote by $H_H(\bar{X}, \bar{b})$ this Hovanskii system, defined by

$H = (h_1, \cdots, h_n)$, or we will abuse notation and denote by H both the system and the tuple of E-polynomials.
With this notion, the closure $ecl^R(B)$ of any subset $B \subseteq R$, that is to say the set of elements of R that are E-algebraic over B, is an E-subring of R, while the closure $ecl^K(B)$ is an E-subfield of K; whereas, as noticed by A.Macintyre, simply being a zero of an E-polynomial would not be a good notion of exponential algebraicity: indeed the sum of two such zeros is not necessarily a zero of another E-polynomial.

Elements of R that are not in $ecl^R(B)$ are said to be E-transcendental or ecl-independant over B in R.

In case we simplify notation by considering systems of E-polynomials in the tensor product, $a \in K$ would be said to be E-algebraic over B if there exists $n \in \mathbb{N}$, $a_1, \cdots, a_n \in K$ with $a = a_1$ and a Hovanskii system $H_H(X_1, \cdots, X_n)$ defined by $H = (h_1, \ldots h_n) \subseteq K[X_1, \ldots, X_n] \otimes_{R[X_1, \ldots, X_n]} R[X_1, \ldots, X_n]^E$, such that $a_1, \cdots, a_n$ is a solution of H_H.

2.1.5 E-ideals

Let (R, E) be an E-ring.

Definition 2.1.14 *$I \subseteq R$ is an E-ideal iff I is an ideal of the ring R such that:*

$$r \in I \to E(r) - 1 \in I$$

If $I \subseteq R$ is an E-ideal we have a well-defined exponential on the quotient R/I, given by:

$$E(r + I) := E(r) + I$$

where $r \in R$. This allows to endow R/I with an E-ring structure.

If $I \subseteq R$ is an ideal, let $I_E \subseteq R$ be the E-ideal generated by I.

Let $P_1, \cdots, P_m \in R[\bar{X}]^E$. Denote $\langle P_1, \cdots, P_m \rangle$ the ideal of the ring $R[\bar{X}]^E$ generated by $P_1, \cdots, P_m$, and consider $\langle P_1, \cdots, P_m \rangle_E \subseteq R[\bar{X}]^E$. The latter is not necessarily finitely generated as an ideal: actually, when the ring R is Noetherian (all its ideals are finitely generated), then the ring of polynomials $R[\bar{X}]$ is Noetherian too, hence every ideal is finitely generated. But this is not true anymore in $R[\bar{X}]^E$. For a counterexample, see [63] p.17, or [39], 3.2 p.204.

Let $(K, R, E) \subseteq (L, R', E)$ be an extension of partial E-fields.

We will say that $I \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ is an E-ideal if I is an ideal of $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ such that for $P \in R[\bar{X}]^E$, $P \in I \Rightarrow E(P) - 1 \in I$.

Let $P_1, \cdots, P_m$ in $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, and $I := \langle P_1, \cdots, P_m \rangle$.
Let $V(I) = V(P_1, \cdots, P_m) \subseteq L^n$ be the set of common zeros of elements of I.

If $W \subseteq L^n$, let $I(W) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ be the ideal of E-polynomials that vanish on all of W, it is an E-ideal as an annulator.

Let us denote $IVI := I(V(I))$.
Like in the algebraic case, $V(I) = V(IVI)$ (to see that $V(IVI) \subseteq V(I)$, notice that $I \subseteq IVI$ and to see that $V(I) \subseteq V(IVI)$, notice that if $\bar{a} \in V(I)$ and $P \in IVI$, then $P(\bar{a}) = 0$).

Remark 2.1.15 *If* $I \subseteq R[\bar{X}]^E$, *then* $V(I) = V(I_E) = V(IVI)$.

Proof. Indeed if $\bar{a} \in V(I_E)$, then $\bar{a} \in V(I)$. Then we have that IVI is an E-ideal containing I_E and that $V(I) = V(IVI) \subseteq V(I_E)$ as I_E is the intersection of all E-ideals of $R[\bar{X}]^E$ containing I. ■

The set $V(I)$ is called an *E-variety*.

Considering a finitely generated ideal I and a variety with presentation $V(I)$, the above equalities 7.1.1: $V(I) = V(I_E) = V(IVI)$, allow us to work like in a Noetherian context.

Note also that we do not have a 'classic' Nullstellensatz as the ring $R[\bar{X}]^E$ is not a Hilbert ring (its prime ideals are not especially intersections of maximal ideals [53]–see also Chapter 5–).

2.1.5.1 Prime E-ideals and irreducible E-varieties

Definition 2.1.16 [63] Let (R, E) be an E-ring. An E-ideal $I \subseteq R$ is a *prime ideal* if the quotient R/I is a domain.

Definition 2.1.17 The E-variety V is called *irreducible* if it cannot be expressed as a proper union of two E-varieties, that is $V \neq A \cup B$, with $A \subsetneq V$ and $B \subsetneq V$.

Let $V := V(P_1, \cdots, P_m)$ be an E-variety and $J := I(V)$. Recall that in the algebraic case, $I(V)$ is prime iff V is irreducible. The proof of the result stated in the following remark follows from its (purely) algebraic analogue in [37], p.25; the E-algebraic proof remains essentially the same–we include it here for completeness–:

Remark 2.1.18 *V is irreducible if and only if its associated E-ideal $J := I(V)$ is prime.*

Proof. If J is not prime, we can find two E-polynomials P, Q such that $P \notin J$, $Q \notin J$ but $PQ \in J$. Let G be the E-ideal generated by J and P, and let H be the E-ideal generated by J and Q. Then let $U := V(G)$ and let $W := V(H)$. Therefore $U \subsetneq V$ and $W \subsetneq V$. Furthermore, $U \cup W = V$. Indeed, $U \cup W \subseteq V$ trivially. Conversely, let $\bar{a} \in V$. Then $PQ(\bar{a}) = 0$ implies $P(\bar{a}) = 0$ or $Q(\bar{a}) = 0$. Hence $\bar{a} \in U$ or $\bar{a} \in W$, proving $V = U \cup W$.

Now suppose $V = U \cup W$ with $U \neq V$ and $W \neq V$; and let $G = I(U)$ and $H = I(W)$. There exist $P \in G$, $P \notin J$ and $Q \in H$, $Q \notin J$. But PQ vanishes on $U \cup W$ and hence lies in J, which is a contradiction. ■

2.1.5.2 E-ideals closed by differentiation

We now recall results from A.Macintyre on E-ideals closed under partial differentiation. Let (R, E) be an E-ring. By (E-)domain, we mean a non-zero (E-)ring in which $ab = 0$ implies $a = 0$ or $b = 0$.

Fact 2.1.19 [39, Theorem 15 p.199.] *Let $i \in \{1, \cdots, n\}$. If R is a characteristic 0 E-domain and $I \subseteq R[\bar{X}]^E$ is an E-ideal which is closed under partial differentiation $\forall j \in \{1, \cdots, n\}$, $\frac{\partial}{\partial X_i} : r \in R \mapsto 0, X_j \mapsto \delta_{ij}$; then either $I = 0$ or I contains a non-zero element of $R[\bar{X} \setminus \{X_i\}]^E$.*

The proof uses Fact 2.1.7 in order to obtain chains of elements (of the ideal I) ordered by the notion of degree on exponential polynomials taking ordinal values of Definition 2.1.5. By well-ordering, for each variable X_i, one obtains an element of minimal degree which is actually an algebraic polynomial. Then by successive use of the usual derivation one obtains an element of I that does not depend on X_i.

Fact 2.1.20 [39, Corollary p.199.] *If R is a characteristic 0 E-ring which is a field, and I is an E-ideal of $R[\bar{X}]^E$ closed under all $\frac{\partial}{\partial X_i}$ then either $I = 0$ or $I = R[\bar{X}]^E$.*

Let (K, R, E) be a partial E-field. Fact 2.1.20 actually shows that ecl^L-independent elements over K do not satisfy any hidden exponential-algebraic relation over K:

Corollary 2.1.21 *Let $\bar{a} := a_1, \cdots, a_n \subseteq L$ be such that $a_1, \cdots, a_n$ are ecl^L-independent over K. Then there is no $m \in \mathbb{N}$, $b_1, \cdots, b_m \in K$ and $P \in \mathbb{Z}[\bar{X}\bar{Y}]^E \setminus \{0\}$ such that $P(\bar{a}\bar{b}) = 0$.*

Proof. Let $m \in \mathbb{N}$, $b_1, \cdots, b_m \in K$ and $P \in \mathbb{Z}[\bar{X}\bar{Y}]^E$ such that $P(\bar{a}\bar{b}) = 0$.
Then for $i = 1, \cdots, n$, $\frac{\partial P}{\partial X_i}(\bar{a}\bar{b}) = 0$ otherwise by definition a_i would belong to $ecl^L(K(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n))$, as we would have a 1×1 Hovanskii system over $K(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ admitting a_i as a zero. Hence the ideal of E-polynomials vanishing on $\bar{a}$ in $\mathbb{Z}[\bar{b}]^E[\bar{X}]^E \cong \mathbb{Z}[\bar{X}\bar{b}]^E$ is an E-ideal (as an annulator) closed by all partial derivations, so by Fact 2.1.20, either it is 0 or it contains a non-zero element of K vanishing on $\bar{a}$ which is impossible, hence $P \equiv 0$. ■

Lemma 2.1.22 *Let $m \in \mathbb{N}$, $b_1, \cdots, b_m \in K$ and $0 \not\equiv P \in \mathbb{Z}[\bar{X}\bar{b}]^E$. Suppose $\emptyset \neq V(P) \subseteq K^{|\bar{X}|}$. Then for all uple $\bar{a} \in V(P)$ there exist $Q \in \mathbb{Z}[\bar{X}\bar{b}]^E$ and $1 \leq i \leq |\bar{a}|$ such that $Q(\bar{a}) = 0$ and $\frac{\partial Q}{\partial X_i}(\bar{a}) \neq 0$.*

Proof. Let $\bar{a} \in V(P)$, and consider also all the partial derivatives of P, $\frac{\partial P}{\partial X_i}$ for $1 \leq i \leq n$. Then there exists a multi-index $\bar{\alpha} := (\alpha_1, \cdots, \alpha_n)$ and $i_0 \in \{1, \cdots, n\}$ such that, if we put

$$\partial^{\alpha} P := \frac{\partial^{\alpha_1 + \cdots + \alpha_n} P}{\partial X_1^{\alpha_1} \cdot \ldots \cdot \partial X_n^{\alpha_n}}$$

then $\partial^\alpha P(\bar{a}) = 0$ and $\frac{\partial \partial^\alpha P}{\partial X_{i_0}}(\bar{a}) \neq 0$. Hence one can let $Q := \partial^\alpha P$.

Suppose not: if P and all its derivatives of type $\partial^\alpha P$ vanish at $\bar{a}$, let I be the ideal generated in $\mathbb{Z}[\bar{b}]^E[\bar{X}]^E$ by P and all its derivatives. By Fact 2.1.20, we obtain a contradiction as $P \not\equiv 0$. ■

Remark 2.1.23 *Let $\bar{a} := a_1, \cdots, a_n \subseteq L$, and consider the (E-)ideal $I(\bar{a})$ of elements of $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. If $I(\bar{a}) \neq \emptyset$, one can consider the set of finite sets (these finite sets are of cardinality less or equal to $|\bar{a}| = n$) of E-polynomials in $I(V(I(\bar{a})))$ that have linearly independent gradients–this is possible by Lemma 2.1.22. This set is partially ordered by inclusion and admits a maximal element $\{G_1, \cdots, G_p\}$.*

2.2 Topological $\mathcal{L}$-structures

2.2.1 Topological extensions

Definition 2.2.1 Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $L = (L, \mathcal{T})$ and $K = (K, \mathcal{V})$ be topological $\mathcal{L}$-fields, where $\mathcal{T}$ and $\mathcal{V}$ are fundamental systems of neighborhoods of 0 of L and K respectively, and such that $L \supseteq K$. Then L is called a *topological $\mathcal{L}$-extension of K* if

1. K is an $\mathcal{L}$-substructure of L and
2. for all $V \in \mathcal{V}$, there exists $W \in \mathcal{T}$, $W \cap K = V$.

Definition 2.2.2 [23, p.573] Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $(K, \mathcal{V})$ and $(L, \mathcal{W})$ be two topological $\mathcal{L}$-fields such that $K \subseteq L$. Let $\mathcal{T} \subseteq \mathcal{W}$. We say that $\mathcal{T}$ satisfies $Comp(K)$ if it satisfies the following conditions:

1. $\forall V \in \mathcal{V}, \exists W \in \mathcal{T}, W \cap K = V$ (hence $\mathcal{T}$ is nonempty)
2. $\forall W \in \mathcal{T}, W \cap K \in \mathcal{V}$
3. $\forall a_0, a_1 \in K, \forall V_0, V_1 \in \mathcal{V}, \forall W_0, W_1 \in \mathcal{T}$ with $W_i \cap K = V_i$ for $i = 0, 1$

$$(a_0 + V_0) \cap (a_1 + V_1) = \emptyset \Rightarrow (a_0 + W_0) \cap (a_1 + W_1) = \emptyset$$

4. for any n-ary function symbol $f \in \mathcal{L}$ and any $a_1, \cdots, a_n \in K$, $V_0, \cdots, V_n \in \mathcal{V}$ such that

$$f(a_1 + V_1, \cdots, a_n + V_n) \subseteq f(a_1, \cdots, a_n) + V_0$$

and any $W_0, \cdots, W_n \in \mathcal{T}$ with $W_i \cap K = V_i$ for $i = 0, \cdots, n$, we have

$$f(a_1 + W_1, \cdots, a_n + W_n) \subseteq f(a_1, \cdots, a_n) + W_0$$

5. $\forall a \in K^\times$, $\forall V_0, V_1 \in \mathcal{V}$, $\forall W_0, W_1 \in \mathcal{W}$ with $W_i \cap K = V_i$ for $i = 0, 1$; if $0 \notin (a + V_1)$ and $(a + V_1)^{-1} \subseteq a^{-1} + V_0$, then

$$(a + W_1)^{-1} \subseteq a^{-1} + W_0$$

Definition 2.2.3 [23, p.573] Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $(K, \mathcal{V}) \subseteq (L, \mathcal{W})$ be two topological $\mathcal{L}$-fields. Let $\mathcal{W}(K) \subseteq \mathcal{W}$ satisfying $Comp(K)$ and $a, b \in L$. Then $a \sim_{\mathcal{W}(K)} b$ (a and b are *infinitely close with respect to* $\mathcal{W}(K)$) iff for every $W \in \mathcal{W}(K)$, $a - b \in W$.

Fact 2.2.4 [23, Lemma 2.11] *Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $(K, \mathcal{V}) \subseteq (L, \mathcal{W})$ be two topological $\mathcal{L}$-fields. Let $\mathcal{W}(K) \subseteq \mathcal{W}$ satisfying $Comp(K)$. Then $\sim_{\mathcal{W}(K)}$ is an equivalence relation.*

Fact 2.2.5 [23, Lemma 2.14] *Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $(K_0, \mathcal{V}_0) \subseteq (K_1, \mathcal{V}_1)$ and $(K_1, \mathcal{V}_1) \subseteq (K_2, \mathcal{V}_2)$ be two topological $\mathcal{L}$-fields extensions. Assume that $\mathcal{V}_1(K_0)$ (resp. $\mathcal{V}_2(K_1)$) satisfies $Comp(K_0)$ (resp. $Comp(K_1)$). Let*

$$\widetilde{\mathcal{V}_2}(K_0) := \{V \in \mathcal{V}_2(K_1) : V \cap K_1 \in \mathcal{V}_1(K_0)\}$$

Then $\widetilde{\mathcal{V}_2}(K_0)$ satisfies $Comp(K_0)$ and moreover for a, $b \in K_1$, $a \sim_{\mathcal{V}_1(K_0)} b$ implies $a \sim_{\widetilde{\mathcal{V}_2}(K_0)} b$.

Fact 2.2.6 [23, p.573] *Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let $(K, \mathcal{V})$ and $(L, \mathcal{W})$ be topological $\mathcal{L}$-fields both endowed with a topology definable by the same $\mathcal{L}$-formula ϕ, and suppose that $(K, \mathcal{V}) \subseteq (L, \mathcal{W})$ is an elementary $\mathcal{L}$-extension (Definition A.0.1). Let $\mathcal{W}(K) := \{\phi(L, \bar{a}) : \bar{a} \subseteq K\}$. Then $\mathcal{W}(K)$ satisfies $Comp(K)$.*

Remark 2.2.7 *Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$, and let M be a $\mathcal{L}$-structure such that $\kappa := |M| > \aleph_0$ and M is endowed with a definable base $\mathcal{V}$ of neighborhoods of* 0*. Fact A.0.8 allow us to construct a topological κ^+-saturated elementary $\mathcal{L}$-extension M^* of M. By Fact 2.2.6, let $\mathcal{V}^*(M) \subseteq \mathcal{V}^*$ a subset satisfying $Comp(M)$. By κ^+-saturation of M^*, there are elements of M^* that 'are in all neighborhoods of zero for the topology of M', namely elements $x \in M$ such that $x \sim_{\mathcal{V}^*(M)} 0$.*

2.2.2 Topological $\mathcal{L}$-partial-E-fields

We want to encompass the notions of $\mathcal{L}$-structures and partial E-field endowed with a topology for which E is continuous.

Definition 2.2.8 Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$. A *topological $\mathcal{L}$-partial-E-field* $(K, R, E, \mathcal{V})$ is a partial E-field (K, R, E) such that (R, τ) is a topological $\mathcal{L}$-structure, where τ is the induced topology on R from $\mathcal{V}$ and $(K, \mathcal{V})$ is a topological $\mathcal{L}_{rings}$-field.

Lemma 2.2.9 *Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$, and let $(K, R, E, \mathcal{V}) \subseteq (L, R', E, \mathcal{W})$ be a topological $\mathcal{L}$-extension. Let $\mathcal{W}(K) \subseteq \mathcal{W}$ satisfying $Comp(K)$. Let $P, Q \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $|\bar{X}| = n$, $\beta, \gamma \in K$ and $\bar{\alpha} \subseteq K^n$ such that $P(\bar{\alpha}) = \beta$ and $Q(\bar{\alpha}) = \gamma$.*

If $t_0, \cdots, t_n \in L$ are such that $t_i \sim_{\mathcal{W}(K)} 0$ for $i = 1, \cdots, n$ then

1. *$P(\bar{\alpha} + \bar{t}) \sim_{\mathcal{W}(K)} \beta$*

2. *If $\beta \neq 0$, then $P(\bar{\alpha} + \bar{t}) \neq 0$ and*

$$Q(\bar{\alpha} + \bar{t}).P(\bar{\alpha} + \bar{t})^{-1} \sim_{\mathcal{W}(K)} \gamma.\beta^{-1}$$

Proof.

1. By induction on the number of iterations of E in P.

   - Suppose $P(\bar{X}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]$. Then P is a polynomial (namely purely algebraic) hence the result is true by Lemma 2.13 of [23], page 575.

- Suppose if $P(\bar{X}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R_k$, then item 1 is true, and let $P(\bar{X}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R_{k+1}$. Thus

$$P(\bar{X}) = \sum_i Q_i(\bar{X})(Q_{0,i}(\bar{X}) + Q_{1,i}(\bar{X})E(Q_{2,i}(\bar{X})))$$

with $Q_{0,i}$, $Q_{1,i}$ and $Q_{2,i}$ in R_k and $Q_i(\bar{X}) \in K[\bar{X}]$. Suppose $Q_{0,i}(\bar{a}) = \beta_{0,i}$, $Q_{1,i}(\bar{a}) = \beta_{1,i}$ and $Q_{2,i}(\bar{a}) = \beta_{2,i}$. By induction hypothesis, $Q_{0,i}(\bar{a} + \bar{t}) \sim_{\mathcal{W}(K)} \beta_{0,i}$, $Q_{1,i}(\bar{a} + \bar{t}) \sim_{\mathcal{W}(K)} \beta_{1,i}$ and $Q_{2,i}(\bar{a}+\bar{t}) \sim_{\mathcal{W}(K)} \beta_{2,i}$. By continuity of $+$, . and E with respect to $\mathcal{W}(K)$ we get the result.

2. By continuity with respect to Comp(K).

■

2.2.3 Ordered abelian groups

Recall that an *ordered abelian group* is an abelian group $(G, +)$ equipped with an ordering $<$ which is compatible with the addition, that is, it satisfies

$$x < y \to x + z < y + z$$

There are several ways of showing that a torsion free abelian group can be ordered, we include here the proofs from [50] for completeness:

Fact 2.2.10 [50, Lemma 26.5, p.113] *A group G is an orderable group if and only if G has a subset S such that:*

1. *$x, y \in S$ implies that $xy \in S$.*
2. *$x^{-1}Sx = S$ for all $x \in G$.*
3. *$1 \notin S$ and if $x \in G$, $x \neq 1$, then either x or x^{-1} belongs to S.*

Proof. Let $(G, <)$ be an ordered group. Then the set $S = \{x \in G : 1 < x\}$ satisfies (1),(2) and (3).
Conversely, suppose that we are given S. We set $x < y$ if and only if $x^{-1}y \in S$. Then condition (1) implies that $<$ is transitive: if $x < y$ and $y < z$, that is $x^{-1}y \in S$ and $y^{-1}z \in S$ then $x^{-1}z \in S$. Condition (2) implies that $<$ is compatible with group multiplication: if $x^{-1}y \in S$ and $z \in G$, then $z^{-1}x^{-1}yz \in S$ hence $xz < yz$. Condition (3) implies antisymmetry and then that $<$ is a strict linear ordering. ■

Fact 2.2.11 [50, Lemma 26.6, p.113] *Any torsion free abelian group can be ordered.*

Proof. [50] Let G be a torsion free abelian group and let $\mathcal{S}$ be the family of all subsets S of G which satisfy the condition $x, y \in S$ implies $xy \in S$ and also $1 \notin S$. It follows easily by Zorn's Lemma that $\mathcal{S}$ contains a maximal member S. This set S satisfies (1), (2) and (3) of Fact 2.2.10: indeed (1) is given and (2) is satisfied because G is abelian. In order to show (3), let $x \in G$, $x \neq 1$ and suppose by way of contradiction that neither x nor x^{-1} belongs to S. If

$$T = S \cup \{sx^n : s \in S, n \geq 1\} \cup \{x^n : n \geq 1\}$$

then clearly $T \supsetneq S$ and T is closed under multiplication. By the maximality of S in $\mathcal{S}$ we have $T \notin \mathcal{S}$ so $1 \in T$. Now G is torsion free so $1 \notin \{x^n : n \geq 1\}$ and therefore for some $s \in S$ and $n \geq 1$, we have $1 = sx^n$ so $x^{-n} \in S$. Replacing x by x^{-1} in this argument we conclude that $x^m \in S$ for some $m \geq 1$. Since S is closed under multiplication this yields

$$1 = (x^m)^n (x^{-n})^m \in S$$

a contradiction. Thus (3) follows and Fact 2.2.10 yields the result. ■

Fact 2.2.12 [49, Neumann's Lemma, p.206] *Let G be an ordered abelian group (written multiplicatively), and let A, B be subsets of G.*

- *If A, B are reverse well-ordered, so is AB; and for each $g \in AB$, there are only finitely many pairs $(a, b) \in A \times B$ such that $ab = g$.*
- *If $A \subseteq G^{<1} := \{g \in G : g < 1\}$ is reverse well-ordered, so is $\cup_n A^n$; and for each $g \in \cup_n A^n$, there are only finitely many tuples $(n, a_1, \cdots, a_n)$ with $a_1, \cdots, a_n \in A$ such that $a_1 \cdots a_n = g$.*

2.2.4 Valued fields

Definition 2.2.13 Let K be a field and G be an ordered abelian group, and extend the ordering and group law of G to $G \cup \{0\}$ by setting: $\forall g \in G$

- $0 \leq g$
- $0.g = g.0 = 0$

A map from K to $G \cup \{0\}$, which satisfies: $\forall a, b \in K$

1. $|a| = 0$ iff $a = 0$
2. $|a.b| = |a|\,|b|$
3. $|a + b| \leq \max\{|a|, |b|\}$

is called an *ultrametric absolute value*, or sometimes an *exponential valuation*, on K; and $(K, |.|)$ is called a *valued field*.

The set $\mathcal{O}_K := \{a \in K \ : \ |a| \leq 1\}$ is then a *valuation ring*, that is is an integral domain such that for every element x of its field of fractions, at least one of x or x^{-1} belongs to it. We denote $\mathbf{m}(\mathcal{O}_K) := \{a \in K \ : \ |a| < 1\}$ its *maximal ideal*. For completeness we recall the notion of valuation and its link with ultrametric absolute values:

Definition 2.2.14 Let K be a field and Γ be an additive ordered abelian group, and extend the ordering and group law of Γ to $\Gamma \cup \{\infty\}$ by setting: $\forall \gamma \in \Gamma$

- $\gamma \leq \infty$
- $\infty + \gamma = \gamma + \infty = \infty$

A map $v : K \to \Gamma \cup \{\infty\}$, such that v satisfies: $\forall a, b \in K$

1. $v(a) = \infty$ iff $a = 0$
2. $v(a.b) = v(a) + v(b)$
3. $v(a + b) \geq \min\{v(a), v(b)\}$

is called a *valuation* on K, and (K, v) is called a valued field.

The set $\mathcal{O}_K := \{a \in K \ : \ v(a) \geq 0\}$ is then a valuation ring with as maximal ideal the set $\mathbf{m}(\mathcal{O}_K) := \{a \in K \ : \ v(a) > 0\}$.

To such a valuation v it is possible to associate an exponential valuation, taking its values in $x^{\Gamma} \cup \{0\}$, x^{Γ} being a multiplicative copy of Γ, by setting for $a \in K$, $|a|_v := x^{-v(a)}$. Thus for $a, b \in K$, $|a|_v \leq |b|_v$ iff $v(b) \leq v(a)$.

The *residue field* k of a valued field (K, v) is the quotient of its valuation ring by its maximal ideal

$$k := \mathcal{O}_K/\mathbf{m}(\mathcal{O}_K)$$

The map $res : \mathcal{O}_K \to \mathcal{O}_K/\mathbf{m}(\mathcal{O}_K)$, $x \mapsto x + \mathbf{m}(\mathcal{O}_K)$ is called the *residue map.*

Definition 2.2.15 A valued field (K, v) is *Henselian* if for $P(X) \in \mathcal{O}_K[X]$ such that $res(P)(X)$ has a simple root in $a \in k$, then $P(X)$ ha a root $a' \in \mathcal{O}_K$ such that $res(a') = a$.

Definition 2.2.16 The valued field $(K, |.|)$ is called *spherically complete* if every (not necessarily countable) collection of nonempty balls that is totally ordered by inclusion has a nonempty intersection.

2.2.5 Completeness and spherical completeness

Let (X, τ) be a topological space. Recall that (X, τ) is called *regular* if it is Haussdorff and whenever A is closed in X and $x \notin A$, then there are disjoint open sets U_x and V_A such that $x \in U_x$ and $A \subseteq V_A$ ([6, Definition 2 and Proposition 11, I.56, 4]).

Fact 2.2.17 [7, Nagata-Smirnov's Theorem, IX, p.109, ex.32] A regular space (X, τ) is *metrizable* iff there is a sequence $(\mathcal{B}_n)_{n\in\mathbb{N}}$ of locally finite families of open sets of X such that $\cup_n \mathcal{B}_n$ is a basis for the topology of X.

Let I be a nonempty set, and let $\mathcal{P}(I)$ denote the set of all subsets of I. Recall that a *filter* D over I is defined to be a set $D \subseteq \mathcal{P}(I)$ such that :

- $I \in D$
- If $X, Y \in D$, then $X \cap Y \in D$
- If $X \in D$ and $X \subseteq Z \subseteq I$, then $Z \in D$.

In metrizable spaces, points admit countable basis of neighborhoods hence countable sequences can be used instead of filters (for more details see for example [7, IX, 6, p.17]):

Definition 2.2.18 *A metrizable space* (X, τ) *is complete if (countable) Cauchy sequences in* X *are convergent in* X*.*

If an ultrametric valued field $(K, |.|)$ is–not especially metrizable but–spherically complete, we have the following characterization using pseudo-Cauchy sequences:

Definition 2.2.19 [26, p.303] Let $(K, |.|)$ be an ultrametric valued field, and let $(k_\alpha)_{\alpha<\lambda}$ be a sequence of elements of K indexed by ordinals $\alpha < \lambda$, where λ is a limit ordinal. It is said to *pseudoconverge* to k, if $|k - k_\alpha|$ is eventually strictly decreasing, that is, for some index α_0 we have

$$|k - k_\sigma| < |k - k_\alpha| \text{ whenever } \sigma > \alpha > \alpha_0$$

We also say in that case that k is a *pseudolimit* of $(k_\alpha)_{\alpha<\lambda}$.

It is called a *pseudo-Cauchy sequence* if, for $\alpha < \beta < \gamma < \lambda$,

$$|k_\gamma - k_\beta| < |k_\beta - k_\alpha|$$

A pseudo-Cauchy sequence may admit several distinct pseudolimits.

Fact 2.2.20 [1, Corollary 3.2.9 p.109] *The ultrametric valued field* $(K, |.|)$ *is spherically complete iff every pseudo-Cauchy sequence in* $(K, |.|)$ *has a pseudolimit in* K*.*

2.3 Examples of (spherically) complete valued partial E-fields

2.3.1 $\mathbb{Q}_p$ and $\mathbb{C}_p$

Recall that the p-adic valuation is defined by:

$$v_p : \begin{array}{ccc} \mathbb{Z}^\times & \to & \mathbb{Z} \\ z = p^n k & \mapsto & n \quad \text{where } gcd(p,k) = 1 \\ 0 & \mapsto & \infty \end{array}$$

This valuation can be extended to $\mathbb{Q}$ by setting $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$, and this defines a p-adic (exponential) absolute value $|.|_p$ on $\mathbb{Q}$:

$$|x|_p = p^{-v_p(x)}$$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the distance induced by this absolute value. The valuation v_p (as well as the absolute value $|.|_p$) extends uniquely to $\mathbb{Q}_p$, the valuation ring of which is denoted $\mathbb{Z}_p$. The latter has maximal ideal $p\mathbb{Z}_p$.
There is a partial exponential map defined on the valuation ring $\mathbb{Z}_p$:

$$\begin{array}{cccll} & \mathbb{Z}_p & \to & \mathbb{Q}_p & \\ E_p : & x & \mapsto & \exp(px) & \text{if } p \neq 2 \\ & x & \mapsto & \exp(p^2 x) & \text{otherwise} \end{array}$$

Where $\exp : z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Let $\mathbb{Q}_p^{alg}$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\mathbb{Q}_p^{alg}$ with respect to the unique extension of $|.|_p$ to $\mathbb{Q}_p^{alg}$. Let $\mathcal{O}_p$ be the valuation ring of $\mathbb{C}_p$ and $\mathcal{M}_p$ its maximal ideal. E_p extends canonically to the valuation ring of $\mathbb{Q}_p^{alg}$ and then to $\mathcal{O}_p$. Note that although being complete, $\mathbb{C}_p$ is not spherically complete.

2.3.2 Hahn series

Let K be a field and G an ordered group. The set of formal power series $K((G))$ is defined to be the set of elements of the form $s = \sum_{g \in G} c_g g$, where $c_g \in K$ and

$$Supp\, s := \{g \in G \ : \ c_g \neq 0\}$$

is reverse well ordered in G, that is the order is total and every nonempty subset of $Supp\, s$ has a greatest element. We will see that it is a field by some results of B.H.Neumann [49] about well ordered sets, an ordered field (see [18]) if K is itself an ordered field, and a topological partial E-field if K is. This section is mainly based on [18].

We follow notations of [18] and denote by $Lm(s)$ the maximum g_0 of $Supp\, s$, in other words its *Leading monomial*, and by $Lc(s)$ its corresponding *Leading coefficient* c_{g_0}. One can define an addition componentwise on $K((G))$ by

$$a + b := \sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and a multiplication by distributing:

$$ab = \sum_{k \in G} c_k k \;\text{ where}\, c_k = \sum_{gh=k} a_g b_h$$

With these operations $K((G))$ is actually a field thanks to Fact 2.2.12–Neumann's Lemma–:

Fact 2.3.1 [49, Part 1, section 4] *With the operations + and . defined above, if K is a ring, then $K((G))$ is a ring, and if K is a field, then $K((G))$ is a field.*

Proof. Indeed, if $a \in K((G))$, $-a \in K((G))$; and if $a, b \in K((G))$, then, as the union of two reverse well-ordered sets is reverse well-ordered in G, $a + b$ is well-defined and by the first item of Neumann's Lemma 2.2.12, ab is well-defined (in particular each c_k is a finite sum). If $\epsilon \in K((G))$ and $Lm(\epsilon) < 1$, each ϵ^i, for $i \in \mathbb{N}$, is well defined and has well ordered support. Then by the second item, $\bigcup_k Supp\{\epsilon^k\}$ has reverse well ordered support hence the infinite sum $1 + \epsilon + \epsilon^2 + \cdots$ is well defined and furthermore is an inverse for $1 - \epsilon$. Therefore if $a \in K((G))$, a can be written $a = Lc(a)Lm(a)(1 + \epsilon)$, where $Lm(-\epsilon) < 1$. Hence $a^{-1} = (1+\epsilon)^{-1}Lm(a)^{-1}Lc(a)^{-1}$. See [49] for a detailed proof. ■

Remark 2.3.2 *The proof also shows that for an element ϵ of $K((G))$ with $Lm(\epsilon) < 1$, then the infinite sum*

$$1 + \epsilon + \frac{\epsilon^2}{2!} + \cdots + \frac{\epsilon^n}{n!} + \cdots =: \exp(\epsilon)$$

is well-defined and belongs to $K((G))$. In other words it is possible to define an exponential for infinitely small elements.

Remark 2.3.3 *Note that K is a subfield of $K((G))$ and that G is a multiplicative subgroup of $K((G))^\times$.*

(The former is seen by identifying an element $k \in K$ with $k.1$, and the latter by identifying $g \in G$ with $1.g$.)

There is a natural exponential valuation on $K((G))$: if $s \in K((G))$, we have let $Lm(s) := \max\ Supp\ s$. Then

$$|.| : K((G))^\times \to G, s \mapsto Lm(s)$$

satisfies $|s_1\, s_2| = |s_1|\,|s_2|$ and $|s_1 + s_2| \leq \max\{|s_1|, |s_2|\}$. By letting $|0| := 0 \in G \cup \{0\}$, and $g > 0$ for all $g \in G$, $|.|$ is an exponential absolute value (non-archimedean, ultrametric); the valued field $(K((G)), |.| = Lm(.))$ has value group G and residue field K.

Then $K((G^{\leq 1})) := \{s \in K((G)) \,:\, |s| \leq 1\}$ is a valuation ring of $K((G))$ with maximal ideal $K((G^{<1}))$; and

$$K((G^{>1})) := \{s \in K((G)) \,:\, \mathit{Supp}\, s > 1\}$$

is an additive subgroup of $K((G))$ as well as K, $K((G^{<1}))$ and $K((G^{\leq 1}))$, as $\mathit{Supp}\, 0 = \emptyset \subseteq G^{>1}$. Then $K((G))$ can be written as a direct sum of K-linear subspaces:

$$K((G)) = K((G^{<1})) \oplus K \oplus K((G^{>1}))$$

Fact 2.3.4 [1, Corollary 2.3.2 p. 76] $(K((G)), Lm(.))$ is spherically complete.

Recall that an abelian group G is *divisible* if for all n in $\mathbb{N}\setminus\{0\}$, $G = nG$.

Fact 2.3.5 [42, Theorem 1] If K is algebraically closed and G is divisible, then $K((G))$ is algebraically closed.

Fact 2.3.6 [2, 6.23, (2) p.218] If K is real closed and G is divisible then $K((G))$ is itself real closed.

If $\mathcal{V}$ is a base of neighborhoods of 0 in K endowing the latter with a topological $\mathcal{L}$-field structure, N.Guzy & F.Point [23] extend the topology of K to $K((G))$, in a way compatible to the natural valuation, in order to make $K((G))$ a topological $\mathcal{L}_{rings}$-extension of K: set

$$W_{V,0} := \{s \in K((G)) \,:\, |s| \leq 1 \text{ and if } |s| = Lm(s) = 1, \text{ then } Lc(s) \in V\}$$

$$W_g := \{s \in K((G)) \,:\, |s| \leq g\}$$

$$\mathcal{T}_{\mathcal{V}} := \{W_{V,0} \,:\, V \in \mathcal{V}\}$$

$$\mathcal{T} := \mathcal{T}_{\mathcal{V}} \cup \{W_g \,:\, g \in G\}$$

Then $\mathcal{T}$ is a base of neighborhoods of 0 in $K((G))$ and by construction if $\mathcal{V}$ is not the discrete topology, then $\mathcal{T}_{\mathcal{V}}$ satisfies Property $Comp(K)$ defined in 2.2.2.

Suppose now that K is equipped with an exponential E, continuous for the topology generated by $\mathcal{V}$. By Remark 2.3.2, we can define an exponential on $K((G^{\leq 1})) = K \oplus K((G^{<1}))$:

$$E' : \begin{cases} K \oplus K((G^{<1})) & \to & K((G^{\leq 1})) \\ r + \epsilon & \mapsto & E(r). \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \end{cases}$$

Lemma 2.3.7 *The projections* $\Pi_K : K((G)) \to K$, $\Pi_{K((G^{<1}))} : K((G)) \to K((G^{<1}))$, $\Pi_{K((G^{>1}))} : K((G)) \to K((G^{>1}))$ *are continuous, and if moreover* E *is continuous on* K *then the exponential* $E' : K((G^{\leq 1})) \to K((G^{\leq 1}))$ *is continuous for the topology induced by* $\mathcal{T}$ *on* $K((G^{\leq 1}))$.

Proof. Let $s = \epsilon + k + a \in K((G)) = K((G^{<1})) \oplus K \oplus K((G^{>1}))$, in other words $\epsilon = \Pi_{K((G^{<1}))}(s)$, $k = \Pi_K(s)$ and $a = \Pi_{K((G^{>1}))}(s)$.

- Let O be open in $K((G^{<1}))$ equipped with the induced topology and such that $\epsilon \in O$. Then there is $U \subseteq K((G))$ open such that $O = U \cap K((G^{<1}))$. As $\epsilon + a + k \in U + a + k$ which is open in $K((G))$,
$$\Pi^{-1}_{K((G^{<1}))}(O) = \bigcup_{k \in K, a \in K((G^{>1}))} U + a + k$$
is open in $K((G))$, so the projection is continuous. The same argument works to show the continuity of Π_K and of $\Pi_{K((G^{>1}))}$.

- Let U be a neighborhood of 0 in $K((G))$. Then $(1+U) \cap K((G^{\leq 1}))$ is a neighborhood of 1 in $K((G^{\leq 1}))$, and
$$V := \exp^{-1}\left((1+U) \cap exp\left(K((G^{<1}))\right)\right) = U$$
because $Lm(\exp(\epsilon) - 1) = Lm(\epsilon)$, hence V is a neighborhood of 0 in $K((G^{\leq 1}))$ (and in $K((G))$).

- Finally $E'(k+\epsilon) := E(k)\exp(\epsilon) = (E \circ \Pi_K).(\exp \circ \Pi_{K((G^{<1}))})(k+\epsilon)$ is continuous on $K((G^{\leq 1}))$ as the composition of the multiplication restricted to $K((G^{\leq 1}))$, the projections, E and exp.

■

Then let $a \in K$, and $U, V \in \mathcal{V}$ such that $E(a+V) \subseteq E(a)+U$ in K. In $K((G^{\leq 1}))$:

$$E(a+W_{V,0}) = (E \circ \Pi_K)(a+W_{V,0}) = E(a+V) \subseteq E(a)+U \subseteq E(a)+W_{U,0}$$

Hence $\mathcal{T}_{\mathcal{V}}$ satisfies Item 4. of Definition 2.2.2 for E.

Corollary 2.3.8 *This turns $(K((G)), K((G^{\leq 1})), E', \mathcal{T})$ into a topological $\mathcal{L}$-partial-E-field which is a topological $\mathcal{L}$-extension of the topological $\mathcal{L}$-partial-E-field $(K, K, E, \mathcal{V})$, for $\mathcal{L} = \mathcal{L}_{rings} \cup \{E\}$. Furthermore, $\mathcal{T}_{\mathcal{V}}$ satisfies $Comp(K)$.*

Let G be an ordered abelian group. The field of Hahn series $\mathbb{R}((G))$ can be ordered by, for $s = \sum c_g g$:

$$s > 0 \text{ iff } s \neq 0 \text{ and } Lc(s) > 0$$

2.3.3 Transseries

We briefly recall the construction of the field of transseries–logarithmic exponential power series– $\mathbb{T} := \mathbb{R}((t))^{LE}$ made in [18] by L.van den Dries, A.Macintyre and D.Marker. It is a topological E-field. We refer the reader to [18] for a detailed construction.

Let $K_{-1} := \mathbb{R}$, $E_{-1} := \exp : \mathbb{R} \to \mathbb{R}^{>0}$, the usual exponential on the reals, and let $(x^{\mathbb{R}}, .)$ be a multiplicative copy of the ordered additive group $(\mathbb{R}, +)$. Denote $G_0 := x^{\mathbb{R}}$, and let $K_0 := K_{-1}((G_0))$.
Set $A_0 := \{s \in K_0 : Supp\, s > 1\}$ and $B_0 := \{s \in K_0 : Supp\, s < 1\}$. Then $K_0 = A_0 \oplus K_{-1} \oplus B_0$. We have a partially defined exponential on K_0:

$$E_0 : \left\{ \begin{array}{ccc} K_{-1} \oplus B_0 & \to & K_0 \\ k+b & \mapsto & \exp(k) \sum_{n \in \mathbb{N}} \frac{b^n}{n!} \end{array} \right.$$

Given $K_n, G_n, A_n, B_n, E_{n-1}$, and an ordered multiplicative copy x^{A_n} of the additive ordered group A_n
(with order-preserving isomorphism $E_M : A_n \to x^{A_n}$, $a \mapsto x^a$),
let $G_{n+1} := G_n \overleftarrow{\times} x^{A_n}$, $K_{n+1} := K_n((x^{A_n})) = K((G_{n+1}))$, $A_{n+1} := \{s \in K_{n+1} : Supp\, s > 1\}$, and $B_{n+1} := \{s \in K_{n+1} : Supp\, s < 1\}$.

Then $K_{n+1} = A_{n+1} \oplus K_n \oplus B_{n+1}$, and

$$E_n : \begin{cases} K_n = A_n \oplus K_{n-1} \oplus B_n & \rightarrow & K_{n+1} \\ a + k + b & \mapsto & E_M(a) E_{n-1}(k) \sum_{n \in \mathbb{N}} \frac{b^n}{n!} \end{cases}$$

For $n \in \mathbb{N}$, the order on $K_n = \mathbb{R}((G_n))$ is defined for $f \in K_n \setminus \{0\}$ by $f >_{K_n} 0$ iff $Lc(f) > 0$ in $(\mathbb{R}, <)$.

Remark 2.3.9 *Reverse well ordered subsets of* $\mathbb{R}$ *(and thus of* $x^{\mathbb{R}}$*) are countable (let* $S \subseteq \mathbb{R}$ *such that* S *is reverse well ordered, let* $s \in S$ *and* $p(s)$ *be the predecessor of* s *then the interval* $(p(s), s)$ *contains a rational; so there is an injection from* S *to* $\{(p(s), s) : s \in S\}$ *to* $\mathbb{Q}$*). Then reverse well ordered subsets of* $\mathbb{R}((x^{\mathbb{R}}))$ *are also countable ([18, p.13], [20]) and then by induction on* $n \in \mathbb{N}$*, and by definition of the order in* K_n*, reverse well ordered subsets of* $\mathbb{R}((G_n))$ *are countable.*

Finally let $\mathbb{R}((t))^E := \bigcup K_n \subsetneq \mathbb{R}((G^E))$, where $G^E = \bigcup G_n$, and let E denote the common extension of the E_n.

The authors in [18] then add the logarithms of the elements: first let

$$\Phi : \begin{array}{ccc} \mathbb{R}((t))^E & \rightarrow & \mathbb{R}((t))^E \\ f & \mapsto & \begin{cases} \sum a_r E(rx) & \text{if} \quad f = \sum a_r x^r \in K_0 \\ \sum \Phi(f_a) E(\Phi(a)) & \text{if} \quad f = \sum f_a E(a) \in K_{n+1} \end{cases} \end{array}$$

It is an order-preserving isomorphism acting like "substituting $E(x)$ for x" and satisfies $\Phi(G_n) \subseteq G_{n+1}$; $\Phi(A_n) \subseteq A_{n+1}$ and $\Phi(K_n) \subseteq K_{n+1}$.

Then let $L_0 := \mathbb{R}((t))^E$ and $\theta_0 := id$. Given L_n and θ_n, take $L_{n+1} \supseteq L_n$ and $\theta_{n+1} : L_{n+1} \rightarrow \mathbb{R}((t))^E$ an isomorphism of ordered E-fields such that for all $z \in L_n$, $\theta_{n+1}(z) = \Phi(\theta_n(z))$.

Finally let $\mathbb{R}((t))^{LE} := \bigcup L_n$, $G^{E,n} := \theta_n^{-1}(G^E)$, $G^{LE} := \bigcup G^{E,n}$.

By letting $G_{m,n} := \theta_m^{-1}(G_n) \subseteq G^{E,m}$, one can write $L_{m,n} := \mathbb{R}((G_{m,n}))$ as a field of generalised power series such that $L_m = \bigcup_n L_{m,n}$.

There is a non-archimedean absolute value on $\mathbb{R}((t))^{LE}$, defined by

$$|f| = \max Supp\, f$$

and a corresponding valuation $v : \mathbb{R}((t))^{LE} \to log(G^{LE}) \cup \{\infty\}$, $f \neq 0 \mapsto v(f) = -log(|f|)$. The valuation group is an ordered additive subgroup of $\mathbb{R}((t))^{LE}$, and the valuation ring is the set: $\{f : |f| \leq 1\}$.

Remark 2.3.10 *The valued field* $(\mathbb{R}((t))^{LE}, |.|)$ *is not spherically complete, otherwise* $\mathbb{R}((t))^{LE}$ *could be written as a Hahn field* $k((G))$*, where* G *is the value group of* $(\mathbb{R}((t))^{LE}, |.|)$ *and* k *its residue field, [54]. However,* $\mathbb{R}((t))^{LE}$ *is obtained in [18, (2.11)] as a proper subfield of* $\mathbb{R}((G^{LE}))$ *and it is shown in [33] that it is not possible to obtain a logarithmic-exponential ordered Hahn field* $k((G))$ *similar to* $\mathbb{R}((t))^{LE}$ *because the ordered additive group of* $k((G))$ *is not isomorphic to its positive multiplicative group when* $G \neq \{1\}$*.*

Nevertheless, for $n, m \in \mathbb{N}$*, by Fact 2.3.4,* $(L_{m,n} := \mathbb{R}((G_{m,n})), Lm(.))$ *is spherically complete; as well as* $(\mathbb{R}((G^{LE})), Lm(.))$*.*

2.4 Model-complete theories of topological partial E-fields

2.4.1 $\mathbb{Q}_p$ as a valued partial E-field

Let $\mathcal{L}_{p,E} := \{+, ., 0, 1, E, P_n, n \in \mathbb{N}\backslash\{0\}\}$, where P_n are unary predicates:

$$P_n(x) \equiv \exists y \; x = y^n$$

and let $T_{\mathbb{Q}_p,E_p}$ denote the theory of the valued field $\mathbb{Q}_p$ with valuation ring (and E-subring) $(\mathbb{Z}_p, E_p)$ in the language $\mathcal{L}_{p,E}$.

In order to show model-completeness of the theory of the valued E-ring $\mathbb{Z}_p$ ([45, p.15]), N.Mariaule actually uses a strategy of L.van den Dries's proof that the structure of the underlying set $\mathbb{R}$ in the language of fields expanded by symbols for the functions exp, cos, sin (restricted to $[0, 1]$) is strongly model-complete [16], as well as ideas from A.Macintyre's [40].

He defines a language $\mathcal{L}_{p,E,C}$ extending $\mathcal{L}_{p,E}$ by some kind of trigonometric functions that could be the p-adic equivalent of $\cos x = \frac{e^{\sqrt{-1}x}+e^{-\sqrt{-1}x}}{2}$ and $\sin x = \frac{e^{\sqrt{-1}x}-e^{-\sqrt{-1}x}}{2\sqrt{-1}}$ within the complex field:

It is known by Krasner's Lemma that the p-adic field $\mathbb{Q}_p$ has finitely many algebraic extensions of a given degree and that any of these extensions is contained in an extension of the type $\mathbb{Q}_p(\beta)$, where β is algebraic over $\mathbb{Q}$. Consequently it is possible to construct a sequence of finite algebraic extensions $K_1 \subseteq K_2 \subseteq \cdots$ such that:

- K_k is the splitting field of $Q_k(X)$ polynomial of some degree N_k with coefficients in $\mathbb{Q}$.
- $K_k = \mathbb{Q}_p(\beta_k)$ for all β_k root of Q_k, and $V_k := \mathcal{O}_{K_k} = \mathbb{Z}_p(\beta_k)$.
- any algebraic extension of degree k is contained in K_k.

Then let M be the Vandermonde matrix of the roots of the minimal polynomial of β_k (note that M is invertible), and let $||.||_k$ be the norm from K_k over $\mathbb{Q}_p$ defined as follows:

Let $P(X) := X^n + a_1 X^{n-1} + \cdots + a_n$ be the minimal polynomial of β_k. Then $||\beta_k||_k := (-1)^n a_n$. For $\alpha \in K_k$, let $||\alpha||_k := ||\beta_k||_k^{[K_k:\mathbb{Q}_p(\alpha)]}$, where $[K_k : \mathbb{Q}_p(\alpha)]$ is the degree of the field extension $K_k/\mathbb{Q}_p(\alpha)$.
Then let

$$c_{i,j,k}(x)_{i<N_k} := ||\det M||_k.M^{-1}.\left(E_p((\beta_k^j)^\sigma x)\right)_{\sigma\in Gal(K_k/\mathbb{Q}_p)}$$

where Gal stands for the Galois group of the extension. The functions $c_{i,j,k}$ are the so-called trigonometric functions. Let

$$\mathcal{L}_{p,E,C} := \mathcal{L}_{p,E} \cup \{c_{i,j,k} : k \in \mathbb{N}, 0 \leq i,j < N_k\}$$

He then shows

Fact 2.4.1 [45, Th. 9.5] *The theory $T_{\mathbb{Z}_p,E_p}$ of the valued E-ring $\mathbb{Z}_p$ in the language $\mathcal{L}_{p,E,C}$ is model-complete.*

Note that $\mathbb{Q}_p$ is interpretable in $\mathbb{Z}_p$, hence its theory as a valued partial E-field is also model-complete:

Let $\phi(\bar{x})$ be a $\mathcal{L}_{p,E,C}$-formula defined over $\mathbb{Q}_p$. Then the formula $\exists\bar{x}\phi(\bar{x})$ is equivalent modulo $T_{\mathbb{Z}_p,E_p}$ to a $\mathcal{L}_{p,E,C}$-formula $\exists\bar{x}\bar{y}\theta(\bar{x}\bar{y})$, where $\theta(\bar{x}\bar{y})$ is a conjunction of E-polynomial equations in $\bar{x}\bar{y}$ defined over $\mathbb{Z}_p$ (using logical equivalences and $x_1 \neq x_2$ iff $\exists z((x_2 - x_1).z - 1 = 0)$, and of a subformula expressing being in a definable open set. Furthermore the valuation is definable using the predicates P_n, by:

$$v(x) \geq 0 \leftrightarrow P_2(1 + px^2) \text{ if } p \neq 2 \text{ or } v(x) \geq 0 \leftrightarrow P_3(1 + px^3) \text{ if } p = 2$$

2.4.2 $\mathbb{C}_p$ as a valued partial E-field

Let (K, v) be an algebraically closed valued field with value group Γ and let $|$ be a binary relation symbol interpretable in the following way: $x|y$ iff $v(x) \leq v(y)$.
Consider the two sorted language:

$$\mathcal{L}_\Gamma := \{+_K, -_K, \cdot_K, 0_K, 1_K, v, |, +_\Gamma, -_\Gamma, 0_\Gamma, \infty_\Gamma, <_\Gamma\}$$

interpreted in K by:

- The first sort is the field K where $+_K, -_K, \cdot_K, 0_K, 1_K$ are interpreted as for the language of rings.
- The second sort is the value group Γ with the point to infinity and $+_\Gamma, -_\Gamma, 0_\Gamma, \infty_\Gamma, <_\Gamma$ are interpreted as for the language of ordered groups.
- The symbol v is a map $K \to \Gamma \cup \{\infty\}$ is interpreted by the valuation and $|$ is a binary relation symbol interpreted as above: $x|y$ iff $v(x) \leq v(y)$.

Fact 2.4.2 *[A.Robinson] The theory $ACVF$ of algebraically closed non-trivially valued fields is axiomatised in $\mathcal{L}_\Gamma$ by*

1. *(K, Γ, v) is a valued field with valuation group Γ.*
2. *There are $x, y \in K \setminus \{0\}$ such that $v(x) < v(y)$.*
3. *K is algebraically closed.*

Fact 2.4.3 *An algebraically closed (non-trivially) valued field has henselian valuation ring, algebraically closed residue field and divisible value group. The converse holds if the residue field has characteristic 0.*

Let $\mathcal{L}_{|,E} := \{+, -, ., 0, 1, |, E\}$.

Fact 2.4.4 [44, Th. 6.2.11] *The $\mathcal{L}_{|,E}$-theory $T_{\mathcal{O}_p,E_p}$ of the valued exponential structure $\mathcal{O}_p$ is model-complete.*

As above, $\mathbb{C}_p$ is interpretable in $\mathcal{O}_p$, and existential formulas in this language are equivalent to existence of zeros of conjunctions of E-polynomial equations and being in definable open sets.

Consider $\mathcal{O}_p\{\{\bar{X}\,\bar{\rho}\}\}_s$ the ring of separated power series in the variables $X_1, \cdots, X_m, \rho_1, \cdots, \rho_n$. For a precise definition of this ring we refer the reader to Paragraph 2 p.78 in [38]. An element $f \in \mathcal{O}_p\{\{\bar{X}, \bar{\rho}\}\}_s$ determines a function from $\mathcal{O}_p^m \times \mathcal{M}_p^n$ to $\mathcal{O}_p$.

Let $\mathcal{L}_{v,M}$ be the 3-sorted language with sorts:

1. $(O, +, ., -, 0, 1)$ where O is a predicate for the valuation ring.
2. $(M, +, ., -, 0, 1)$ where M is a predicate for its maximal ideal.
3. $|C|$, the valuation group, with the language of ordered groups.

and a symbol $|.|$ for the function $O \to |C|$. Let $\mathcal{L}_{an}$ be the language $\mathcal{L}_{v,M}$ expanded by function symbols for each element in $\mathcal{O}_p\{\{\bar{X}, \bar{\rho}\}\}_s$. Then set division symbols on the valuation ring:

$$D_0(x, y) : O^2 \to O : (x, y) \mapsto \begin{cases} x/y & \text{if } |x| \leq |y| \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$D_1(x, y) : O^2 \to M : (x, y) \mapsto \begin{cases} x/y & \text{if } |x| < |y| \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{L}_{an}^D := \mathcal{L}_{an} \cup \{D_0, D_1\}$. L.Lipschitz showed the following:

Fact 2.4.5 [38, Th. 3.8.2] $(\mathcal{O}_p, \mathcal{M}_p, |\mathbb{C}_p|)$ *admits elimination of quantifiers in $\mathcal{L}_{an}^D$.*

J.Denef and L.van den Dries showed:

Fact 2.4.6 [14, Theorem (1.1) p.90] *The theory of the $\mathcal{L}_{an}^D$-structure $\mathbb{Z}_p$ admits elimination of quantifiers.*

2.4.3 $\mathbb{R}$ as an ordered E-field

Now denote by $\mathcal{L}_{or,E} := \{+, -, ., 0, 1, E, <\}$ the language of ordered rings that are E-rings; actually $<$ is definable by $x_1 < x_2$ iff $\exists z((x_2 - x_1).z^2 - 1 = 0)$, and let $\mathcal{L}_{or,\epsilon} := \{+, -, ., 0, 1, \epsilon, <\}$.

Let $T_{\mathbb{R},\exp}$ be the $\mathcal{L}_{or,E}$-theory of the ordered exponential field of real numbers; and $T_{\mathbb{R},\epsilon}$ be the $\mathcal{L}_{or,\epsilon}$-theory of the ordered field of real numbers where the unary function symbol ϵ is interpreted by the restricted exponential function–that we call ϵ too– on $]0,1[$:

$$\forall x(0 < x < 1 \to \epsilon(x) = \exp(x) \wedge (x \leq 0 \vee x \geq 1) \to \epsilon(x) = 0)$$

Fact 2.4.7 [57] *$T_{\mathbb{R},\exp}$ is recursively axiomatized over $T_{\mathbb{R},\epsilon}$, via a recursive set of axioms expressing the fact that* exp *is a strictly increasing isomorphism from* $(\mathbb{R}, +, 0)$ *to* $(\mathbb{R}^{>0}, ., 1)$ *which eventually dominates every polynomial.*

Fact 2.4.8 [65, 2d main Th.] *The $\mathcal{L}_{or,E}$-theory of the ordered exponential field of real numbers $T_{\mathbb{R},\exp}$ is model-complete.*

Let $\phi(\bar{x})$ be a $\mathcal{L}_{or,E}$-definable formula, then the formula $\exists\bar{x}\phi(\bar{x})$ is equivalent modulo $T_{\mathbb{R},\exp}$ to a $\mathcal{L}_{or,E}$-formula $\exists\bar{x}\bar{y}\theta(\bar{x}\bar{y})$, where $\theta(\bar{x}\bar{y})$ is a conjunction of E-polynomial equations in $\bar{x}\bar{y}$ (using logical equivalences and $x_1 \neq x_2$ iff $\exists z((x_2 - x_1).z - 1 = 0)$.
Then it is possible to consider E-polynomial equations with only one iteration of E by adding variables; hence actually by Fact 2.4.8 a $\mathcal{L}_{or,E}$-formula $\exists\bar{x}\Phi(\bar{x})$ is equivalent to the existential formula

$$\exists\bar{x}\exists\bar{y}F(\bar{x}\bar{y}e^{\bar{x}}e^{\bar{y}}) = \bar{0}$$

where $F := (f_1, \cdots, f_r)$, $f_i \in \mathbb{Z}[\bar{x}\bar{y}e^{\bar{x}}e^{\bar{y}}]$.

Fact 2.4.9 [30]*Let $g_1, \cdots, g_m$ be functions on $\mathbb{R}^{m+n}$ defined on $\mathbb{R}$ by E-polynomial terms with only one iteration of E. Let $G := (g_1, \cdots, g_m)$. Then there is $N \in \mathbb{N}$ such that for any $\bar{b} \in \mathbb{R}^n$ the set*

$$\{\bar{a} \in \mathbb{R}^m : G(\bar{a}\bar{b}) = 0 \text{ and } \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix} (\bar{a}\bar{b}) \neq 0\}$$

contains at most N elements.

Now let $\mathbb{R}\{\{X_1, \cdots, X_m\}\}$ be the ring of all real power series in $X_1, \cdots, X_m$ that converge in a neighborhood of I^m, with $I := [-1, 1]$.
For $f \in \mathbb{R}\{\{X_1, \cdots, X_m\}\}$, let $\tilde{f} : x \mapsto f(x)$ if $x \in I^m$, $x \mapsto 0$ otherwise.
Let $\mathcal{L}_{an}$ be the language of ordered rings augmented by a new function symbol for each function $\tilde{f}$, and let

$$D(x, y) : I^2 \to I : (x, y) \mapsto \begin{cases} x/y & \text{if } |x| \leq |y| \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{L}_{an}^D := \mathcal{L}_{an} \cup \{D\}$, and $\mathcal{L}_{an,E} := \mathcal{L}_{an} \cup \{E\} = \mathcal{L}_{an} \cup \mathcal{L}_{or,E}$. Let T_{an} (resp. T_{an}^D, $T_{an,\exp}$) be the $\mathcal{L}_{an}$-theory (resp. $\mathcal{L}_{an}^D$-theory, $\mathcal{L}_{an,E}$-theory) of the $\mathcal{L}_{an}$-structure (resp. $\mathcal{L}_{an}^D$-structure, $\mathcal{L}_{an,E}$-structure) $\mathbb{R}$.

Fact 2.4.10 [19, Corollary 2.11] *As $\mathcal{L}_{an}$-structures, $\mathbb{R} \preccurlyeq \mathbb{R}((t))^{LE}$.*

Let $K \models T_{an,\exp}$. Then the constructions of Subsection 2.3.3 can be carried out to construct a structure $K((t))^{LE}$ [17, 2.11].

Fact 2.4.11 [17, 2.11] *Let $K \models T_{an,\exp}$. Then $K \preccurlyeq K((t))^{LE}$.*

Fact 2.4.12 [14, Theorem (4.6) p.125] *T_{an}^D admits elimination of quantifiers.*

Fact 2.4.13 [17, Theorem 1.1, p.417] *$T_{an,\exp}$ is axiomatized by T_{an} together with, for each $n \in \mathbb{N} \setminus \{0\}$, the following axioms: for all x, for all y*

1. $E(x + y) = E(x)E(y)$
2. $x < y \to E(x) < E(y)$
3. $x > 0 \to \exists y E(y) = x$
4. $x > n^2 \to E(x) > x^n$
5. $\forall x(-1 \leq x \leq 1 \to E(x) = \mathcal{E}(x))$

where $\mathcal{E}$ is the function symbol of $\mathcal{L}_{an}$ corresponding to the exponential power series $\sum_{n \geq 0} \frac{1}{n!} X^n \in \mathbb{R}\{\{X\}\}$.

Let log be another unary function symbol, $\mathcal{L}_{an,E,\log} := \mathcal{L}_{an,E} \cup \{\log\}$, and let $T_{an,\exp,\log}$ be $T_{an,\exp}$ extended by the axiom:

$$\forall x (x > 0 \to E(\log x) = x) \wedge (x \leq 0 \to \log(x) = 0)$$

Fact 2.4.14 [19, Corollary 4.5] *$T_{an,\exp,\log}$ (resp. $T_{an,\exp}$) admits elimination of quantifiers in $\mathcal{L}_{an,E,\log}$.*

Consequently to Fact 2.4.14, $\mathbb{R} \preccurlyeq \mathbb{R}((t))^{LE}$ as $\mathcal{L}_{an,E,\log}$-structures.

Chapter 3

Preliminaries

In this Chapter we recall a few analytic and algebraic results linked to the topological notions of completeness, definable completeness, or spherical completeness: the Implicit Function Theorem, Newton-Kantorovich Theorem, and Hensel's Lemmas. We also set a complete context in which we will work later on: $(K[[x^{\mathbb{Z}}]], |.|)$, where K is a field of characteristic 0 endowed with the trivial topology, $(x^{\mathbb{Z}}, .)$ a multiplicative copy of the additive group $(\mathbb{Z}, +)$, and $K[[x^{\mathbb{Z}}]]$ the valuation ring of the valued field of Hahn series $(K((x^{\mathbb{Z}})), |.|)$. Then, starting from a topological E-field $(K, E, \mathcal{V})$ we construct a topological $\mathcal{L}$-extension the domain of which contains $K((x^{\mathbb{Z}}))$.

3.1 Implicit function Theorem

Let K be a field and $(V, ||.||_V)$, $(W, ||.||_W)$ be normed K-vector spaces. Let $\mathbf{L}(V, W)$ denote the set of (continuous) linear applications from V to W. Recall that if $f \in \mathbf{L}(V, W)$, then

$$||f|| := \sup_{u \in V} \frac{||f(u)||_W}{||u||_V}$$

defines a norm on $\mathbf{L}(V, W)$.

Definition 3.1.1 Let $U \subseteq V$ open. A mapping $f : U \to W$ is said to be *Fréchet differentiable* at $\bar{a} \in U$ if there is a bounded linear map

$A : V \to W$ such that:

$$\lim_{h \to 0} \frac{||f(\bar{a}+\bar{h}) - f(\bar{a}) - A\bar{h}||_W}{||\bar{h}||_V} = 0$$

In that case, A is usually denoted by $Df_{\bar{a}}$.

The mapping f is said to be *Fréchet differentiable on* U if it is differentiable on all $\bar{a} \in U$. This defines a mapping

$$Df : \begin{array}{ccc} U & \to & \mathbf{L}(V,W) \\ \bar{a} & \mapsto & Df_{\bar{a}} \end{array}$$

Definition 3.1.2 Let $r \in \mathbb{N}$. Let $U \subseteq V$ open. A mapping $f : U \to W$ is said to be *of class* C^0 if f is continuous. If $r \geq 1$, it is said to be *of class* C^r if it is Fréchet differentiable on U and if the mapping $Df : U \to \mathbf{L}(V,W)$ is of class C^{r-1}. It is said to be *of class* C^∞, if it is of class C^r for all $r \geq 1$.

The Fréchet derivative in finite dimensional spaces is the usual derivative.

Let $Jf_{\bar{a}}$ be the $m \times n$ matrix whose rows are the vectors $\nabla f_1(\bar{a}), \cdots, \nabla f_m(\bar{a})$ (the Jacobian matrix of f at $\bar{a}$). If $\bar{a} = \bar{x}\bar{y}$, let $J_{(\bar{0},\bar{y})}f_{\bar{a}}$ be the submatrix of $Jf_{\bar{a}}$ of partial derivatives $(\frac{\partial f_i}{\partial x_j}(\bar{a}))$, $i = 1, \cdots, m$, $j = |\bar{x}|+1, \cdots, |\bar{x}\bar{y}|$.

If A is a squared invertible matrix, let A^{-1} denote its inverse. We first recall the Implicit function Theorem for Banach spaces:

Definition 3.1.3 [8, p.9] Let $(K, |.|)$ be $(\mathbb{R}, |.|)$, $(\mathbb{C}, |.|)$, or a commutative characteristic 0 complete non-discrete valued field with an ultrametric absolute value $|.| : K \to \mathbb{R}^{\geq 0}$. We call a complete normed K-vector space a *Banach space*.

Fact 3.1.4 (Implicit Function Theorem) [8, instance of 5.6.7] *Let* $(K, |.|)$ *be* $(\mathbb{R}, |.|)$, $(\mathbb{C}, |.|)$, *or a commutative characteristic* 0 *complete non-discrete valued field with an ultrametric absolute value* $|.| : K \to \mathbb{R}^{\geq 0}$*; and* $(V, ||.||_V)$, $(W, ||.||_W)$ *and* $(Z, ||.||_Z)$ *be Banach spaces on* K*. Let* $m, n, p \in \mathbb{N} \setminus \{0\}$ *and suppose* $\dim V = n$, $\dim W = m$, $\dim Z = p$ *as* K*-vector spaces. Let the mapping* $f = (f_1, \cdots, f_m) : V \times W \to Z$ *be of class* C^r *for* $r \geq 1$*. If* $(\bar{x}_0, \bar{y}_0) \in V \times W$, $f(\bar{x}_0, \bar{y}_0) = 0$*, and* $y \mapsto J_{(\bar{0},\bar{y})}f_{\bar{x}_0\bar{y}_0}$ *is an* *isomorphism from* W *onto* Z*, then there exist neighborhoods* O *of* $\bar{x}_0$ *and* O' *of* $\bar{y}_0$ *and a class* C^r *function* $g : O \to O'$ *such that* $f(\bar{x}_0, g(\bar{x}_0)) = \bar{0}$ *and such that for all* $(\bar{x}, \bar{y}) \in O \times O'$*,* $f(\bar{x}, \bar{y}) = \bar{0}$ *if and only if* $\bar{y} = g(\bar{x})$*. Furthermore, for* $\bar{x} \in O$*,* $\det J_{(\bar{0},\bar{y})} f_{\bar{x},g(\bar{x})} \neq 0$ *and*

$$Jg_{\bar{x}} = -(J_{(\bar{0},\bar{y})} f_{\bar{x},g(\bar{x})})^{-1} J f_{\bar{x},g(\bar{x})}$$

Remark 3.1.5 *If such neighborhoods* O *and* O'*–that can be chosen as box neighborhoods– are fixed and the topology is definable, then the uniqueness given by the fact that*

$$\textit{for all} (\bar{x}, \bar{y}) \in O \times O' \ f(\bar{x}, \bar{y}) = \bar{0} \ \textit{iff} \ \bar{y} = g(\bar{x})$$

gives us the definability of g *(see for example* [65, 4.1-4.3 p.1063], *where it is shown that Fact 3.1.4 is true in any model of* $T_{\mathbb{R},\exp}$*; this also applies to our unordered context). Consequently in the above conditions a valued field that is elementary equivalent to a complete valued field holds an implicit function theorem.*

Fact 3.1.4 is also true in the context of definably complete structures [60, Theorem 2.2.8 p.19]:

Definition 3.1.6 [61, Definition 1] Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{<\}$. A *definably complete structure* R (in the language $\mathcal{L}$) is an $\mathcal{L}$-expansion of an ordered field, such that every definable subset of the domain of R which is bounded from above, has a least upper bound.

Note that definable completeness is first-order expressible [60, p.12]:

for every $\mathcal{L}$-formula $\phi(\bar{x}, y)$ in $n+1$ variables, $n \in \mathbb{N}$,

$$\forall \bar{x} \ (\quad \exists z \forall y \ (\phi(\bar{x}, y) \to y \leq z)$$
$$\to \exists z(\ \forall y(\phi(\bar{x}, y) \to y \leq z) \wedge \forall t \forall y(\phi(\bar{x}, y) \to y \leq t) \to z \leq t \ ) \quad)$$

and that it is a weak version of Dedekind completeness.

For example, given $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{<\}$, any $\mathcal{L}$-expansion of the ordered field of reals (by Dedekind completeness) and its elementary $\mathcal{L}$-extensions;

hence any $\mathcal{L}_{or,E}$-structure $(K, E, <) \models T_{\mathbb{R},\exp}$ is definably complete. Nevertheless, the class of definably complete structures strictly contains the class of o-minimal expansions of the ordered field of reals (see [60]).

Note that if K is a definably complete $\mathcal{L}$-structure, then, as an expansion of an ordered field, K can be equipped with an absolute value: define $|.| : K \to K^{\geq 0}; x \mapsto -x$ if $x < 0$, $x \mapsto x$ otherwise. Then K is a topological $\mathcal{L}_{rings}$-field for the topology of the absolute value.
Moreover, recall that for $n \in \mathbb{N} \setminus \{0\}$, and $V = K^n$,

$$||.||_V := ||.||_\infty : \bar{x} = x_1, \cdots, x_n \mapsto \max_{i=1}^{n} |x_i|$$

defines a norm on the K-vector space K^n (the 'maximum' norm). Then let $n, m, p \in \mathbb{N} \setminus \{0\}$, $V = K^n$, $W = K^m$, $Z = K^p$ and consider $(V, ||.||_V)$, $(W, ||.||_W)$ and $(Z, ||.||_Z)$ in the hypotheses of Fact 3.1.4.

In order to encompass all these situations, we define Hypotheses $\mathcal{D}'$ and $\mathcal{I}m$:

Definition 3.1.7 *Let $\mathcal{L} \supseteq \mathcal{L}_{rings}$ and $(K, \mathcal{V})$ be a topological $\mathcal{L}$-structure. We say that $(K, \mathcal{V})$ satisfies Hypothesis $\mathcal{D}'$ if $\mathcal{V}$ is given by an absolute value $|.| \to \mathbb{R}^{\geq 0}$ or an exponential valuation $|.| \to G \cup \{0\}$, where G is an ordered abelian group.*

Recall that a function f is said to be analytic, if f is infinitely differentiable and for all x in the domain of f, the Taylor series of f at x does converge to f in a neighborhood of x.

Definition 3.1.8 Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$. Let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field that satisfies Hypothesis $\mathcal{D}'$. We will say that it satisfies *Hypothesis $\mathcal{I}m$* if, given $n, m, p \in \mathbb{N} \setminus \{0\}$, and normed K-vector spaces $(K^n, ||.||_\infty)$, $(K^m, ||.||_\infty)$, and $(K^p, ||.||_\infty)$, and a mapping $f = (f_1, \cdots, f_p) : K^n \times K^m \to K^p$ where for $i = 1, \cdots, p$, $f_i \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$;
if there is $U \subseteq K^n \times K^m$ open such that $f : U \to K^p$ is of class C^∞ (resp. analytic), and $(\bar{x}_0, \bar{y}_0) \in U$ such that $f(\bar{x}_0, \bar{y}_0) = 0$, and $y \mapsto J_{(\bar{0},\bar{y})} f_{\bar{x}_0 \bar{y}_0}$ is a K-vector space isomorphism from K^m onto K^p, then

1. there exist open neighborhoods $O \subseteq K^n$ of $\bar{x}_0$ and $O' \subseteq K^m$ of $\bar{y}_0$ such that $O \times O' \subseteq U$,

2. there is a function $g : O \to O'$ such that $f(\bar{x}_0, g(\bar{x}_0)) = \bar{0}$ and such that for all $(\bar{x}, \bar{y}) \in O \times O'$, $f(\bar{x}, \bar{y}) = \bar{0}$ if and only if $\bar{y} = g(\bar{x})$. Furthermore, $\det J_{(\bar{0},\bar{y})} f_{\bar{x},g(\bar{x})} \neq 0$.

3. The function g is of class C^∞ (resp. analytic), and for $\bar{x} \in O$,

$$Jg_{\bar{x}} = -(J_{(\bar{0},\bar{y})} f_{\bar{x},g(\bar{x})})^{-1} J f_{\bar{x},g(\bar{x})}$$

3.1.0.1 Examples

- The topological exponential field $(\mathbb{C}, \exp, |.|)$, by Fact 3.1.4.

- If $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{<, E\}$, and $(K, R, E, \mathcal{V})$ is a definably complete $\mathcal{L}$-structure, and a topological partial E-field the topology of which is the order topology. Then by [60, Theorem 2.2.8 p.19], $(K, R, E, \mathcal{V})$ satisfies Hypothesis $\mathcal{I}m$. Moreover, one can choose O, O' definable, and obtain that the function $g : O \to O'$ is definable. In particular, any $\mathcal{L}_{or,E}$-structure $(K, E, <) \models T_{\mathbb{R},\exp}$ satisfies Hypothesis $\mathcal{I}m$.

- The $\mathcal{L}_{E,p,C}$-structure $(\mathbb{Q}_p, \mathbb{Z}_p, E_p, \mathcal{V}_p)$ (resp. the $\mathcal{L}_{|,E}$-structure $(\mathbb{C}_p, \mathcal{O}_p, E_p, \mathcal{V}_p)$), where $\mathcal{V}_p$ is a base of neighborhoods of 0 in $\mathbb{Q}_p$ (resp. in $\mathbb{C}_p$) for the valuation $|.|_p$, and E_p is analytic on $\mathbb{Z}_p$ (resp. on $\mathcal{O}_p$), satisfies Hypothesis $\mathcal{I}m$ by Fact 3.1.4 applied to the Banach spaces $(K^n, ||.||_\infty)$, $(K^m, ||.||_\infty)$, and $(K^p, ||.||_\infty)$ over $(\mathbb{Q}_p, |.|_p)$ (resp. over $(\mathbb{C}_p, |.|_p)$). Here again, one can choose O, O' definable, and obtain that the function $g : O \to O'$ is definable.

- Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$. Let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field that satisfies Hypotheses $\mathcal{D}$ and $\mathcal{D}'$, and such that K is the field of fractions of R.
  If $\mathcal{L} = \mathcal{L}_{E,p,C}$ and $(R, E) \models T_{\mathbb{Z}_p,E_p}$ (resp. if $\mathcal{L} = \mathcal{L}_{|,E}$ and $(R, E) \models T_{\mathcal{O}_p,E_p}$), and if we are given an open definable set U and $f : U \subseteq K^n \times K^m \to K^p$ analytic (for example $f \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$) and $(\bar{x}_0, \bar{y}_0) \in U$ such that $f(\bar{x}_0, \bar{y}_0) = 0$, and $y \mapsto J_{(\bar{0},\bar{y})} f_{\bar{x}_0\bar{y}_0}$ is an isomorphism from K^m onto K^p); then, as the topology is definable, by Remark 3.1.5 and the above item, the obtained definable function $g : O \to O'$ is continuously differentiable on O, hence by the formula giving Jg it is analytic. Consequently $(K, R, E, \mathcal{V})$ satisfies Hypothesis $\mathcal{I}m$.

In the conditions of Hypothesis $\mathcal{I}m$, let us denote:

$$\begin{array}{rccc} \widehat{\ }: & C^0(O \times O', Z) & \to & C^0(O, Z) \\ & f & \mapsto & \widehat{f} : \bar{x} \mapsto f(\bar{x}, g(\bar{x})) \end{array}$$

Where $C^0(., Z)$ denote the continuous functions from . to Z.

In particular for $i = 1, \cdots, p$, $\widehat{f_i} \equiv 0$, hence for $j = 1, \cdots, |\bar{x}|$, $\frac{\partial \widehat{f_i}}{\partial x_j} \equiv 0$.

Fact 3.1.9 Let h be a continuously differentiable scalar function defined on $O \times O'$ such that $h(\bar{x}_0, \bar{y}_0) = 0$, and with $m = |\bar{y}_0|$. Then $\nabla f_1(\bar{x}_0, \bar{y}_0), \cdots, \nabla f_m(\bar{x}_0, \bar{y}_0), \nabla h(\bar{x}_0, \bar{y}_0)$ are linearly independent if and only if $D\widehat{h}_{\bar{x}_0} \not\equiv 0$.

Proof. The proof is word to word the same than in [65, Lemma 4.7 p.1065]. Let $f_{m+1} := h$ and suppose that $\sum_{i=1}^{m+1} a_i.\nabla f_i(\bar{x}_0, \bar{y}_0) = 0$ and that there is i such that $a_i \neq 0$. If $a_{m+1} = 0$ then $\sum_{i=1}^{m} a_i.\nabla f_i(\bar{x}_0, \bar{y}_0) = 0$ hence all a_i's are zeros as $\nabla f_1(\bar{x}_0, \bar{y}_0), \cdots, \nabla f_m(\bar{x}_0, \bar{y}_0)$ are linearly independent as $Df_{(\bar{x}_0, \bar{y}_0)}$ is surjective. Consequently $a_{m+1} \neq 0$. Let us write $g' : \bar{x} \mapsto (\bar{x}, g(\bar{x}))$. By derivation we have that

$$\frac{\partial \widehat{f_i}}{\partial x_j}(x_0) = \sum_{k=1}^{|\bar{x}_0|+|\bar{y}_0|} \frac{\partial f_i}{\partial x_k}(\bar{x}_0, \bar{y}_0).\frac{\partial g'_k}{\partial x_j}(\bar{x}_0) \quad (D)$$

for $j = 1, \cdots, |\bar{x}_0|$ and $i = 1, \cdots, m+1$. Furthermore, for $j = 1, \cdots, |\bar{x}_0|$,

$$\begin{aligned} \frac{\partial \widehat{f_{m+1}}}{\partial x_j}(\bar{x}_0) &= a_{m+1}^{-1}.\sum_{i=1}^{m+1} a_i \frac{\partial \widehat{f_i}}{\partial x_j}(\bar{x}_0) \quad \text{as for } i = 1, \cdots, m, \ \frac{\partial \widehat{f_i}}{\partial x_j}(\bar{x}_0) = 0 \\ &= a_{m+1}^{-1}.\sum_{k=1}^{|\bar{x}_0|+|\bar{y}_0|} \left(\frac{\partial g'_k}{\partial x_j}(\bar{x}_0) \sum_{i=1}^{m+1} a_i \frac{\partial f_i}{\partial x_k}(\bar{x}_0, \bar{y}_0) \right) \\ &= 0 \end{aligned}$$

as $\sum_{i=1}^{m+1} a_i \frac{\partial f_i}{\partial x_k}(\bar{x}_0, \bar{y}_0) = 0$ by hypothesis. Therefore $D\widehat{h}_{\bar{x}_0} \equiv 0$.

Conversely, suppose that $\nabla f_1(\bar{x}_0, \bar{y}_0), \cdots, \nabla f_m(\bar{x}_0, \bar{y}_0), \nabla f_{m+1}(\bar{x}_0, \bar{y}_0)$ are linearly independent. Let A be the $(|\bar{x}_0| + |\bar{y}_0|) \times (m+1)$ matrix with columns ∇f_i, for $1 \leq i \leq m+1$. By linear independence of the columns, rank $A = m+1$ so ker A is of dimension $|\bar{x}_0|+|\bar{y}_0|-(m+1) = |\bar{x}_0|-1$. By the derivation chain rule (D) and because for $i = 1, \cdots, m$, $\frac{\partial \widehat{f_i}}{\partial x_j}(\bar{x}_0) = 0$, one obtains:

$$\left(\frac{\partial g'_1}{\partial x_j}(\bar{x}_0), \cdots, \frac{\partial g'_{|\bar{x}_0|+|\bar{y}_0|}}{\partial x_j}(\bar{x}_0)\right) A = \left(0, \cdots, 0, \frac{\partial \widehat{f_{m+1}}}{\partial x_j}(\bar{x}_0)\right)$$

for $j = 1, \cdots, |\bar{x}_0|$. As $\frac{\partial g'_i}{\partial x_j} = \delta_{ij}$ for $1 \leq i, j \leq |\bar{x}_0|$, the vectors

$$\left(\frac{\partial g'_1}{\partial x_j}(\bar{x}_0), \cdots, \frac{\partial g'_{|\bar{x}_0|+|\bar{y}_0|}}{\partial x_j}(\bar{x}_0)\right)$$

for $j = 1, \cdots, |\bar{x}_0|$ are linearly independent hence they cannot all be in ker A. Consequently there is some j in $\{1, \cdots, |\bar{x}_0|\}$ for which $\frac{\partial \widehat{f_{m+1}}}{\partial x_j}(\bar{x}_0) \neq 0$. ■

3.1.0.2 Desingularization of E-polynomial functions

This subsubsection will be used only in Section 6.3.

Let K be a field endowed with a definable topology, and let σ_n be a non-empty collection of non-empty definable open subsets of K^n which is closed under finite intersection. Denote by $\mathcal{D}^{(n)}(\sigma_n)$ the set of equivalence classes, or germs, of pairs $[f, U]$ where $U \in \sigma_n$ and $f : U \to K$ is the restriction to U of an E-polynomial function defined on K (thus is infinitely differentiable): $[f_1, U_1]$ and $[f_2, U_2]$ are said to be equivalent if there exists $U \in \sigma_n$, $U \subseteq U_1 \cap U_2$, and for all $x \in U$, $f_1(x) = f_2(x)$.
It is a ring when equipped with the natural operations of addition and multiplication.

Fact 3.1.10 ['Lack of flat functions'] Let $\mathcal{L} = \mathcal{L}_{rings} \cup \{<, E\}$, and $(K, R, E, \mathcal{V})$ be a definably complete $\mathcal{L}$-structure, and a topological partial E-field the topology of which is the order topology. (resp. $\mathcal{L} = \mathcal{L}_{E,p,C}$ and $(R, E) \models T_{\mathbb{Z}_p,E_p}$ and $(K, R, E, \mathcal{V})$ is a topological partial E-field) (resp. $\mathcal{L} = \mathcal{L}_{|,E}$ and $(R, E) \models T_{\mathcal{O}_p,E_p}$ and $(K, R, E, \mathcal{V})$ is a topological partial E-field).
Let M be a Noetherian subring of $\mathcal{D}^{(n)}(\sigma_n)$ closed under differentiation.

Let $I \subseteq M$ be an ideal also closed under differentiation. Then $V(I) = \emptyset$ or $V(I) = K^n$.

The proof is in [65, proof of Theorem 4.9], (resp. uses the fact that the functions are analytic, as it is done in [44, proof of Proposition 5.1.4]: indeed, if I is closed under differentiation and $V(I) \neq \emptyset$, let $[g, U] \in I$ where g is analytic and $\bar{a} \in V(I)$. Then all the partial derivatives of g vanish at $\bar{a}$ so by analyticity $g \equiv 0$ on a neighborhood of $\bar{a}$, hence on $V(I)$, and then $I = \{0\}$ so $V(I) = K^n$.)

Let $(K, R, E, \mathcal{V})$ be a topological partial E-field.
Let $P(\bar{X}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. The E-polynomial P corresponds to a term t of the language $\mathcal{L}_{rings} \cup \{E\}$. This term has been constructed by induction after a finite number of steps. Let us slightly abuse notation and call subterms of 'P' the terms appearing in this iterative construction.
Now let $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. By Fact 2.1.9, endowing $R[\bar{x}]^E$ with the usual partial derivations, there is an isomorphism between E-polynomials and E-polynomial functions. Let $U \in \mathcal{V}$, where $\mathcal{V}$ is a base of neighborhoods of 0 in K such that $(K, R, E, \mathcal{V})$ is a topological partial E-field. The ring M of germs $[f, U]$ generated by functions associated to subterms of $P_1, \cdots, P_p$ is finitely generated and closed by differentiation.

- Let $(K, E, <) \models T_{\mathbb{R},\exp}$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}]^E$, $\bar{a} \in V(P) \subseteq K^{|\bar{X}|}$. Let U be a neighborhood of $\bar{a}$ for the order topology, and $F := \{\exp\}$, or

- Let $(K, R, E, \mathcal{V}) \models Th(\mathbb{Q}_p, \mathbb{Z}_p, E_p, |.|_p)$ (resp. $(K, R, E, \mathcal{V}) \models Th(\mathbb{C}_p, \mathcal{O}_p, E_p, |.|_p)$). Let $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $\bar{a} \in V(P) \subseteq K^{|\bar{X}|}$. Let U be a neighborhood of $\bar{a}$, and $F := \{E_p\}$.

Then, as seen above, the ring M of germs $[f, U]$ generated by functions associated to subterms of $P_1, \cdots, P_p$ on U is finitely generated and closed by differentiation.
Consequently Proofs of [44, Proposition 5.1.4] and [65, Theorem 4.9] go through, we write it for completeness:

If $\bar{a} \in V(f_1, \cdots, f_m)$ and $\nabla f_1(\bar{a}), \cdots, \nabla f_m(\bar{a})$ are linearly independent, we say that $\bar{a} \in V^{reg}(f_1, \cdots, f_m)$.

Fact 3.1.11 Let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial E-field satisfying Hypothesis $\mathcal{I}m$ and 'Lack of flat functions' (3.1.10). Let F be a family of E-polynomial functions $K^n \to K$, $n \in \mathbb{N} \setminus \{0\}$.
Let $\bar{a} \in K^n$ and M a Noetherian subring of the ring $\mathcal{D}^{(n)}(\sigma_n)$ generated by germs associated to F on an open neighborhood U of $\bar{a}$ such that M is closed under differentiation. Let $m \in \mathbb{N}$ and $[f_1, U_1], \cdots, [f_1, U_1]$ be germs in M. Suppose $\bar{a} \in V(f_1, \cdots, f_m)$ and the gradients of $f_1, \cdots, f_m$ at $\bar{a}$, $\nabla f_1(\bar{a}), \cdots, \nabla f_m(\bar{a})$, are linearly independent. Then, exactly one of the following is true:

1. $n = m$; or
2. $m < n$ and for all $[h, W] \in M$ such that $h(\bar{a}) = 0$, h vanishes on $U \cap V^{reg}(f_1, \cdots, f_m)$ for some open neighborhood U of $\bar{a}$.
3. $m < n$ and for some $[h, W] \in M$, $h(\bar{a}) = 0$ and $\nabla f_1(\bar{a}), \cdots, \nabla f_m(\bar{a})$, $\nabla h(\bar{a})$, are linearly independent.

Proof. Suppose $m < n$ and let $d = n - m$. The gradients $\nabla f_1(\bar{a}), \cdots,$ $\nabla f_m(\bar{a})$, are linearly independent, so we can suppose without loss of generality that $J_{(\bar{0}_d, \bar{y}_m)} f_{\bar{a}}$ is invertible, where $|\bar{0}_d| = d$ and $\bar{y}_m = y_1, \cdots, y_m$. Let $\lambda : \bar{x} \mapsto \det J_{(\bar{0}_d, \bar{y}_m)} f_{\bar{x}}$. There is a neighborhood U of $\bar{a}$ on which λ is invertible, and $\Lambda := [\lambda, U] \in \mathcal{D}^{(n)}(\sigma_n)$ is invertible.
Let $M^* := M[\Lambda^{-1}]$; denote $\bar{a}_d \bar{a}_m := \bar{a}$, where $\bar{a}_d := a_1, \cdots a_d$ and $\bar{a}_m := a_{d+1}, \cdots, a_n$. Consider $\widehat{M^*}$, where $\widehat{\ }$ is the map defined in the conditions of Hypothesis $\mathcal{I}m$. Note that $\widehat{M^*}$ is a Noetherian subring of $\mathcal{D}^{(d)}(\sigma_d)$.
By Hypothesis $\mathcal{I}m$, we have that:

$$\begin{aligned} Jg_{\bar{x}_d} &= -(J_{(\bar{0}_d, \bar{y}_m)} f_{\bar{x}_d, g(\bar{x}_d)})^{-1} J f_{\bar{x}_d, g(\bar{x}_d)} \\ &= -\lambda^{-1}\, {}^t com(J_{(\bar{0}_d, \bar{y}_m)} f_{\bar{x}_d, g(\bar{x}_d)}) J f_{\bar{x}_d, g(\bar{x}_d)} \end{aligned}$$

Consequently the partial derivatives of g belong to M^*, and then by the chain rule of derivations, $\widehat{M^*}$ is closed under differentiation.

Then let $I := \{g \in \widehat{M^*} : g(\bar{a}_d) = 0\}$.

- If $I = \{0\}$. Let $g = [h, W] \in M$, such that $h(\bar{a}) = 0$. Then $\widehat{g}(\bar{a}_d) = 0$, hence $\widehat{g} \in I$, hence $\widehat{g} = 0$. The latter meaning that h vanishes on $U \cap V^{reg}(f_1, \cdots, f_m)$ for some open neighborhood U of $\bar{a}$.

- If $I \neq \{0\}$. By Fact 3.1.10, I is not closed under differentiation, thus there is $g \in M^*$ and $1 \leq i \leq d$ such that $\widehat{g} \in I$ and $\frac{\partial \widehat{g}}{\partial x_i} \notin I$. Nevertheless, there is a power of Λ, say Λ^k, such that $\Lambda^k.g \in M$. Let $h = \Lambda^k.g$. Then $h(\bar{a}) = 0$ and

$$\frac{\partial \widehat{h}}{\partial x_i}(\bar{a}_d) = \left(k\widehat{\Lambda}^{k-1}(\bar{a}_d)\frac{\partial \widehat{\Lambda}}{\partial x_i}\widehat{g}(\bar{a}_d) \right) + \left(\widehat{\Lambda}^k(\bar{a}_d)\frac{\partial \widehat{g}}{\partial x_i}(\bar{a}_d) \right) \neq 0$$

  Consequently $D\widehat{h}_{\bar{a}} \not\equiv 0$, thus by Fact 3.1.9, $\nabla f_1(\bar{a}), \cdots, \nabla f_m(\bar{a})$, $\nabla h(\bar{a})$, are linearly independent.

■

3.2 Newton-Kantorovich's theorem

Fact 3.2.1 (Newton-Kantorovich's Theorem) [22, Theorem p.10], [60, Theorem 1.4.1, p.13] *Let $(V, ||.||_V)$, $(W, ||.||_W)$ $(Z, ||.||_Z)$ be either Banach spaces over $(\mathbb{R}, |.|)$, or of the form $(K^n, ||.||_\infty)$, $(K^m, ||.||_\infty)$, and $(K^p, ||.||_\infty)$, where K is a definably complete $\mathcal{L}$-structure, for $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{<, E\}$, $m, n, p \in \mathbb{N} \setminus \{0\}$. Let U be an open convex subset of V. Let $f : U \to W$ be Fréchet differentiable on U with*

$$||Df_x - Df_y|| = \sup_{u \in V} \frac{||Df_x(u) - Df_y(u)||_W}{||u||_V} \leq \lambda ||x - y||_V \quad (*)$$

for some $\lambda > 0$ and for $x, y \in U$. Assume that $x_0 \in U$ is such that:

1. *$Df_{x_0}^{-1} : W \to V$ exists*
2. *there is $\xi > 0$, $||Df_{x_0}^{-1}|| \leq \xi$*
3. *there is $\epsilon > 0$, $||Df_{x_0}^{-1} f(x_0)|| \leq \epsilon$*
4. *$h = 2\xi\lambda\epsilon < 1$*

5. $B(x_0, t^*) := \{x : ||x - x_0||_V < t^*\} \subseteq U$, *where* $t^* := \frac{2}{h}(1 - \sqrt{1-h})\epsilon$

In the case where K *is a definably complete* $\mathcal{L}$*-structure, assume also that* f *is definable.*

Then

1. *there is* $x^* =$ *such that* $f(x^*) = 0$ *and* $x^* \in \bar{B}(x_0, t^*)$
2. x^* *is the only solution of* $f(x) = 0$ *in the set* $B(x_0, t') \cap U$, *where* $t' := \frac{2}{h}(1 + \sqrt{1-h})\epsilon$

Lemma 3.2.2 *Let* $\mathbb{L} := \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{R}((t))^{LE}$, *and let* $\mathcal{W}$ *be a base of neighborhoods of* 0 *in* $\mathbb{L}$ *for the topology of the absolute value* $|.|$. *Let* $(K, \exp) \subsetneq (\mathbb{L}, \exp)$ *be an* E*-subfield such that* $\mathbb{Q} \subseteq K$ *and let* $\mathcal{V}$ *be the base of neighborhoods of* 0 *in* K *induced by* $\mathcal{W}$.

Let $Q(\bar{X}) = (Q_1(\bar{X}), \cdots, Q_m(\bar{X}))$, *where for* $i = 1, \cdots, m$, $Q_i(\bar{X}) \in K[\bar{X}]^E$. *Let* $\bar{a} \in K^n$ *be a regular zero of* Q.

Then there is $t^* \in \mathbb{R}, t^* > 0$ *and a neighborhood* $W \in \mathcal{W}$, *such that for any elements* $t_1, \cdots, t_d$ *in* W *ecl-independent over* K, *where* $d \in \mathbb{N}$ *and* $d + m \leq n$, *letting*

$$\bar{a}_0 := a_1 + t_1, \cdots, a_d + t_d, a_{d+1}, \cdots, a_n$$

then there is a zero of Q *in* $ecl^{\mathbb{L}}(K(t_1, \cdots, t_d)) \cap B(\bar{a}_0, t^*) \subseteq \mathbb{L}$.

Proof. Let $\bar{t}_d := t_1, \cdots, t_d \subseteq B(0, 1) \subseteq \mathbb{L}$ be an *ecl*-independent tuple over K, $r := n - d - m$, and let $\bar{a}_d := a_1, \cdots, a_d$, $\bar{a}_r := a_{d+1}, \cdots, a_r$, and $\bar{a}' := a_{r+1}, \cdots, a_n$.

Consider

$$Q'(\bar{X}') := (Q_1(\bar{a}_d, \bar{a}_r, \bar{X}'), \cdots, Q_{2n-d-r}(\bar{a}_d, \bar{a}_r, \bar{X}'))$$

$$Q'^t(\bar{X}') := (Q_1(\bar{a}_d + \bar{t}_d, \bar{a}_r, \bar{X}'), \cdots, Q_{2n-d-r}(\bar{a}_d + \bar{t}_d, \bar{a}_r, \bar{X}'))$$

where $\bar{X}' := X_{r+1}, \cdots, X_n$.

Let $U = B(\bar{a}', 1) \subseteq \mathbb{L}^{2n-d-r} = \mathbb{L}^m$.

1. Because $\bar{a}$ is a regular zero of Q, we have that det $DQ'_{\bar{a}'} \neq 0$. By continuity, there is $0 < \delta_1 \leq \frac{1}{2}$ such that $||\bar{t}_d||_{\mathbb{L}^d} < \delta_1$ implies det $DQ'^t_{\bar{a}'} \neq 0$. Hence $DQ'^t_{\bar{a}'}$ is invertible.

2. $||[DQ'_{\bar{a}'}]^{-1}|| := \sup_{u \in \mathbb{L}^{2n-d-r}} \dfrac{||[DQ'_{\bar{a}'}]^{-1}(u)||}{||u||}$
   By continuity there is $0 < \delta_2 \leq \delta_1$ such that $||\bar{t}_d||_{\mathbb{L}^d} < \delta_2$ implies
   $$||[DQ'^t_{\bar{a}'}]^{-1}|| < 2||[DQ'_{\bar{a}'}]^{-1}||$$
   Let $\xi := \max\{1, 2||[DQ'_{\bar{a}'}]^{-1}||\}$.

3. DQ' is a continuous linear operator, hence is Lipschitz continuous– equivalently satisfies assumption $(*)$ of Fact 3.2.1–. Let λ_1 be the Lipschitz constant for DQ' on $\bar{U}$. By continuity , there is $0 < \delta_3 \leq \delta_2$ such that $||\bar{t}_d||_{\mathbb{L}^d} < \delta_3$ implies that DQ'^t satisfies $(*)$ for the Lipschitz constant $2\lambda_1$, hence
   $$\sup_{\bar{x}' \neq \bar{u}' \in \bar{U}} \frac{||DQ'^t_{\bar{x}'} - DQ'^t_{\bar{u}'}||}{||\bar{x}' - \bar{u}'||} \leq 2\lambda_1$$
   Let $\lambda := \max\{1, 2\lambda_1\}$.

4. Let $\epsilon := \frac{\delta_1}{2\lambda\xi}$. Recall that $\bar{t}_d$ do not appear in Q' which is defined on K. By continuity there is $0 < \delta_4 \leq \delta_3$ such that $||\bar{t}_d||_{\mathbb{L}^d} < \delta_4$ implies
   $$||Q'^t(\bar{a}')[DQ'^t_{\bar{a}'}]^{-1} - Q'(\bar{a}')[DQ'_{\bar{a}'}]^{-1}|| < \epsilon$$

5. Take $\bar{t}_d \subseteq B(\bar{0}, \delta_4)$ Then $h = 2.\xi.\lambda.\epsilon = \delta_1 \leq \frac{1}{2}$.

6.
   $$\begin{aligned} t^* &= \frac{2}{h}(1 - \sqrt{1-h})\epsilon \\ &= \frac{2}{\delta_1}(1 - \sqrt{1-\delta_1})\frac{\delta_1}{2\lambda\xi} \end{aligned}$$
   We have that $-\sqrt{1-\delta_1} \leq -(1-\delta_1)$ and then that $t^* \leq \frac{\delta_1}{\lambda\xi} \leq \delta_1$.

Consequently the hypotheses of the theorem are satisfied and we can let $W := B(\bar{0}, \delta_4)$. Using Newton-Kantorovich Theorem 3.2.1, we then find a point $\bar{a}'^*$ such that $Q'^t(\bar{a}'^*) = 0$, hence the point $(\bar{a}_d + \bar{t}_d, \bar{a}_r, \bar{a}'^*)$ is in A. ■

3.3 Hensel's Lemma in Laurent series

3.3.1 Laurent series

Let $(K, E, \mathcal{V})$ be a topological E-field, and let $(x^{\mathbb{Z}}, .)$ be a multiplicative copy of the additive abelian group $(\mathbb{Z}, +)$.

Set $t := x^{-1}$ and $K((t)) := K((x^{\mathbb{Z}}))$, field of Hahn series as constructed in Subsection 2.3.2. Hence for $s = \sum s_k t^{-k} = \sum s_k x^k \in K((x^{\mathbb{Z}}))$,

$$Supp\, s = \{x^k \in x^{\mathbb{Z}} : s_k \neq 0\}$$

$$Lm(s) = \max\, Supp\, s$$

$$Lc(s) = s_{k_0} \;\text{ where }\; x^{k_0} = Lm(s)$$

Then one constructs a topological partial E-field $(K((t)) := K((x^{\mathbb{Z}})), E, \mathcal{W})$, where $\mathcal{W}$ is defined as in subsection 2.3.2:

$$W_{V,0} := \{s \in K((x^{\mathbb{Z}}))\; :\; Lm(s) \leq 1 \text{ and if } Lm(s) = 1, \text{ then } Lc(s) \in V\}$$

$$W_{x^k} := \{s \in K((x^{\mathbb{Z}}))\; :\; Lm(s) \leq x^k\}$$

$$\mathcal{W}(K) := \{W_{V,0}\; :\; V \in \mathcal{V}\}$$

$$\mathcal{W} := \mathcal{W}(K) \cup \{W_{x^k}\; :\; x^k \in x^{\mathbb{Z}}\}$$

Recall that if $\mathcal{V}$ is not discrete then $\mathcal{W}(K)$ satisfies $Comp(K)$, which induces an equivalence relation $\sim_{\mathcal{W}(K)}$ on $K((t))$ with in particular:

$$t \sim_{\mathcal{W}(K)} 0$$

When $\mathcal{V}$ is the discrete topology on K, then $\mathcal{W}$ corresponds to the topology given by the canonic ultrametric absolute value of $K((t))$:

$$|.| : \left\{ \begin{array}{ccc} K((x^{\mathbb{Z}})) & \to & x^{\mathbb{Z}} \cup \{0\} \\ s & \mapsto & |s| = Lm(s) \end{array} \right.$$

With valuation ring

$$K[[x^{\mathbb{Z}}]] = \{s : Lm(s) \leq 1\}$$

and maximal ideal

$$\mathbf{m}(K[[x^{\mathbb{Z}}]]) = \{s : Lm(s) < 1\}$$

Equivalently

$$v : \begin{cases} K((t)) & \to & \mathbb{Z} \cup \{\infty\} \\ s = \sum_i k_i t^i & \mapsto & v(s) = \min\{i : k_i \neq 0\} \end{cases}$$

With valuation ring $K[[t]] = \{s = \sum k_i t^i : i \geq 0\}$ and maximal ideal $\mathbf{m}(K[[t]]) = \{s = \sum k_i t^i : i > 0\}$.

We will abuse notation and denote also v by $|.|$.

For $k, l \in K((t))$, set $k \sim_K l$ iff $|k - l| < 1$ that is $k - l \in \mathbf{m}(K[[t]])$. Note that $t \sim_K 0$ in $K((t))$.

Remark 3.3.1 *Starting from a topological field $(K, \mathcal{V})$, let us consider both defined topologies on $K((t))$: the one given by $\mathcal{W}$ and the one given by $|.|$ that is trivial on K. Let $s \in K((t))$. Then*

$$s \sim_K 0 \;\; \textit{iff}\; s \sim_{\mathcal{W}(K)} 0$$

Indeed if $s \sim_K 0$, trivially $s \in V$ for all $V \in \mathcal{W}(K)$. The converse holds too because we have supposed the topology given by $\mathcal{V}$ on K to be Hausdorff.
We will use this equivalence mainly in Section 7.

Endow $x_1^{\mathbb{Z}} \times \cdots \times x_n^{\mathbb{Z}}$ with the antilexicographic order and consider

$$K((x_1^{\mathbb{Z}})) \cdots ((x_n^{\mathbb{Z}})) \cong K((x_1^{\mathbb{Z}} \times \cdots \times x_n^{\mathbb{Z}}))$$

First notice that one can define a topology on $K((x_1^{\mathbb{Z}})) \cdots ((x_n^{\mathbb{Z}}))$ by recurrence, considering, for $i = 2, \cdots, n$, $K((x_1^{\mathbb{Z}})) \cdots ((x_i^{\mathbb{Z}}))$ as a field of Hahn series over $K((x_1^{\mathbb{Z}})) \cdots ((x_{i-1}^{\mathbb{Z}}))$ and endowing it by $\mathcal{W}_i$, with $\mathcal{W}_0 := \mathcal{V}$ and $\mathcal{W}_1$ constructed as $\mathcal{W}$ above, thus

$$Lm_i : K((x_1^{\mathbb{Z}})) \cdots ((x_i^{\mathbb{Z}})) \to K((x_1^{\mathbb{Z}})) \cdots ((x_{i-1}^{\mathbb{Z}}))$$

For $s \in K((x_1^{\mathbb{Z}})) \cdots ((x_n^{\mathbb{Z}}))$, let

$$Lm(s) := (Lm_1 (Lc_2 (\cdots (Lc_n(s)))), \cdots, Lm_{n-1}(Lc_n(s)), Lm_n(s))$$

$$\begin{aligned} W_{n,V,0} := \{s \in K((x_1^{\mathbb{Z}})) \cdots ((x_n^{\mathbb{Z}})) \; : Lm(s) \leq \bar{1} \text{ and if } \\ Lm(s) = \bar{1}, \text{ then } Lc(s) \in V\} \end{aligned}$$

$$W_{n,\bar{x}^{\bar{k}}} := \{s \in K((x_1^{\mathbb{Z}}))\cdots((x_n^{\mathbb{Z}})) \ : \ Lm(s) \leq (x_1^{k_1}, \cdots, x_n^{k_n})\}$$
$$\mathcal{W}_n(K) := \{W_{n,V,0} \ : \ V \in \mathcal{V}\}$$
$$\mathcal{W}_n := \mathcal{W}_n(K) \cup \{W_{n,\bar{x}^{\bar{k}}} \ : \ \bar{x}^{\bar{k}} \in x_1^{\mathbb{Z}} \times \cdots \times x_n^{\mathbb{Z}}\}$$

where $\bar{x}^{\bar{k}} = (x_1^{k_1}, \cdots, x_n^{k_n})$.

Now let us denote $K((\bar{t})) := K((t_1))\cdots((t_n))$
(resp. $K[[\bar{t}]] := K[[t_1]]\cdots[[t_n]]$).

Let $\bar{i} := (i_1, \cdots, i_n)$, $\bar{t} := (t_1, \cdots, t_n)$, $k_{\bar{i}} := k_{i_n,\cdots,i_1}$, and

$$\sum_{\bar{i}} k_{\bar{i}} \bar{t}^{\bar{i}} := \sum_{i_n} \left(\sum_{i_{n-1}} \left(\cdots \sum_{i_1} k_{i_n,\cdots,i_1} t_1^{i_1} \cdots \right)_{n-1} t_{n-1}^{i_{n-1}} \right)_n t_n^{i_n}$$
$$= \sum_{i_n} \cdots \sum_{i_1} k_{i_n,\cdots,i_1} t_1^{i_1} \cdots t_n^{i_n}$$

If $\mathcal{V}$ is trivial on K, one can similarly define an absolute value $|.|$ on $K((t_1))\cdots((t_n))$ by recurrence, considering, for $i = 2, \cdots, n$, $K((t_1))\cdots((t_i))$ as a field of Hahn series over $K((t_1))\cdots((t_{i-1}))$ and endowing it by $|.|_i$:

$$|.| : \begin{cases} K((x_1^{\mathbb{Z}}))\cdots((x_n^{\mathbb{Z}})) & \to & (x_1^{\mathbb{Z}} \times \cdots \times x_n^{\mathbb{Z}}) \cup \{0\} \\ s & \mapsto & (|Lc_2(\cdots(Lc_n(s)))|_1, \cdots, |Lc_n(s)|_{n-1}, |s|_n) \end{cases}$$

Or equivalently by $|.| : K((t_1))\cdots((t_n)) \to (\mathbb{Z} \times \cdots \times \mathbb{Z}) \cup \{\infty\}$.

Lemma 3.3.2 $(K[[\bar{t}]], |.|)$, *is complete.*

Proof. Let $(s_p := \sum_{\bar{i}} k_{p,\bar{i}} \bar{t}^{\bar{i}})$ be a Cauchy sequence. For $h \in \mathbb{N}$, $s_{p+h} - s_p$ converges to 0 in $K[[\bar{t}]]$ when $p \to \infty$, hence for each $\bar{i}$, $k_{p+h,\bar{i}} - k_{p,\bar{i}}$ converges to 0 in $K[[\bar{t}]]$. As each $k_{p,\bar{i}} \in K$, for each $\bar{i}$ there is $N_{\bar{i}}$ such that if $p \geq N_{\bar{i}}$, $k_{p+h,\bar{i}} - k_{p,\bar{i}} = 0$. In other words, the sequence $(k_{p,\bar{i}})_p$ is stationnary after a finite number of steps, thus converges to some $k_{\bar{i}} \in K$.

Let $s := \sum_{\bar{i}} k_{\bar{i}} \bar{t}^{\bar{i}}$, and let $r \in \mathbb{N}$.

$$s_p - s = \sum_{\bar{i}} (k_{p,\bar{i}} - k_{\bar{i}}) \bar{t}^{\bar{i}} = \sum_{i_1,\cdots,i_n=0}^{r} (k_{p,\bar{i}} - k_{\bar{i}}) \bar{t}^{\bar{i}} + \sum_{i_1,\cdots,i_n \geq r+1} (k_{p,\bar{i}} - k_{\bar{i}}) \bar{t}^{\bar{i}}$$

Hence

$$|s_p - s| \geq \min\left\{\left|\sum_{i_1,\cdots,i_n=0}^{r}(k_{p,\bar{i}} - k_{\bar{i}})\bar{t}^{\bar{i}}\right|, |\bar{t}^{r+1}|\right\}$$

where $\bar{t}^{r+1} := (t_1^{r+1}, \cdots, t_n^{r+1})$.

Let $N_r := \max_{0\leq i_1,\cdots,i_n\leq r} N_{\bar{i}}$. For $p \geq N_r$,

$$\left|\sum_{i_1,\cdots,i_n=0}^{r}(k_{p,\bar{i}} - k_{\bar{i}})\bar{t}^{\bar{i}}\right| = |\bar{0}| = \infty$$

hence $|s_p - s| \geq (r+1, \cdots, r+1) > (r, \cdots, r)$. Consequently,

$$\forall r \exists N, p \geq N \rightarrow |s_p - s| > \bar{r}$$

Thus (s_p) converges to s in $K[[\bar{t}]]$. ■

From now on we will work with the exponential valuation.

Now let $m \in \mathbb{N}\setminus\{0\}$, and $\bar{x} \in K[[\bar{t}]]^m$. Define $||\bar{x}|| := \max_{i=1}^m |x_i|$, and for $A = (a_{ij})_{i,j=1,\cdots,m}$, where each $a_{i,j} \in K[[\bar{t}]]$, $||A|| := \max_{i,j=1}^m |a_{ij}|$. This makes $K[[\bar{t}]]^m$ and $\mathcal{M}_{m,m}(K[[\bar{t}]])$, the set of $m \times m$ matrices with coefficients in $K[[\bar{t}]]$ equipped with the addition of matrices, complete normed vector spaces over $K[[\bar{t}]]$. If K is equipped with an exponential E, let us denote, for $\bar{k} := k_1, \cdots, k_m \in K[[\bar{t}]]^m$, $E(\bar{k}) := (E(k_1), \cdots, E(k_m))$, and in the particular case where $\bar{k} \in \mathbf{m}(K[[\bar{t}]])^m$, by Remark 2.3.2,

$$\exp(\bar{k}) = \sum_i \frac{\bar{k}^i}{i!} = (\sum_i \frac{k_1^i}{i!}, \cdots, \sum_i \frac{k_m^i}{i!})$$

Lemma 3.3.3 *Let $m := |\bar{X}| \in \mathbb{N}\setminus\{0\}$, and H_H defined by*

$$H(\bar{X}) := (H_1(\bar{X}), \cdots, H_m(\bar{X})) \in K[[\bar{t}]][\bar{X}]^E$$

be a Hovanskii system. Then for $\bar{a}, \bar{h} \in K[[\bar{t}]]^m$, $\bar{h} \sim_K \bar{0}$, and $H(\bar{a}) \sim_K \bar{0}$,

$$||H(\bar{a} + \bar{h}) - H(\bar{a}) - \bar{h}.JH_{\bar{a}}|| \leq ||\bar{h}||^2$$

Proof. First note that if $\bar{a} := \bar{a}_0 + \bar{a}_1$, where $\bar{a}_1, \bar{h} \in \mathbf{m}(K[[\bar{t}]]^m)$ and $\bar{a}_0 \in K^m$, then $E(\bar{a}+\bar{h}) = E(\bar{a}_0)E(\bar{a}_1+\bar{h}) = E(\bar{a}_0)\sum_k \frac{(\bar{a}_1+\bar{h})^k}{k!}$. Then by [12, Remark 4.5.2 & Proposition 4.5.3], for $i = 1, \cdots, m$, H_i is analytic as an iteration of compositions of E and polynomial functions. Hence

$$|H_i(\bar{a}+\bar{h}) - H_i(\bar{a}) - h_i.\nabla H_i(\bar{a})| \leq |h_i|^2$$

Therefore:

$$||H(\bar{a}+\bar{h}) - H(\bar{a}) - \bar{h}.JH_{\bar{a}}|| \leq ||\bar{h}||^2$$

■

3.3.2 Hensel's lemma

Proposition 3.3.4 *Let (K, R, E) be a partial E-field, and $H \subseteq K[[\bar{t}]][\bar{X}] \otimes_{R[[\bar{t}]][\bar{X}]} R[[\bar{t}]][\bar{X}]^E$ defining a $n \times n$ Hovanskii system. Let $\bar{a} \in K[[\bar{t}]]^n$ such that $H(\bar{a}) \sim_K \bar{0}$ and $\det JH_{\bar{a}} \nsim_K 0$. Then there is $\bar{b} \in K[[\bar{t}]]^n$ such that $H(\bar{b}) = \bar{0}$, $\bar{b} \sim_K \bar{a}$, and $\det JH_{\bar{b}} \nsim_K 0$.*

Proof. Let $\bar{a}_0 := \bar{a}$, and consider the order one Taylor development of H at $\bar{a}$, namely:

$$T_{H,\bar{a},1}(\bar{h}) := H(\bar{a}) + JH_{\bar{a}}.\bar{h}$$

Then $T_{H,\bar{a},1}(\bar{h}) = \bar{0}$ is a linear system of equations in $\bar{h}$, solvable as $\det JH_{\bar{a}} \neq 0$:

$$\begin{pmatrix} \frac{\partial H_1}{\partial x_1}(\bar{a}) & \cdots & \frac{\partial H_1}{\partial x_n}(\bar{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial H_n}{\partial x_1}(\bar{a}) & \cdots & \frac{\partial H_n}{\partial x_n}(\bar{a}) \end{pmatrix} . \begin{pmatrix} -h_1 \\ \vdots \\ -h_n \end{pmatrix} = \begin{pmatrix} H_1(\bar{a}) \\ \vdots \\ H_n(\bar{a}) \end{pmatrix}$$

We call $\bar{b}_1$ its solution:

$$\begin{aligned} \bar{b}_1 &= -(JH_{\bar{a}})^{-1}H(\bar{a}) \\ &= \frac{1}{\det JH_{\bar{a}}}\ {}^t com(JH_{\bar{a}})H(\bar{a}) \end{aligned}$$

where ${}^t com(JH_{\bar{a}}) := (c_{i,j})$ is the tranpose of the cofactor matrix of $JH_{\bar{a}}$. For each $i, j = 1, \cdots, n$, $|c_{i,j}| = 1$ as $c_{i,j}$ is a minor of $\det JH_{\bar{a}}$, hence

$||\bar{b}_1|| \leq ||H(\bar{a})||$, as $|\det JH_{\bar{a}}| = 1$:

$$\begin{aligned} |b_{1,i}| &= |(\det JH_{\bar{a}})^{-1}|. \left| \sum_j c_{i,j}.H_j(\bar{a}) \right| \\ &\leq 1. \max_j |c_{i,j}|.|H_j(\bar{a})| \\ &= \max_j |H_j(\bar{a})| \\ &= ||H(\bar{a})|| < 1 \end{aligned}$$

Consequently $\bar{b}_1 \sim_K \bar{0}$. Let $\bar{a}_1 := \bar{a}_0 + \bar{b}_1$. By continuity, $H(\bar{a}_1) \sim_K \bar{0}$ and $\det JH_{\bar{a}_1} \nsim_K 0$. Therefore we can reiterate and find a solution of $T_{H,\bar{a}_1,1}(\bar{h}) = \bar{0}$ which we call $\bar{b}_2$. By Lemma 3.3.3, as $H(\bar{a}) = -JH_{\bar{a}}.\bar{b}_1$, we have

$$|H_i(\bar{a}_1)| = |H_i(\bar{a}_1) - H_i(\bar{a}) + \nabla H_i(\bar{a}).b_{1,i}| \leq |b_{1,i}|^2 \leq ||\bar{b}_1||^2 < 1$$

And $||H(\bar{a}_1)|| = \max_i |H_i(\bar{a}_1)| \leq ||\bar{b}_1||^2$.

Then for $k \in \mathbb{N}$, let $\bar{a}_{k+1} := \bar{a}_k + \bar{b}_{k+1}$. Following the induction process, one obtains for each $i = 1, \cdots, n$ and $k \in \mathbb{N}$:

$$|a_{k+1,i} - a_{k,i}| = |b_{k+1,i}| \leq ||H(\bar{a}_k)|| \leq ||\bar{b}_k||^2 \leq ||\bar{b}_1||^{2^k} \leq ||H(\bar{a})||^{2^k}$$

The sequences $(||\bar{b}_k||)_k$ and $(||H(\bar{a}_k)||)_k$ converge to $\bar{0}$, hence the sequences $(\bar{b}_k)_k$ and $(H(\bar{a}_k))_k$ converge to $\bar{0}$ too. Moreover, the sequences $(a_{k,i})_k$ are Cauchy sequences: indeed,

$$a_{k+h,i} - a_{k,i} = a_{k+h,i} - a_{k+h-1,i} + a_{k+h-1,i} - a_{k+h-2,i} + \cdots - a_{k,i}$$

hence

$$|a_{k+h,i} - a_{k,i}| \leq \max_{j=1}^{h} \{|a_{k+j,i} - a_{k+j-1,i}|\} = |a_{k+1,i} - a_{k,i}| \leq ||H(\bar{a})||^{2^k}$$

By completeness of $K[[\bar{t}]]$, $(a_{k,i})_{k\geq 1}$ converges to some c_i in $K[[\bar{t}]]$. Then, one gets that $|c_i - a_i| \leq 1$, and furthermore that $c_i \sim_K a_i$. As each H_i is continuous, the sequence $(H(\bar{a}_k))_k$ converges to $H(\bar{c})$, so $H(\bar{c}) = \bar{0}$. Furthermore as $\bar{c} - \bar{a} \sim_K \bar{0}$, we have that $\det JH_{\bar{c}} \nsim_K 0$ by continuity (otherwise $\det JH_{\bar{a}} \sim_K 0$) hence it is different from 0. ■

3.3.3 An E-field containing Laurent series

In this section we construct a topological unordered E-field of underlying set $K((t))^E$–hence which contains Laurent series $K((t))$–starting from a topological unordered E-field $(K, E, \mathcal{V})$. We follow and adapt the 'exponential part' of the iterative construction of the field $\mathbb{R}((t))^{LE}$ in [18]. Note that it is then possible to iterate the construction to obtain an E-field $K((\bar{t}))^E := ((K((t_1))^E)\cdots)((t_n))^E$ containing $K((t_1))\cdots((t_n))$.

3.3.3.1 Orderable groups and Hahn series

We begin by recalling that as K is a field of characteristic 0, its additive group, being abelian and torsion-free, is orderable by Fact 2.2.11, and that we can choose the order with $0 < 1$.

Definition 3.3.5 Let K be a field of characteristic 0 such that $(K, +, <_K$ $)$ is an ordered additive group, let $(G, ., <_G)$ be an ordered abelian group and consider the field of generalized series $K((G))$. For $a, b \in (K((G)), +)$, set $a < b$ iff $Lc(b - a) >_K 0$.

Remark 3.3.6 *The order $<$ on $K((G))$ is implicitly defined from $<_G$ as well as from $<_K$, as for $s \in K((G))$, $Lc(s)$ is determined by $|s| = Lm(s) = \max Supp(s) \in G$.*

Lemma 3.3.7 *The order $<$ on $(K((G)), +)$ of Definition 3.3.5 is a well-defined order which is compatible with the group law and which extends both $<_K$ and $<_G$. Furthermore $(K((G^{\leq 1})), +, <)$ is convex in $(K((G)), +, <)$.*

Proof. It is obvious that $<$ defines an order on $K((G))$ compatible with the group law (the set $S = \{s : s > 0\}$ satisfies Items 1, 2, and 3 of Fact 2.2.10), and that this order extends $<_K$. To see that $<$ extends $<_G$, first note that for $g \in G$, $Lc(g) = 1$. Hence for $g_1, g_2 \in G$, $Lc(g_2 - g_1) = 1$ iff $g_2 >_G g_1$, $Lc(g_2 - g_1) = -1$ iff $g_2 <_G g_1$, and $Lc(g_2 - g_1) = 0$ iff $g_2 = g_1$. In other words we have that $g_1 < g_2$ iff $Lc(g_2 - g_1) >_K 0$ iff $Lc(g_2 - g_1) = 1$ iff $g_1 <_G g_2$.

Next we show that $(K((G^{\leq 1})), +, <)$ is convex in $(K((G)), +, <)$:
Consider a, b two elements in $K((G^{\leq 1}))$, and $c \in K((G))$ such that $a < c < b$. Suppose by way of contradiction that $c \notin K((G^{\leq 1}))$. Then $|c| > 1$ but $|a|, |b| \leq 1$. Hence $|c| > |a|$ and $|c| > |b|$, thus:

1. $Lc(c-a) = Lc(c) > 0$ because $a < c$.
2. $Lc(b-c) = Lc(-c) = -Lc(c)$ because $|b| < |c|$ and $-Lc(c) > 0$ because $c < b$.

which is a contradiction –if $c \neq 0$– so $c \in K((G^{\leq 1}))$. ■

We adapt the notion of 'pre-exponential' ordered field defined by L.van den Dries, A.Macintyre and D.Marker in [18] to:

Definition 3.3.8 We define a quintuple $(F, A, \mathcal{O}, E, \mathcal{V})$, where F is a field of characteristic 0 equipped with a base $\mathcal{V}$ of neighborhoods of 0 making it a topological field, and where $(F, +, <)$ is an ordered group, to be a *pre-exponential topological field* if

1. $(A, +, <)$ and $(\mathcal{O}, +, <)$ are ordered subgroups of $(F, +, <)$ and $\mathcal{O}$ is convex in $(F, +, <)$,
2. $F = A \oplus \mathcal{O}$, and
3. $E : (\mathcal{O}, +, 0) \to (F^{\times}, ., 1)$ is a homomorphism of abelian groups, continuous for the topology induced by $\mathcal{V}$ on $\mathcal{O}$.

Note that E is asked to be continuous for the topology $\mathcal{V}$ of F, and not especially for the order topology of the additive group.

3.3.3.2 Iterative construction of $K((t))^E$ as a union of Hahn fields

Starting from a topological E-field $(K, E, \mathcal{V})$, we follow the construction of [18] that we adapt to our topological unordered context.
To begin let $K_{-1} := K$, $\mathcal{V}_{-1} := \mathcal{V}$, and order the additive group $(K, +)$ by $<$ by Fact 2.2.11: indeed, K is torsion free as the underlying additive group of a characteristic 0 field. Let $K_0 := K((x^K))$;

$$\mathcal{O}_0 := K((x^{K^{\leq 1}})) = \{s \in K_0 \,:\, |s| \leq 1\}$$

$$A_0 := K((x^{K^{>1}})) = \{s \in K_0 \,:\, \mathit{Supp}\, s > 1\}$$

By Lemma 3.3.7, $(\mathcal{O}_0, +, <)$ is convex in $(K_0, +, <)$. Define $\mathcal{V}_0$ as in Subsection 2.3.2 on Hahn series.

$$W_{V,0} := \{s \in K((x^K)) \,:\, Lm(s) \leq 1 \text{ and if } Lm(s) = 1, \text{ then } Lc(s) \in V\}$$

$$W_{x^k} := \{s \in K((x^K)) \,:\, Lm(s) \leq x^k\}$$
$$\mathcal{V}_0(K) := \{W_{V,0} \,:\, V \in \mathcal{V}_{-1}\}$$
$$\mathcal{V}_0 := \mathcal{W}(K) \cup \{W_{x^k} \,:\, x^k \in x^K\}$$

As seen in Subsection 2.3.2 on Hahn series it is possible to extend E to a homomorphism $E_0 : \mathcal{O}_0 \to K_0^{\times}$:

$$E_0 : \begin{cases} \mathcal{O}_0 = K_{-1} \oplus K((x^{K<1})) & \to & K_0^{\times} \\ k+\epsilon & \mapsto & E(k).\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \end{cases}$$

K_0 is a topological $\mathcal{L}_{rings} \cup \{E\}$-extension of K_{-1} by Corollary 2.3.8, $Im\ E_0 \subseteq \mathcal{O}_0$, and E_0 is continuous on $\mathcal{O}_0$ by Lemma 2.3.7, consequently:

Remark 3.3.9 $(K_0, A_0, \mathcal{O}_0, E_0, \mathcal{V}_0)$ *is a pre-exponential topological field.*

Remark 3.3.10 *Notice that, as* $K \supseteq \mathbb{Q}$*,*

$$K((x^K)) =: K_0 \supseteq K((x^{\mathbb{Z}}))$$

Hence $K((t))$ *is a subfield of* $K((x^K))$*.*
Moreover, if $k \in K$*, then* $|k| = |k.x^0| < |x|$*, and likewise* $|k| > |x^{-1}| = |t|$*. In other words,* x *is infinitely large relatively to the elements of* K*, while* t *is infinitely small relatively to the elements of* K*.*

Now suppose $n \geq 0$ and that we have a pre-exponential topological field $(K_n = K((G_n)), A_n, \mathcal{O}_n, E_n, \mathcal{V}_n)$. We take a multiplicative copy x^{A_n} of A_n, and set $x^k >_{x^{A_n}} 1$ iff $k >_{A_n} 0$. Let $M_n : A_n \to x^{A_n}$ be an order preserving isomorphism. We then consider $K_{n+1} := K_n((x^{A_n}))$, and we set:

- $A_{n+1} := \{s \in K_n((x^{A_n})) \,:\, Supp\ s >_{x^{A_n}} 1\}$
- $\mathcal{O}_{n+1} := \{s \in K_n((x^{A_n})) \,:\, |s| \leq_{x^{A_n}} 1\}$, that is the valuation ring of $K_n((x^{A_n}))$
- $G_{n+1} := G_n \overleftarrow{\times} x^{A_n}$ as a direct product, on which we put an antilexicographic order.

We denote $(K((G_n)))\,((x^{A_n}))$ simply by $K((G_n))((x^{A_n}))$.
Notice that G_n is a convex ordered subgroup of G_{n+1}. Consequently there is an isomorphism of $K((G_n))$-algebras between $K((G_{n+1}))$ and $K((G_n))((x^{A_n}))$:

$$\sum_{g_{n+1}\in G_{n+1}} c_{g_{n+1}} g_{n+1} \mapsto \sum_{g_A \in x^{A_n}} \left(\sum_{g_n \in G_n} c_{g_A,g_n} g_n \right) g_A$$

Hence the notions of 'leading monomial' and 'leading coefficient' are well-defined for an element in $K((G_{n+1}))$: in particular the exponential absolute value at step $n+1$,

$$|.|_{n+1} : K((G_{n+1})) \to G_{n+1} = G_n \overleftarrow{\times} x^{A_n}$$

can be constructed from $Lm(.) : K((G_{n+1})) \to G_{n+1} = G_n \overleftarrow{\times} x^{A_n}$,
or from
$|.| : K((G_n))((x^{A_n})) \to x^{A_n}$, $Lc(.) : K((G_n))((x^{A_n})) \to K((G_n))$,
and $|.|_n : K((G_n)) \to G_n$:

$$|s|_{n+1} := Lm(s) = (|Lc(s)|_n, |s|)$$

One can easily check that $|.|_{n+1}$ is an exponential absolute value with value group G_{n+1} and valuation ring $K((G_{n+1}^{\leq 1}))$.

Fact 3.3.11 [18, Lemma 2.2 p.9]

$$\begin{aligned} G_n &= x^K \overleftarrow{\times} x^{A_0} \overleftarrow{\times} \cdots \overleftarrow{\times} x^{A_{n-1}} \\ &= x^K \overleftarrow{\times} x^{A_0 \oplus \cdots \oplus A_{n-1}} \end{aligned}$$

with $x^K \cap x^{A_0 \oplus \cdots \oplus A_{n-1}} = \{1\}$

By Lemma 3.3.7, one extends the order of K_n by setting, for $a, b \in (K_{n+1}, +)$:

$$b >_{K_{n+1}} a \text{ iff } Lc(b-a) >_{K_n} 0$$

This makes $\mathcal{O}_{n+1}$ convex in K_{n+1}.
Then $K_{n+1} = A_{n+1} \oplus \mathcal{O}_{n+1}$ and

$$\mathcal{O}_{n+1} = K_n \oplus \mathbf{m}(\mathcal{O}_{n+1}) = A_n \oplus \mathcal{O}_n \oplus \mathbf{m}(\mathcal{O}_{n+1})$$

We use the multiplicative copy of A_n to extend the exponential to the 'infinitely large' elements of A_n:

$$E_{n+1} : \left\{ \begin{array}{ccc} A_n \oplus \mathcal{O}_n \oplus \mathbf{m}(\mathcal{O}_{n+1}) & \to & K_{n+1}^{\times} \\ s = a + o + \epsilon & \mapsto & E_{n+1}(s) = M_n(a) E_n(o) \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \end{array} \right.$$

We extend the topology as in Subsection 2.3.2 by setting as neighborhoods of 0 in $K_n((x^{A_n}))$:

$$W_{V,0} := \{ s \in K_n((x^{A_n})) \,:\, |s| \leq 1 \text{ and if } |s| = Lm(s) = 1, \text{ then } Lc(s) \in V \}$$

$$W_{M(a)} := \{ s \in K_n((x^{A_n})) \,:\, |s| \leq M(a) \}$$

$$\mathcal{T}_{\mathcal{V}_n} := \{ W_{V,0} \,:\, V \in \mathcal{V}_n \}$$

$$\mathcal{V}_{n+1} := \mathcal{T}_{\mathcal{V}_n} \cup \{ W_{M(a)} \,:\, a \in A_n \}$$

Lemma 3.3.12 *The application sending A_n to its mutiplicative copy x^{A_n},*

$$M : A_n \subseteq K_n((x^{A_n})) \to x^{A_n} \subseteq K_n((x^{A_n})), a \mapsto s = M(a)$$

is continuous on A_n.

Proof. Actually, to be precise we should write:

$$M : \begin{array}{ccc} A_n.1 \subseteq K_n((x^{A_n})) & \to & K_n((x^{A_n})) \\ a = Lc(a).1 & \mapsto & s = 1.|s| = 1.M\left(Lc(a)\right) \end{array}$$

and consider the continuity of M at a point

$$a_0 \in A_n.1 \subseteq K_n.1 \subseteq K_n((x^{A_n}))$$

that is $a_0 = Lc(a_0).1$, with $Supp(Lc(a_0)) >_{K_n} 1$ in K_n. As this makes the notations quite heavy, we do it only for the proof. Let $U \in \mathcal{V}_{n+1}$ and $s_0 = 1.M(Lc(a_0))$. Then $s_0 + U$ is an open neighborhood of s_0 in $K_n((x^{A_n}))$. By Fact 2.0.2, item $(TM1)$, there is $O \in \mathcal{V}_{n+1}$ such that $s_0 O \subseteq U$, that is $s_0(1 + O) \subseteq s_0 + U$.
Then, as $(s_0 + s_0 O) \cap x^{A_n} = \{s_0\}$, one obtains

$$M^{-1}\left((s_0 + s_0 O) \cap x^{A_n}\right) = \{a_0\} = \{Lc(a_0).1\}$$

Furthermore, $\{a_0\}$ is open for the induced topology on $A_n.1$. Indeed, $Supp\,(Lc(a_0)) > 1$, hence let $V \in \mathcal{V}_n$ be such that $A_n \cap V = \{0\}$. Then $W_{V,0} \cap A_n.1 = \{0\}$, and $(a_0 + W_{V,0}) \cap A_n.1 = \{a_0\}$ which is well the trace of an open set of K_{n+1}. ■

$Im\, E_{n+1} \subseteq K_{n+1}^{\times}$, and E_{n+1} is continuous on $\mathcal{O}_{n+1}$ by Lemma 2.3.7 and Lemma 3.3.12. This makes K_{n+1} is a topological $\mathcal{L}_{rings} \cup \{E\}$-extension of K_n by Corollary 2.3.8, and more generally of K_i, for $-1 \leq i \leq n$.

Finally, we let $K((t))^E := \cup_n K_n$ and E the common extension of the E_n. We have a topology on $K((t))^E$ [23, p.576] given by the following set $\mathcal{W}$ of neighborhoods of 0:

$$\left\{ \bigcup_{\beta \leq \alpha \leq \omega} W_\alpha \;:\; W_\beta \in \mathcal{V}_\beta \,\text{and}\, W_{\alpha+1} \in \mathcal{T}_{\mathcal{V}_\alpha} \,\text{with}\, W_{\alpha+1} \cap K_\alpha = W_\alpha \right\}$$

which endow $K((t))^E$ with the structure of a topological field, and by [23, Lemma 2.16], it is then a topological $\mathcal{L}_{rings}$-extension of each $(K_n, \mathcal{V}_n)$, $n \geq 0$. Moreover E is continuous on $K((t))^E$ hence $(K((t))^E, E, \mathcal{W})$ is a topological $\mathcal{L}_{rings} \cup \{E\}$-extension of $(K, E, \mathcal{V})$–the latter being a topological E-field, contrary to the $(K_n, \mathcal{O}_n, E_n, \mathcal{V}_n)$, $n \geq 1$, that are only topological partial E-fields–.

Remark 3.3.13 *In this section, starting from a characteristic* 0 *unordered topological field* K*, we have arbitrarily ordered its underlying additive group* $(K, +)$*. Note that one only needs to order the latter so that all the* $(K_n, +)$ *are ordered. Independently, one can also consider as valued groups the underlying additive groups of the Hahn fields appearing in the construction:*
$(K((x^k)), +, Lm(.))$, $(K_n((x^{A_n})), +, Lm(.))$, $(K((G_n)), +, Lm(.))$.
Still independently, we have constructed $(K((t))^E, E, \mathcal{W})$ *a topological* $\mathcal{L}_{rings} \cup \{E\}$*-extension of* $(K, E, \mathcal{V})$*, where* $\mathcal{W}$ *is neither the order topology nor the valuation topology.*

Chapter 4

E-algebraicity and E-derivations

Derivations over a ring are used in the study of transcendental extensions in field theory. The use of E-derivations over an E-ring (R, E) in the study of E-transcendental extensions has been investigated by J.Kirby in [28]. Recall that when K is an E-field, there is a closure operator on K, ecl^K, introduced by A.Macintyre (Definition 2.1.13).

In [66], A.Wilkie shows for $K = \mathbb{C}$ or $K = \mathbb{R}$, using analysis techniques and o-minimality, that ecl^K is equal to another closure operator on K defined by derivations, which implies that ecl^K is a *pregeometry*, that is a closure operator with finite character satisfying the Steinitz exchange property; that is, for any subsets C, B of K and $a, b \in K$, we have the following:

1. $C \subseteq ecl^K(C)$
2. $B \subseteq C \Rightarrow ecl^K(B) \subseteq ecl^K(C)$
3. $ecl^K(ecl^K(C)) = ecl^K(C)$
4. $ecl^K(C) = \bigcup\{ecl^K(C_0) | C_0 \text{ is a finite subset of } C\}$
5. "*Exchange*" $a \in ecl^K(C \cup \{b\}) \setminus ecl^K(C) \Rightarrow b \in ecl^K(C \cup \{a\})$

J.Kirby then proves that for any E-field K of characteristic 0, the E-algebraic closure operator ecl^K is a pregeometry, with the important consequence that there is a good notion of dimension associated to E-algebraicity. His proof also works by showing that this pregeometry corresponds in fact to a pregeometry defined using derivations, therefore generalizing the work of A.Wilkie; but to extend derivations on partial E-field extensions, he uses techniques from algebra together with a result of J.Ax [3] (Fact 4.3.1) and the technique of amalgamation of strong extensions created by E.Hrushovski in [24].

Let $(K, R, E) \subseteq (L, R', E)$ be an inclusion of partial E-fields. Let $B \subseteq L$, let $Der(L/B)$ denote the set of all derivations on L which vanish on B, and $EDer(L/B)$ the subset of $Der(L/B)$ composed by E-derivations (with respect to R'). Note that $Der(L/B)$ and $EDer(L/B)$ are L-vector spaces.

Fact 4.0.1 [28, Propositions 4.7 and 7.1] *Let $a \in L$. Then $a \in ecl^L(B)$ iff for every $D \in EDer(L/B)$, $Da = 0$.*

Definition 4.0.2 Let $\bar{a} := a_1, \cdots, a_n \in L^n$. The ecl^L-dimension of $\bar{a}$ over K, $ecl^L-\dim_K \bar{a}$, is the maximum cardinality of an ecl^L-independent subset of $\{a_1, \cdots, a_n\}$ over K.

Suppose there is an E-derivation D on (K, R, E). In this chapter we show how to extend it on (L, R', E) using the two characterizations of E-algebraicity as well as other results of J.Kirby on strong extensions. On our way we show links between ecl-dimension of some tuples of elements and linear dimensions of associated spaces of E-derivations; we then use these results in Chapter 6 when constructing points in E-varieties that are 'maximally' ecl-independent.

4.1 Linear dimension of spaces of E-derivations and ecl-dimension

Throughout this section, let $(K, R, E) \subseteq (L, R', E)$ be an inclusion of partial E-fields. Fact 4.0.1 implies that:

Lemma 4.1.1 *Let $a_1, \cdots, a_n \in L$ be ecl^L-independent over K. There are $D_1, \cdots, D_n$ in $EDer(L/K)$ such that $D_i(a_j) = \delta_{ij}$; in other words, there are n E-derivations of $EDer(L/K)$ that are L-linear independent.*

Proof. Suppose that $a_1, \cdots, a_n$ are ecl^L-independent over K. Then for $1 \leq i \leq n$, $a_i \notin ecl^L(K \cup \{a_1, \cdots, a_n\} \setminus \{a_i\})$, hence by Fact 4.0.1, there is $D_i \in EDer(L/K(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n))$ such that $D_i(a_i) \neq 0$ and $D_i(x) = 0$ for $x \in ecl^L(K \cup \{a_1, \cdots, a_n\} \setminus \{a_i\})$. Consequently it is possible to construct n E-derivations D'_i such that D'_i is trivial on K and $D'_i(a_j) = \delta_{ij}$, by setting:

$$D'_i := \frac{1}{D_i(a_i)} D_i$$

■

Let $a_1, \cdots, a_n \in L$ be ecl^L-independent over K. Let $L_0 := K(\bar{a}, E(\bar{a}))$ and $L_1 := ecl^L(K(\bar{a}))$. Then $K \subseteq L_0 \subseteq L_1 \subseteq L$ as fields. The free family $D_1, \cdots, D_n$ of $EDer(L/K)$ from Lemma 4.1.1, seen as a family of $EDer(L_0/K)$, also generates the latter as a L_0-vector space:

Lemma 4.1.2 *Let $a_1, \cdots, a_n \in L$ be ecl^L-independent over K; $D_1, \cdots, D_n$ in $EDer(L/K)$ such that $D_i(a_j) = \delta_{ij}$, and let $D'_i := D_{i|L_0}$ for $1 \leq i \leq n$. Then $D'_1, \cdots, D'_n$ generates $EDer(L_0/K)$ as a L_0-vector space.*

Proof. Let $D \in EDer(L_0/K)$, and for $1 \leq i \leq n$, let $\alpha_i := Da_i$. To simplify notations, let us identify D_i and $D_{i|L_0}$. Then for $1 \leq j \leq n$, $(D - \sum_{i=1}^n \alpha_i D_i)(a_j) = \alpha_j - \alpha_j = 0$. ■

Corollary 4.1.3 *If $\bar{b} \in R' \subseteq L$, the linear dimension of $EDer(K(\bar{b}, E(\bar{b}))/K)$ over $K(\bar{b}, E(\bar{b}))$ is $ecl^L - dim_K(\bar{b})$.*

Proof. Let $n := |\bar{b}|$, and $d := ecl^L - dim_K(\bar{b})$. Suppose w.l.o.g. $b_1, \cdots, b_d$ are ecl^L-independent over K and set $\bar{b}_d := b_1, \cdots, b_d$. Let D, $D_1, \cdots, D_d$ in $EDer(K(\bar{b}, E(\bar{b}))/K)$ such that for $1 \leq i, j \leq d$, $D_i(b_j) = \delta_{ij}$, which is possible by Lemma 4.1.1, and let $\alpha_i := D(b_i)$. Simple calculations show that $(D - \sum_{i=1}^d \alpha_i D_i) \in EDer(K(\bar{b}, E(\bar{b}))/K(\bar{b}_d))$, and thus for $d + 1 \leq k \leq n$, $(D - \sum_{i=1}^d \alpha_i D_i)(b_k) = 0$ by Fact 4.0.1 because $b_k \in ecl^L(K(\bar{b}_d)) = ecl^L(K(\bar{b}))$. ■

Corollary 4.1.4 *Let $(K, R, E, \mathcal{V})$ be a topological partial E-field where $\mathcal{V}$ is a definable topology.*
Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$ and assume that $(K, R, E, \mathcal{V})$ is a topological $\mathcal{L}$-partial-E-field.

For all $d \in \mathbb{N}$, there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{V}')$ of $(K, R, E, \mathcal{V})$ that contains $K \cup \{t_1, \cdots, t_d\}$, where $t_1, \cdots, t_d$ are $ecl^{K((t_1))\cdots((t_d))}$-independent over K, and for $i = 1, \cdots, d$, $t_i \sim_K 0$. Moreover for all $1 \leq i \leq d$, and $V \in \mathcal{V}'(K)$, $t_i \in V$, where $\mathcal{V}'(K)$ satisfies $Comp(K)$.

Proof. By Fact A.0.8, one can construct iteratively, starting from $\mathcal{M}_0 := (K, R, E, \mathcal{V})$ a chain $(\mathcal{M}_i)_{0 \leq i \leq d}$, where $\mathcal{M}_{i+1}$ is a topological κ_i^+-saturated elementary $\mathcal{L}$-extension of $\mathcal{M}_i$, for $\kappa_i \geq |M_i|$. Let $0 \leq i \leq d$. By Remark 2.2.7, there is $t_{i+1} \in M_{i+1}$ such that for all $V \in \mathcal{V}_{i+1}(M_i)$, $t_{i+1} \in V$, that is $t_{i+1} \sim_{\mathcal{V}_{i+1}(M_i)} 0$ in L. Consequently, by Remark 3.3.1, t_{i+1} is also in all neighborhoods of 0 of $K((t_1))...((t_i))$. By Corollary 4.1.3, to prove the result it suffices to construct d E-derivations of $EDer(K((t_1))\cdots((t_d))/K)$ that are linearly independent. Let $i, j = 1, \cdots, d$. For commodity, let us denote

$$K_i := K((t_1))...((t_i)) = K_{i-1}((t_i))$$

Let us define D_j by $D_j(t_i) = \delta_{ij}$, and $D_{j \upharpoonright K_{j-1}} = 0$.

For $i \geq j$, suppose $D_{j \upharpoonright K_{i-1}}$ has been defined and let $s_i := \sum a_k t_i^k \in K_{i-1}((t_i))$, where $a_k \in K_{i-1}$. We construct a derivation–that commutes with infinite sums– by setting:

$$D_j(s_i) := \sum_k D_j(a_k t_i^k) = \sum_k k a_k D_j(t_i) t_i^{k-1} + \sum_k D_j(a_k) t_i^k$$

Consequently $D_j(s_i) = \sum k a_k t_i^{k-1}$ for $i = j$ and $D_j(s_i) = \sum D_j(a_k) t_i^k$ for $i > j$. Therefore D_j defines a derivation, as additivity and Leibniz rules are satisfied: indeed, if $r_i = \sum b_k t^k$ and $v_i = \sum (\sum_{l+j=k} a_l b_j) t^k$, then by construction,

$$D_j(v_i) = D_j(s_i) r_i + s_i D_j(r_i)$$

Recall that there is an exponential defined by the series on $K_{i-1}[[t_i]]$, the valuation ring of K_i for the canonical absolute value. To check that D_j defines an E-derivation on K_i for $i \geq j$, let $\epsilon_i = \sum a_k t_i^k \in K_{i-1}[[t_i]]$.

$D_j(\epsilon_i^n) = nD_j(\epsilon_i)\epsilon_i^{n-1}$. If $i \geq j$,

$$\begin{aligned} D_j(E(\epsilon_i)) &= D_j(\sum_{n\geq 0} \frac{\epsilon_i^n}{n!}) \\ &= \sum \frac{n}{n!} D_j(\epsilon_i)\epsilon_i^{n-1} \\ &= D_j(\epsilon_i) \sum_{n\geq 1} \frac{\epsilon_i^{n-1}}{(n-1)!} \\ &= D_j(\epsilon_i) \sum_{n\geq 0} \frac{\epsilon_i^n}{n!} \\ &= E(\epsilon_i)D_j(\epsilon_i). \end{aligned}$$

If $i < j$, $D_j(\epsilon_i) = 0$ hence

$$D_j(E(\epsilon_i)) = D_j\left(\sum_{n\geq 0} \frac{\epsilon_i^n}{n!}\right) = \sum_{n\geq 0} \frac{n}{n!} D_j(\epsilon_i)\epsilon_i^{n-1} = 0 = E(\epsilon_i)D_j(\epsilon_i)$$

Hence we have constructed d E-derivations of $EDer(K_d/K)$ that are linearly independent over K_d. ■

4.2 Extensions of E-derivations on E-algebraic elements

We extend here a result of S. Lang on solutions of Hovanskii systems of algebraic polynomials [37, Theorem 3 p.185]. Our generalization consists in replacing algebraic polynomials by E-polynomials.

Proposition 4.2.1 *Let $(K, R, E) \subseteq (L, R', E)$ be partial E-fields, where (K, R, E) is endowed with an E-derivation D. Let $a \in ecl^L(K)$. Then there is a unique E-derivation extending D on $K(a, E(a))$, where $K(a, E(a))$ is the field generated by K, a and $E(a)$.*

Proof. There exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in L$ and $h_1, \ldots, h_n \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ such that $a = a_1$ and $(a_1, \ldots, a_n)$ is a solution to the Hovanskii system H_H given by $H = (h_1, \cdots, h_n)$:

$$h_i(a_1, \ldots, a_n) = 0 \text{ for } i = 1, \ldots, n$$

and

$$\begin{vmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \frac{\partial h_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial X_1} & \cdots & \frac{\partial h_n}{\partial X_n} \end{vmatrix} (a_1, \ldots, a_n) \neq 0$$

Without loss of generality one can suppose there is $0 \leq p \leq n$, $a_i \in R'$ iff $i \leq p$. Let us construct another Hovanskii system H_G, where $G = (g_1, \cdots, g_{n+p})$, by letting, for $i = 1, \cdots, n$, $i' = 1, \cdots, p$:

$$g_i(X_1, \cdots, X_n, X_{n+1}, \cdots, X_{n+p}) := h_i(X_1, \cdots, X_n)$$

$$g_{n+i'} := X_{n+i'} - E(X_{i'})$$

Let $\bar{\bar{X}} := (X_1, \cdots, X_{n+p})$. Then for $i = 1, \cdots, n$, $i' = 1, \cdots, p$ and $j = 1, \cdots, n+p$

$$\frac{\partial g_i}{\partial X_j}(\bar{\bar{X}}) = \frac{\partial h_i}{\partial X_j}(\bar{X})$$

and,

$$\frac{\partial g_{n+i'}}{\partial X_j}(\bar{\bar{X}}) = \delta_{n+i',j} - \delta_{i',j} E(X_{i'})$$

Let $\pi_i : L^n \to L$ be the projection on the ith coordinate. The Jacobian matrix of G, JG, is now:

$$JG = \begin{pmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \cdots & \frac{\partial h_1}{\partial X_n} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\partial h_n}{\partial X_1} & \cdots & \cdots & \frac{\partial h_n}{\partial X_n} & 0 & \cdots & \cdots & 0 \\ -E \circ \pi_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \vdots & 0 & & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & \ddots & & 0 & \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & -E \circ \pi_p & 0 & \cdots & 0 & 1 \end{pmatrix}$$

As $0 \neq \det JH_{a_1 \cdots a_n} = \det JG_{a_1 \cdots a_n E(a_1) \cdots E(a_p)}$, $(a_1, \cdots, a_n, E(a_1), \cdots, E(a_p))$ is a regular solution of the system G. By Lemma 2.1.8, we have an E-derivation on E-polynomials, thus we can write, for $i = 1, \cdots, n+p$,

$$Dg_i(\bar{\bar{X}}) = g_i^D(\bar{\bar{X}}) + \sum_{i=1}^{|\bar{\bar{X}}|} \frac{\partial g_i}{\partial X_i}(\bar{\bar{X}}) DX_i$$

And an E-derivation on a_i should satisfy

$$Dg_i(\bar{\bar{a}}) = g_i^D(\bar{\bar{a}}) + \sum_{i=1}^{|\bar{\bar{a}}|} \frac{\partial g_i}{\partial a_i}(\bar{\bar{a}}) Da_i$$

Hence we obtain the linear system S with unknowns $Y_1, \cdots, Y_{n+p}$:

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial h_1}{\partial X_i}(\bar{a})Y_i & = & -h_1^D(a_1, \cdots, a_n) \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} \frac{\partial h_n}{\partial X_i}(\bar{a})Y_i & = & -h_n^D(a_1, \cdots, a_n) \\ Y_{n+1} - E(a_1)Y_1 & = & 0 \\ \vdots & \vdots & \vdots \\ Y_{n+p} - E(a_p)Y_p & = & 0 \end{cases}$$

The determinant of which is different from 0 as $\bar{\bar{a}}$ is a regular solution of G. Hence S admits a unique solution $b_1, \cdots, b_{n+p}$. Let us set $Da_1 := b_1, \cdots, Da_n := b_n, D(E(a_1)) := b_{n+1}, \cdots, D(E(a_p)) := b_{n+p}$; then D is an E-derivation on $a_1, \cdots, a_n$, and it is uniquely determined.

Now, in order to see that this is well defined, suppose that a is a coordinate of two different tuples $\bar{a}$ and $\bar{a}'$ solutions of two different Hovanskii systems H and H', giving rise to Da_1 and $D'a_1$. Then $Da_1 - D'a_1 = (D - D')a_1$, with $D - D' = 0$ on K hence $D - D' \in EDer(K(\bar{a}\bar{a}')/K)$ so as $a_1 \in ecl^{K(\bar{a}\bar{a}')}(K)$,then $(D - D')a_1 = 0$ hence $Da_1 = D'a_1$. ■

Remark 4.2.2 • *If a is a zero of an E-polynomial $P(X)$, then if $\frac{\partial P}{\partial X}(a) \neq 0$, we have a Hovanskii system. If $\frac{\partial P}{\partial X}(a) = 0$, then $P^D(a) = 0$, and the equation $DP(a) = 0$ does not define Da.*

- *The field $K(a, E(a))$ is neither an E-field nor defines a partial E-field in the sense of Definition 2.1.1.*

Let $\bar{b} \subseteq L$, $d := ecl^L - dim_K(\bar{b})$. Suppose w.l.o.g. $b_1, \cdots, b_d$ are ecl^L-independent over K. Proposition 4.2.1 shows an E-derivation on $ecl^L(K(\bar{b}))$ is uniquely determined by a given E-derivation on $K(\bar{b}_d)$, hence the free family $D_1, \cdots, D_d$ of Corollary 4.1.3, seen as a family of $EDer(ecl^L(K(\bar{b}))/K)$, also generates the latter as a $ecl^L(K(\bar{b}))$-vector space:

Corollary 4.2.3 *Let* $\bar{b} \subseteq L$, $d := ecl^L - dim_K(\bar{b})$. *Suppose w.l.o.g.* $b_1, \cdots, b_d$ *are* ecl^L*-independent over* K.

- *Let* $D_1, \cdots, D_d$ *in* $EDer(K(\bar{b}, E(\bar{b}))/K)$ *such that for* $1 \leq i, j \leq d$, $D_i(b_j) = \delta_{ij}$. *Then* $D_1, \cdots, D_d$ *generates* $EDer(ecl^L(K(\bar{b}))/K)$ *as a* $ecl^L(K(\bar{b}))$*-vector space.*

- *If* $\bar{b} \in R' \subseteq L$, *the linear dimension of* $EDer(ecl^L(K(\bar{b}))/K)$ *over* $ecl^L(K(\bar{b}))$ *(resp. of* $EDer(K(\bar{b}, E(\bar{b}))/K)$ *over* $K(\bar{b}, E(\bar{b}))$*) is* $ecl^L - dim_K(\bar{b})$.

4.2.0.1 *ecl* is closed under D

Proof of Proposition 4.2.1 already shows that if $\bar{a} \in ecl^L(K)$, then $D\bar{a} \in ecl^L(K)$: indeed E-polynomials with coefficients in K evaluated at $\bar{a}$ are in $K[\bar{a}]^E$ and its field of fractions $\langle K[\bar{a}]^E \rangle \subseteq ecl^L(K(\bar{a})) = ecl^L(K)$ as $\bar{a} \in ecl^L(K)$. Nevertheless, in this section, we show it again by simultaneous use of the two definitions of E-algebraicity proved to be equivalent by J.Kirby. Consequently this section is not necessary to the following chapters and can be skipped by the reader.

Suppose there is an E-derivation on L such that

$$(K, R, E, D) \subseteq (L, R', E, D)$$

is a differential partial E-fields extension. We first want to show that if $a \in ecl^L(K)$, then $Da \in ecl^L(K)$; hence $(ecl^L(K), D)$ is a differential field.

We have $a \in ecl^L(K)$ iff $a = a_1$ and $\bar{a} = (a_1, \cdots, a_n) \subseteq L$ is a solution of a Hovanskii system $H = (h_1(\bar{X}), \cdots, h_n(\bar{X}))$ over K iff for every $d \in EDer(L/K)$, $da = 0$. (note that we in fact have $da_i = 0$ for all $i = 1, \cdots, n$.)

Lemma 4.2.4 *Let* $a \in ecl^L(K)$, $d \in EDer(L/K)$ *and* $P(\bar{X}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $|\bar{X}| = n$. *Then*

$$dDP(\bar{a}) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).dDa_i$$

Proof. By Lemma 2.1.8,

$$DP(\bar{X}) = P^D(\bar{X}) + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{X}).DX_i$$

As $D_{|K}$ is a ring morphism, the coefficients of $P^D(\bar{X})$ are in K hence $(P^D)^d(\bar{X}) = 0$. Similarly, for each i, $\frac{\partial P}{\partial X_i}(\bar{X})$ has coefficients in K, hence $\left(\frac{\partial P}{\partial X_i}\right)^d(\bar{X}) = 0$, thus

$$\begin{aligned} dDP(\bar{X}) &= \sum_{i=1}^{n} \frac{\partial P^D}{\partial X_i}(\bar{X})dX_i + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{X}).dDX_i \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 P}{\partial X_j \partial X_i}(\bar{X}).DX_i dX_j \end{aligned}$$

As $da_i = 0$, we finally get:

$$dDP(\bar{a}) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).dDa_i$$

■

Proposition 4.2.5 *Let $a \in ecl^L(K)$. Then $Da \in ecl^L(K)$.*

Proof. There exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in L$ and $h_1, \ldots, h_n \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ such that $a = a_1$ and $(a_1, \ldots, a_n)$ is a solution to the Hovanskii system H given by the h_i's:

$$h_i(a_1, \ldots, a_n) = 0 \text{ for } i = 1, \ldots, n$$

and

$$\begin{vmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \frac{\partial h_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial X_1} & \cdots & \frac{\partial h_n}{\partial X_n} \end{vmatrix} (a_1, \ldots, a_n) \neq 0$$

Furthermore if $d \in EDer(L/K)$ then $da_i = 0$. Let $\bar{a} = (a_1, \cdots, a_n)$. By Lemma 4.2.4, for each $i = 1, \cdots, n$,

$$0 = dDh_i(\bar{a}) = \sum_{i=1}^{n} \frac{\partial h_i}{\partial X_i}(\bar{a}).dDa_i$$

We thus get a linear system in the dDa_i's, namely:

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial h_1}{\partial X_i}(\bar{a}).dDa_i = 0 \\ \vdots \\ \sum_{i=1}^{n} \frac{\partial h_n}{\partial X_i}(\bar{a}).dDa_i = 0 \end{cases}$$

the determinant of which is different from 0, as $\bar{a}$ is a solution of H. Finally we get that for all $i = 1, \cdots, n$, for all $d \in EDer(L/K)$, $dDa_i = 0$, hence $Da_i \in ecl^L(K)$. ■

4.3 Strong extensions

Recall that the transcendence degree td of a field extension L/K, or dimension of L over K as fields, is the greatest cardinality of an algebraically independent subset of L over K. Such a subset is called a transcendence base of L over K. If $\bar{a} \subseteq L$, let $td(\bar{a}/K) := td(K(\bar{a})/K)$. If $B, \bar{x} \subseteq L$, $\mathrm{ldim}_{\mathbb{Q}}(\bar{x}/B)$ is the linear dimension of the quotient

$$\frac{\langle B, \bar{x}\rangle_{\mathbb{Q}}}{\langle B\rangle_{\mathbb{Q}}}$$

where $\langle B, \bar{x}\rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space generated by $B \cup \bar{x}$ and $\langle B\rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space generated by B.

Recall that Schanuels's conjecture for $\mathbb{C}$ states that if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree at least n over $\mathbb{Q}$.

Schanuel's conjecture allows to avoid hidden exponential-algebraic relations. When $\bar{x}$ is algebraic (over $\mathbb{Q}$), Schanuel's conjecture is the Lindemann Weierstrass Theorem, stating that:

$$td(e^{\bar{x}}/\mathbb{Q}) \geq ldim_{\mathbb{Q}}(\bar{x})$$

where $td(e^{\bar{x}}/\mathbb{Q})$ is the transcendence degree of the field extension $\mathbb{Q}(e^{\bar{x}})/\mathbb{Q}$. When $\bar{x}$ is $ecl^{\mathbb{C}}$-independent over $\mathbb{Q}$, the following theorem can be used:

Fact 4.3.1 [3, Th.3] *Let F be a field of characteristic 0, let Δ be a set of derivations on F and let $C = \bigcap_{\partial \in \Delta} ker\partial$ be the field of constants. Suppose $x_1, \cdots, x_n, y_1, \cdots, y_n \in F$ satisfy $\partial y_i = y_i \partial x_i$ for each $i = 1, \cdots, n$*

and each $\partial \in \Delta$. Then

$$td(\bar{x}\bar{y}/C) \geq ldim_{\mathbb{Q}}(\bar{x}/C) + rk(\partial x_i)_{\partial \in \Delta, i=1,\cdots,n}$$

where $rk(.)$ is the rank of the matrix.

Fact 4.3.1, together with Lindemann Weierstrass' Theorem, reduces Schanuel's conjecture to elements in $ecl^{\mathbb{C}}(\mathbb{Q})$ that are not algebraic over $\mathbb{Q}$. Indeed, let $\Delta := EDer(\mathbb{C}/\mathbb{Q})$. Then the field of constants is $C = ecl^{\mathbb{C}}(\mathbb{Q})$ and by Fact 4.0.1 of J.Kirby and Lemma 4.1.1, $rk(\partial x_i)_{\partial \in \Delta, i=1,\cdots,n} = n$ if $\bar{x} \subseteq \mathbb{C} \setminus C$ is such that $x_1, \cdots, x_n$ are $ecl^{\mathbb{C}}$-independent over C. In that case, $td(\bar{x}, e^{\bar{x}}) \geq td(\bar{x}, e^{\bar{x}}/C) \geq 2n$.
Now let $\langle K, A(K), E \rangle$ be a partial E-field in the sense of Definition 2.1.3– in particular $A(K)$ is considered as a $\mathbb{Q}$-vector space and not especially as endowed with a ring structure–. If $C \subseteq K$, set $E(C) := E(C \cap A(K))$. The above proof for complex numbers is actually an instance of the proof of the following fact:

Fact 4.3.2 [28, Corollary 5.2] *Suppose $C \subseteq K$ is such that $ecl^K(C) = C$, and let $\bar{a} := a_1, \cdots, a_n \in K$. Then*

$$td(\bar{a}, E(\bar{a})/C, E(C)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) \geq ecl^K - dim_C \bar{a}$$

Corollary 4.3.3 *Let $(K, R, E) \subseteq (L, R', E)$ be an extension of partial E-fields, and let $\bar{a} := a_1, \cdots, a_n \in L$ ecl^L-independent over K. Then*

$$td(\bar{a}, E(\bar{a})/ecl^L(K)) \geq 2n$$

In [24], E.Hrushovski has introduced a notion of predimension, while creating the technique of 'amalgamation of strong extensions':

Let $\bar{x} \subseteq R$ and $B \subseteq R$. The *predimension* of $\bar{x}$ over B is:

$$\delta(\bar{x}/B) := td(\bar{x}, E(\bar{x})/B, E(B)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/B)$$

In the context of partial E-domains (as in Definition 2.1.3), J.Kirby has defined a notion of strong extensions:

Definition 4.3.4 [28, Definition 5.3 p.8] *An embedding $\langle R_1, A(R_1), E \rangle \subseteq \langle R_2, A(R_2), E \rangle$ of partial E-domains is said to be strong iff for every tuple $\bar{x}$ from $A(R_2)$, we have $\delta(\bar{x}/A(R_1)) \geq 0$.*

As we want to use his results to extend E-derivations, but as our definition of partial E-fields is different (Definition 2.1.1), we state an intermediate definition that we will use only in this section:

Definition 4.3.5 A *partial E-Domain* is a two-sorted structure

$$\langle K, R; +_K, ._K, +_R, \alpha, E_R \rangle$$

where $\langle K, +_K, ._K \rangle$ is an integral domain, namely a commutative ring with no non-zero zero divisors, $\langle R, +_R \rangle$ is a subgroup of $\langle K, +_K \rangle$,

$$\alpha : \langle R, +_R \rangle \to \langle K, +_K \rangle$$

is an injective homomorphism of additive groups and

$$E_R : \langle R, +_R \rangle \to \langle K \setminus \{0\}, ._K \rangle$$

is a group morphism. R is identified with its image under α, $._K$ is written . and $+_R$ and $+_K$ are both written $+$.

We will denote partial E-Domains $\langle K, R; +_K, ._K, +_R, , \alpha, E_R \rangle$ by

$$\langle K, A(K), E \rangle$$

where by $A(K)$ we mean the domain of definition of E_R; we will call them 'partial E-Domains' with a 'D' instead of the 'd' of J.Kirby's partial E-domains.

We will consider extensions of partial E-Domains

$$\langle R_1, A(R_1), E \rangle \subseteq \langle R_2, A(R_2), E \rangle$$

for which $A(R_1)$ is a pure subgroup of $A(R_2)$, namely for which, for all $n \in \mathbb{Z}$,

$$nA(R_1) = A(R_1) \cap nA(R_2)$$

(whenever an element of $A(R_1)$ has an n^{th} root in $A(R_2)$, it also has one in $A(R_1)$)
instead of asking for $A(R_1)$ and $A(R_2)$ to be both endowed with a $\mathbb{Q}$-vector space structure, and adapt J.Kirby's results to that setting [52].

Definition 4.3.6 Let $\langle K, A(K), E\rangle$ be a partial E-domain (resp. a partial E-Domain) such that K is a field. The field K is said to be *exponential-graph-generated* iff K is generated as a field by $A(K) \cup E(A(K))$.

Remark 4.3.7 *Let (K, R, E) be a partial E-field–this time in the sense of Definition 2.1.1–such that K is the field of fractions of R. Let $A(K)$ be the additive underlying group of R, then K is exponential-graph-generated as a field by $A(K) \cup E(A(K))$.*

Let us recall some results of J.Kirby.

Fact 4.3.8 [28, Lemma 5.5]*For ordinals α, let R_α be partial E-domains. Then the following conditions hold:*

1. *The identity $R_1 \hookrightarrow R_1$ is strong.*
2. *The composite of strong extensions is strong.*
3. *Suppose λ is an ordinal, $(R_\alpha)_{\alpha<\lambda}$ is a λ-chain of strong extensions (that is, for each $\alpha \leq \beta < \lambda$ there is a strong extension $f_{\alpha,\beta}$ of R_α into R_β and for all $\alpha \leq \beta \leq \gamma$, we have $f_{\beta,\gamma} \circ f_{\alpha,\beta} = f_{\alpha,\gamma}$ and $f_{\alpha,\alpha}$ is the identity on R_α), and R is the union of the chain. Then the embedding $R_\alpha \subseteq R$ is strong for each α.*
4. *Suppose $(R_\alpha)_{\alpha<\lambda}$ is a λ-chain of strong extensions with union R, and the embedding $R_\alpha \subseteq S$ is strong for each α. Then the embedding $R \subseteq S$ is strong.*

Fact 4.3.9 [28, Proposition 5.6] *Let $\langle F_0, A(F_0), E\rangle \subseteq \langle F, A(F), E\rangle$ be a strong embedding of partial E-domains, suppose F_0 and F are fields and exponential-graph-generated. Then for some ordinal λ there is a chain $(F_\alpha)_{\alpha\leq\lambda}$ of partial E-domains such that , for all ordinals $0 \leq \alpha \leq \beta \leq \lambda$, the following conditions hold:*

1. *$F = F_\lambda$,*
2. *F_α is exponential-graph-generated,*
3. *for an ordinal limit β, $F_\beta = \cup_{\alpha<\beta} F_\alpha$,*
4. *$td(F_{\beta+1}/F_\beta)$ is finite,*

5. the embedding $F_\alpha \subseteq F_\beta$ is strong.

Proof. We write J.Kirby's proof for completeness. Let λ be the smallest ordinal of cardinality $|A(F)|$, and enumerate $A(F)$ as $(r_\alpha)_{\alpha<\lambda}$. Let us construct by induction F_β satisfying 1-5 and such that $r_\beta \in F_{\beta+1}$ and the embedding $F_\beta \subseteq F$ is strong.
For ordinal limit β, let $A(F_\beta) := \bigcup_{\alpha<\beta} A(F_\alpha)$. Let F_β be the partial E-subfield of F generated by $A(F_\beta)$, in order that 2 and 3 hold. Item 5 holds by Item 3 of Fact 4.3.8, and the embedding $F_\beta \subseteq F$ is strong by Item 4 of Fact 4.3.8.
Given $\beta < \lambda$, if $r_\beta \in A(F_\beta)$, let $F_{\beta+1} := F_\beta$. Otherwise, by induction the embedding $F_\beta \subseteq F$ is strong hence for any finite tuple $\bar{x}$ from $A(F)$, we have $\delta(\bar{x}/F_\beta) \geq 0$. Choose a tuple $\bar{x}$ containing r_β and such that $\delta(\bar{x}/F_\beta)$ is minimal. Let $A(F_{\beta+1}) := \langle A(F_\beta), \bar{x}\rangle_{\mathbb{Q}}$ a $\mathbb{Q}$-subspace of $A(F)$, and let $F_{\beta+1}$ be the partial E-subfield of F generated by $A(F_{\beta+1})$. By the minimality of $\delta(\bar{x}/F_\beta)$, the embedding $F_{\beta+1} \subseteq F$ is strong. For any $\alpha \leq \beta$, because the embedding $F_\alpha \subseteq F$ is strong, one obtains that the embedding $F_\alpha \subseteq F_{\beta+1}$ is strong. Furthermore $td(F_{\beta+1}/F_\beta) \leq 2|\bar{x}|$, which is finite, hence Item 4 holds. Eventually, $\bigcup_{\alpha<\lambda} A(F_\alpha) = A(F)$ and then $F = F_\lambda$. ■

Fact 4.3.10 [52] *Let $\langle F_0, A(F_0), E\rangle \subseteq \langle F, A(F), E\rangle$ be a strong embedding of partial E-Domains, suppose F_0 and F are fields and exponential-graph-generated. Then for some ordinal λ there is a chain $(F_\alpha)_{\alpha\leq\lambda}$ of partial E-Domains such that , for all ordinals $0 \leq \alpha \leq \beta \leq \lambda$, the following conditions hold:*

1. *$F = F_\lambda$,*
2. *F_α is exponential-graph-generated,*
3. *for an ordinal limit β, $F_\beta = \cup_{\alpha<\beta} F_\alpha$,*
4. *$td(F_{\beta+1}/F_\beta)$ is finite,*
5. *the embedding $F_\alpha \subseteq F_\beta$ is strong.*

Proof. To extend Fact 4.3.9 to the setting of partial E-Domains, one needs in the proof to assume by induction that $A(F_\beta)$ is a pure subgroup of $A(F)$. Then one sets $A(F_{\beta+1})$ to be the divisible hull in $A(F)$ of the subgroup generated by $A(F_\beta)$ and $\bar{x}$, where r_β belongs to $\bar{x}$ and $\delta(\bar{x}/F_\beta)$ is minimal. ■

Let $F_0 \subseteq F$ be an extension of fields, $\partial \in Der(F_0)$, and

$$\Omega(F) = \langle db : b \in F \rangle$$

$$I := \left\langle \sum a_i db_i | a_i, b_i \in F_0, \sum a_i \partial b_i = 0 \right\rangle$$

Then I is a sub-F-module of $\Omega(F)$, let $\Omega(F/\partial)$ denote the quotient $\Omega(F)/I$. We naturally have a quotient map :

$$\Omega(F) \twoheadrightarrow \Omega(F/\partial)$$

Now let $J := \left\langle \overline{db} | b \in F_0 \right\rangle \subseteq \Omega(F/\partial)$. J is a sub-F-module of $\Omega(F/\partial)$, and the quotient $\Omega(F/\partial)/J$ is isomorphic to $\Omega(F/F_0)$, the quotient of $\Omega(F)$ by the relations relations $dc = 0$ for each $c \in F_0$. We naturally have the two quotient maps :

$$\Omega(F) \twoheadrightarrow \Omega(F/\partial) \twoheadrightarrow \Omega(F/F_0)$$

Let $Der(F/\partial) := \{\eta \in Der(F) | (\exists \lambda \in F) \eta_{\restriction F_0} = \lambda \partial\}$.

Fact 4.3.11 [28, Lemma 6.2] *$Der(F/\partial)$ is the dual space of $\Omega(F/\partial)$, that is the set of linear maps f from $\Omega(F/\partial)$ to F.*

Fact 4.3.12 [28, Theorem 6.3] *Let $\langle F_0, A(F_0), E \rangle \subseteq \langle F, A(F), E \rangle$ be a strong embedding of partial E-domains, suppose F_0 and F are fields and exponential-graph-generated. Then every E-derivation on F_0 extends to F.*

Proof. We write J.Kirby's proof for completeness, and we detail some parts. Let F' be the partial E-subfield of F generated by the graph of exponentiation of F, that is $F' := \langle A(F) \cup exp(A(F)) \rangle$. Then every E-derivation on F' extends to F, as only the field operations must be respected and the characteristic is zero. Consequently we assume $F = F'$. By Fact 4.3.9, it is enough to prove the theorem for strong embeddings of exponential-graph-generated partial E-fields $F_1 \subseteq F_2$ such that $td(F_2/F_1)$ is finite. Let ∂ be an E-derivation on F_1, and let

$$EDer(F_2/\partial) := Der(F_2/\partial) \cap EDer(F_2)$$

The embedding $F_1 \subseteq F_2$ is strong hence $td(F_2/F_1) \geq ldim_{\mathbb{Q}}(A(F_2)/A(F_1))$, thus as $td(F_2/F_1)$ is finite, so is $ldim_{\mathbb{Q}}(A(F_2)/A(F_1))$. It is then possible to consider a $\mathbb{Q}$-basis $a_1, \ldots, a_n$ for $A(F_2)$ over $A(F_1)$. Then let

$$w_i := de^{a_i}/e^{a_i} - da_i \in \Omega(F_2)$$

Now let Λ be the F_2-subspace of $\Omega(F_2)$ generated by $\omega_1, \ldots, \omega_n$. Recall that by the universal property, for each derivation $\partial \in Der(F_2)$, there is a unique F_2-linear map $\partial^* : \Omega(F_2) \to F_2$, $da \mapsto \partial a$ such that the following diagram commutes:

$$\begin{array}{ccc} F_2 & \xrightarrow{d} & \Omega(F_2) \\ & \overset{\partial}{\searrow} & \downarrow \partial^* \\ & & F_2 \end{array}$$

Moreover by Fact 4.3.11, the space of derivations $Der(F_2)$ is the dual space of $\Omega(F_2)$, hence we can consider the annihilator of Λ in it:

$$\begin{aligned} Ann(\Lambda) &= \{\partial^* : \Omega(F_2) \to F_2 | \forall f \in \Lambda = \langle \omega_1, \ldots, \omega_n \rangle_{F_2}, \partial^* f = 0\} \\ &= \{\partial \in Der(F_2) | \forall f \in \Lambda, \partial^* f = 0\} \end{aligned}$$

Note that by definition of Λ, we have $EDer(F_2/F_1) = Der(F_2/F_1) \cap Ann(\Lambda)$ and $EDer(F_2/\partial) = Der(F_2/\partial) \cap Ann(\Lambda)$.
Indeed, let $\partial \in Der(F_2/F_1) \cap Ann(\Lambda)$. Then for each $i = 1, \ldots, n$,

$$\partial^*(\frac{de^{a_i}}{e^{a_i}} - da_i) = 0 \text{ meaning that}$$

$$(\frac{\partial e^{a_i}}{e^{a_i}} - \partial a_i) = 0 \text{ as } \partial^* \text{ is } F_2\text{-linear}$$

that is $\partial \in EDer(F_2/F_1)$. The converse is clear as the relations of E-derivations are verified in $EDer(F_2/F_1)$ and the proof for $EDer(F_2/\partial)$ is similar.

We now want to get information about the dimension of $EDer(F_2/F_1)$ (resp. of $EDer(F_2/\partial)$) relatively to $Der(F_2/F_1)$ (resp. to $Der(F_2/\partial)$), through the codimension of $Ann(\Lambda)$ in them.

For this, first let $\hat{\omega}_i$ be the image of ω_i under the natural quotient map $\Omega(F_2) \twoheadrightarrow \Omega(F_2/F_1)$.

Fact 4.3.13 [28, Fact 6.4] If the differentials $\hat{\omega}_1, \dots, \hat{\omega}_n$ are F_2-linearly dependant in $\Omega(F_2/F_1)$, then there is a non-zero $\mathbb{Z}$-linear combination $b = \sum_{i=1}^{n} m_i a_i$ such that b and e^b are both algebraic over F_1. (F_1, F_2 exponential-graph-generated fields and $\Omega(F_2/F_1)$, as a module over the field F_2, is an F_2-vector space).

For a proof of the claim, see an intermediate step in the proof of Ax's theorem, or [29, Proposition 3.7], where the group S is $\mathbb{G}_m$.
By Fact 4.3.13, if the $\hat{\omega}_i$ are F_2-linearly dependant, then, for some such b, we have :

$$\begin{aligned}\delta(b/F_1) &= td(b, e^b/A(F_1) \cup exp(A(F_1))) - ldim_{\mathbb{Q}}(b/A(F_1))\\ &= td(b, e^b/F_1) - ldim_{\mathbb{Q}}(b/A(F_1))\\ &= 0 - 1 < 0\end{aligned}$$

This contradicts the fact that the embedding $F_1 \subseteq F_2$ is strong, hence the $\hat{\omega}_i$ are F_2-linearly independant in $\Omega(F_2/F_1)$.
We have thus seen that the image of the subspace Λ has dimension n in $\Omega(F_2/F_1)$, consequently it has also dimension n in the intermediate space $\Omega(F_2/\partial)$.
The subspaces $Der(F_2/F_1)$ and $Der(F_2/\partial)$ of $Der(F_2)$ are dual to the quotients $\Omega(F_2/F_1)$ and $\Omega(F_2/\partial)$ of $\Omega(F_2)$, hence $Ann(\Lambda)$ has codimension n in $Der(F_2/F_1)$ and also in $Der(F_2/\partial)$ (Indeed, recall that for vector spaces V, W, there is an isomorphism between the dual of the quotient W/V and $Ann(V)$ (hence the codimension of $Ann(V)$ in W is the dimension of V)).

We are now ready to use the information about the dimension of $EDer(F_2/F_1)$ (resp. of $EDer(F_2/\partial)$) relatively to $Der(F_2/F_1)$ (resp. to $Der(F_2/\partial)$), through the codimension of $Ann(\Lambda)$ in them: if $\partial = 0$, then the result is trivial. Otherwise,

$$\dim Der(F_2/\partial) = \dim Der(F_2/F_1) + 1$$

Hence, because of the previous paragraph,

$$\dim EDer(F_2/\partial) = \dim EDer(F_2/F_1) + 1$$

Thus there is $\eta \in EDer(F_2/\partial) \setminus EDer(F_2/F_1)$. Then $\eta_{\restriction F_1} = \lambda\partial$ for some non-zero λ. Let $\eta' = \lambda^{-1}\eta$. Then η' extends ∂ to F_2. ■

Note that J.Kirby's proof uses Fact 4.3.1 of J.Ax.

Fact 4.3.14 [52] *Let $\langle F_0, A(F_0), E\rangle \subseteq \langle F, A(F), E\rangle$ be a strong embedding of partial E-Domains, suppose F_0 and F are fields and exponential-graph-generated. Then every E-derivation on F_0 extends to F.*

Proof. To extend Fact 4.3.12 to the setting of partial E-Domains, one needs to assume that $A(F_0)$ is pure in $A(F)$. Then in the proof, instead of choosing a $\mathbb{Q}$-linear base $a_1, \cdots, a_n$ of $A(F_2)$ over $A(F_1)$, one chooses $a_1, \cdots, a_n \in A(F_2) \setminus A(F_1)$ maximal $\mathbb{Z}$-independent over $A(F_1)$ and generating $A(F_2)$ over $A(F_1)$ in the following way:
for any $b \in A(F_2)$, there are $z_1, \cdots, z_n \in \mathbb{Z}$, $u \in A(F_1)$ and $k \in \mathbb{N} \setminus \{0\}$ such that

$$kb = \sum_{i=1}^{n} z_i a_i + u$$

Note that if $\sum_{i=1}^{n} z_i a_i \in \langle A(F_1)\rangle_{\mathbb{Q}}$, then $\sum_{i=1}^{n} z_i a_i \in A(F_1)$:
indeed if $\sum_{i=1}^{n} z_i a_i \in \langle A(F_1)\rangle_{\mathbb{Q}}$, then for some $k \in \mathbb{N} \setminus \{0\}$, $k\sum_{i=1}^{n} z_i a_i \in A(F_1)$. Hence, because $A(F_1)$ is pure in $A(F_2)$ and $\sum_{i=1}^{n} z_i a_i \in A(F_2)$, one obtains that $\sum_{i=1}^{n} z_i a_i \in A(F_1)$.
Consequently in the intermediate claim of the proof [28, Fact 6.4], $b \notin \langle A(F_1)\rangle_{\mathbb{Q}}$, hence $ldim_{\mathbb{Q}}(b/A(F_1)) = 1$ thus $\delta(b/F_1) < 0$. ∎

We are now ready to deal with extensions of partial E-fields $(K, R, E) \subseteq (L, R', E)$, where (K, R, E), (L, R', E) are as defined in Definition 2.1.1.

Fact 4.3.15 [52] *Let $(K, R, E) \subseteq (L, R', E)$ be an extension of partial E-fields, such that there is an E-derivation D on $ecl^L(K)$. Then D extends to an E-derivation D^* on L.*

Proof. Let us consider the partial E-Domain $\langle L, A(L), E\rangle$, where $A(L)$ is the additive underlying group of R'. Let F_1 be the subfield generated by $(A(L) \cap ecl^L(K)) \cup E(A(L) \cap ecl^L(K))$. Note that $F_1 \subseteq ecl^L(K)$ is a partial E-subfield of L. We will show that the embedding of partial E-Domains

$$\langle F_1, A(F_1), E\rangle \subseteq \langle L, A(L), E\rangle$$

is strong, so that we can apply Fact 4.3.14.

Let $\bar{a} \in A(L)$. By Fact 4.3.2 (equivalently by Fact 4.3.1 and Corollary 4.1.3),

$$td(\bar{a}, E(\bar{a})/ecl^L(K) \geq ecl^L - dim_{ecl^L(K)}(\bar{a}) + ldim_{\mathbb{Q}}(\bar{a}/ecl^L(K))$$

Then note that:

$$ecl^L - dim_{ecl^L(K)}(\bar{a}) = ecl^L - dim_K(\bar{a}) = ecl^L - dim_{F_1}(\bar{a})$$

Furthermore, $td(\bar{a}, E(\bar{a})/A(F_1) \cup E(A(F_1))) \geq td(\bar{a}, E(\bar{a})/ecl^L(K))$.

Then suppose we have a $\mathbb{Q}$-linear combination u of elements of $\bar{a}$ belonging to $ecl^L(K)$. There is $n \in \mathbb{N} \setminus \{0\}$, $nu \in A(L)$. Hence $nu \in A(L) \cap ecl^L(K)$ and so $nu \in F_1 \cap A(L) = A(F_1)$. Therefore, $ldim_{\mathbb{Q}}(\bar{a}/ecl^L(K)) = ldim_{\mathbb{Q}}(\bar{a}/A(F_1))$, hence eventually $\delta(\bar{a}/F_1) \geq 0$. ■

Corollary 4.3.16 *Let $(K, R, E) \subseteq (L, R', E)$ an extension of partial E-fields, such that there is an E-derivation D on K. Then D extends to an E-derivation D^* on L.*

Proof. By Proposition 4.2.1, D extends as an E-derivation on $K_0 := ecl^L(K)$. Then by Fact 4.3.15, it extends to L. ■

Chapter 5

Nullstellensätze

This chapter is a joint work with Françoise Point [53] and is independent from the Preliminaries Chapter. We show here a version of Strong and Weak Nullstellensätze for partial E-fields (K, R, E). In this setting, we deal with the fact that the E-ring $R[\bar{X}]^E$ is not a Hilbert–or Jacobson–ring, that is any prime ideal is not an intersection of maximal ideals. Furthermore the notion of *ecl*-closure involves tuples of elements, which is not the case of algebraic closure. In the algebraic versions of the Nullstellensätze, these two properties are involved: for a field K and $M \subseteq K[X]$ a maximal ideal, the field $K[X]/M$ is algebraic over K.
By induction on $n \in \mathbb{N} \setminus \{0\}$, for $M \subseteq K[X_1, \cdots, X_n]$ a maximal ideal, $K[X_1, \cdots, X_n]/M$ is algebraic over K $(*)$.
From this, and given K algebraically closed the "Strong Nullstellensatz" follows:

there is a 1-1 correspondance between points in the n-dimensional affine space over K and maximal ideals of $K[\bar{X}]$, given by

$$\Phi : (a_1, \cdots, a_n) \mapsto (X_1 - a_1, \cdots, X_n - a_n)$$

thus if $I \subseteq K[\bar{X}]$ and $Q \in K[\bar{X}]$ vanishes on $V(I)$, then $Q(\bar{X})$ is in every maximal ideal that contains I.

Hence as $K[\bar{X}]$ is a Hilbert ring, $Q \in \sqrt{I}$, the *radical of* I, which is the set of polynomials $P \in K[\bar{X}]$ such that there exists $n \in \mathbb{N} \setminus \{0\}$ with $P^n \in I$, and it is the intersection of prime ideals containing I.

Fact 5.0.1 • **Weak Nullstellensatz** *If I is proper then $V(I) \neq \emptyset$*

- **Strong Nullstellensatz** $IVI = \sqrt{I}$

Furthermore, recall that K needs to be algebraically closed: indeed $M := \langle X^2 + 1 \rangle \subseteq \mathbb{R}[X]$ is maximal as $\mathbb{R}[X]/M \cong \mathbb{C}$ is a field but $X^2 + 1$ has no root in $\mathbb{R}$.

Now let (K, R, E) be a partial E-field such that K is algebraically closed, $P_1, \cdots, P_m \in R[\bar{X}]^E$, $I := \langle P_1, \cdots, P_m \rangle \subseteq R[\bar{X}]^E$ the ideal generated by $P_1, \cdots, P_m$, and $V(I) \subseteq K^n$.

- Does $V(I) \neq \emptyset$ iff I is proper?
- Does $G \in IVI$ imply $G \in \sqrt{I}$?

The first question is not true in general. Let C_1 be the subset of $\mathbb{C}[Z]^E$ with maximum one iteration of E. Let

$$F(Z) := E(Z) - c \in C_1, \text{ and } G(Z) := E(iZ) - 1$$

where c is not a power of e^π, and let $I := \langle F(Z), G(Z) \rangle \subseteq C_1$. P. D'Aquino, G. Terzo, A. Fornasiero, and A. Macintyre show that

$$\emptyset = V(I) \subseteq \mathbb{C} \text{ but } I \neq C_1$$

(Indeed let $(C, E) \supseteq (\mathbb{C}, E)$ be an E-field extension and let $a \in V_C(I)$, the set of zeros of I that belong to C. Suppose towards a contradiction that the kernel of E in C, $\ker_C(E)$, is not bigger than the kernel of E in $\mathbb{C}$, $\ker_{\mathbb{C}}(E)$:

$$\ker_C(E) = \ker_{\mathbb{C}}(E)$$

Then as $E(ia) = 1$, $ia \in \ker_C(E) = \ker_{\mathbb{C}}(E)$, hence $a \in 2\pi\mathbb{Z}$; $E(2\pi z) = c$ for some $z \in \mathbb{Z}$; $E(\pi)^{2z} = c$, which contradicts the fact that c is not a power of e^π.)

Here we first show a version of $(*)$:

Proposition 5.0.2 *Let (F, E) be an E-field, and let $M \subseteq F[X]^E$ an ideal. Suppose that M is both an E-ideal and a maximal ideal, and let (L, E) be an E-field extension of (F, E) containing the E-field $(F[X]^E/M, E)$.*
Then $F[X]^E/M \subseteq ecl^L(F)$.

Proof. Let $u = X + M \in F[X]^E/M$. We have $P(u) = 0$, hence by Corollary 2.1.21, $u \in ecl^L(F)$. ■

Proposition 5.0.2 raises the questions:

- If I is maximal as an E-ideal, is it maximal as an ideal?
- If I is maximal as an ideal, is it an E-ideal?
- Is it possible to construct an ideal that is both an E-ideal and a maximal ideal?

Let (R, E) be an E-ring and let $I \subseteq R[\bar{X}]^E$ be a proper ideal. For I to be embeddable in an E-ideal $J \subseteq R[\bar{X}]^E$, I must satisfy, as a set:

$$\bigcup_{\substack{P\in I\\ E(P)-1\notin I\\ L\notin I}} \{L\left(E(P)-1\right)\} \cap (U+I) = \emptyset$$

where U is the set of invertible elements of $R[\bar{X}]^E$. But this is not possible if I is a maximal ideal, unless I is already an E-ideal.

Then let $n \in \mathbb{N}$, and let R_n be the ring of E-polynomials with coefficients in R and maximum n iterations of E the construction of which has been recalled in Subsection 2.1.2. Let $I_n \subseteq R_n$ be a proper ideal of R_n. For I_n to be embeddable in an ideal J_{n+1} of R_{n+1} that contains $E(I_n) - 1$, I_n must satisfy:

$$\bigcup_{\substack{P\in I_n\\ E(P)-1\notin I_n.R_{n+1}\\ L\notin I_n.R_{n+1}}} \{L\left(E(P)-1\right)\} \cap (U_{n+1}+I_n.R_{n+1}) = \emptyset$$

where U_{n+1} is the set of invertible elements of R_{n+1}, and $I_n.R_{n+1}$ is the ideal generated by I_n in R_{n+1}. Suppose $I_n.R_{n+1} \cup (E(I_n) - 1)$ is proper, and let

$$J_{n+1} \supseteq I_n.R_{n+1} \cup (E(I_n) - 1)$$

Then $J_{n+1} \cap R_n \supseteq I_n$, thus if I_n is maximal, $J_{n+1} \cap R_n$ equals either I_n or R_{n+1}. But if J_{n+1} is proper, it cannot contain R_n thus the intersection with R_n equals I_n. Hence it satisfies

$$\bigcup_{\substack{P\in J_{n+1}\cap R_n\\ E(P)-1\notin J_{n+1}\\ L\notin J_{n+1}}} \{L\,(E(P)-1)\}\cap (U_{n+1}+J_{n+1})=\emptyset$$

Consequently we would like to construct a sequence $(I_n \subseteq R_n)_n$ of maximal ideals that are 'semi E-ideals':

$$\begin{cases} P\in I_{n-1}\Rightarrow E(P)-1\in I_n \quad (\dagger_n)\\ \quad I_n\cap R_{n-1}=I_{n-1} \quad (\dagger\dagger_n)\end{cases}$$

Note that for $n\in\mathbb{N}$, if I_n is a proper ideal of R_n, then $I_n\cap R_{-1}=\{0\}$, as $U_n\supseteq R_{-1}=R$ by Fact 2.1.4.

Example 5.0.3 *Let $I\subseteq R_n$ be an ideal and suppose there is an extension (R',E) of (R,E) and a tuple $\bar{a}\subseteq R'$ such that $\bar{a}\in V(I)$. Then let*

$$I_{\bar{a}} := \{P\in R_n : P(\bar{a})=0\}$$

By definition $I_{\bar{a}}$ is a prime ideal containing I and such that for all $Q\in R_{n-1}\cap I$, then $E(Q)-1\in I_{\bar{a}}$.

Example 5.0.4 *Let K be a field, and consider (X), the ideal generated by X in $K[X]$. It is a maximal ideal. Let K_1 be the group ring*

$$(K[X])\,[\exp\,(XK[X])]$$

and J the ideal generated in K_1 by X and $E(XK[X])-1$. Let us show that J is a maximal ideal of K_1: let $P\in K_1\setminus J$, we show that $P+J$ is invertible:

$$P=\sum_{i=1}^{l} P_i(X)\exp(X^{k_i}K_i(X))$$

where $P_i, K_i\in K[X]$ and $k_i\in\mathbb{N}$. Because $P\notin J$,

$$P\neq\sum_{i=1}^{l} X^{n_i}Q_i(x)+P_i(X)(\exp(X^{k_i}K_i(X))-1)$$

where $Q_i, P_i, K_i \in K[X]$, *and* $n_i, k_i \in \mathbb{N}$ *are such that* $\sum_{i=1}^{l} X^{n_i} Q_i(x) \neq 0$.

$$\begin{aligned} P + J &= \sum_{i=1}^{l} P_i(X) \exp(X^{k_i} K_i(X)) + J \\ &= -\sum_{i=1}^{l} P_i(X) + J \\ &= \sum_{i=1}^{l} a_i + X P_i'(X) + J \\ &= \sum_{i=1}^{l} a_i \end{aligned}$$

where $XP_i'(X) + a_i = -P_i(X)$, *and where* $\sum_{i=1}^{l} a_i \in K \setminus \{0\}$ *as* $P \notin J$.
Hence $\sum_{i=1}^{l} a_i = k \in K \setminus \{0\}$ *and* $P + J$ *is invertible.*
The ideal J *thus satisfies* $(\dagger_1)$ *and* $(\dagger\dagger_1)$.

We will show:

Proposition 5.0.5 *Let* $P_1, \cdots, P_m, Q \in R_n$, $n \in \mathbb{N}$ *and*

$$I := \langle P_1, \cdots, P_m \rangle \subseteq R_n$$

be such that $P \in I \cap R_{n-1} \Rightarrow E(P) - 1 \in I$.

- ***"Weak Nullstellensatz"*** $P_1, \cdots, P_m$ *have a common zero in an* E*-field extending* (R, E) *iff the ideal* I *is proper.*
- ***"Strong Nullstellensatz"*** *Assume the ideal* I *is proper and* Q *vanishes at each common zero of* I *in any partial exponential field containing* (R, E)*. Then* $Q \in \sqrt{I}$.

5.1 Hilbert rings and rings of E-polynomials

As written above in the introduction, a Hilbert–or Jacobson–ring is a ring in which every prime ideal is an intersection of maximal ideals. A ring R is Hilbert [31, Theorem 1] iff $R[X]$ is Hilbert iff for every maximal ideal M of $R[X]$, $M \cap R$ is a maximal ideal of R iff for any ideal I of R, $\sqrt{I} = J(I)$, where

$$\sqrt{I} := \{u \in R : u^n \in I \text{ for some } n \in \omega\}$$

and $J(I)$ is the Jacobson radical of I, namely the intersection of all maximal ideals of R containing I.

$$J(I) := \{u \in R : \forall z \exists y (1 + u.z).y - 1 \in I\}$$

As seen in the introduction, because $\sqrt{I} = J(I)$ in a Hilbert ring, Hilbert Nullstellensätz holds.

The Rabinowitsch Spectrum of R can also be defined: $Spec(R)$ is the set of prime ideals of R of the form $R \cap M$, where M is a maximal ideal of $R[Y]$. Then ([27, Proposition 1.11]) for an ideal $I \subseteq R$, where R is a Hilbert ring:

$$\sqrt{I} = \bigcap_{P \in Spec(R), I \subseteq P} P$$

Let G be an abelian group. Recall that a subset $\{g_\alpha\} \subseteq G$ is linearly independent if for any finite sum of elements of $\{g_\alpha\}$ with coefficients $n_\alpha \in \mathbb{Z}$,

$$\sum n_\beta g_\beta = 0 \Rightarrow \forall \beta, n_\beta = 0$$

and that the *torsion-free rank of* G is the cardinality of a maximal linearly independent subset of G, equivalently the dimension of the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$.

Example 5.1.1 *Let* (R, E) *be an* E*-ring,* $n \in \mathbb{N} \backslash \{0\}$*,* $\bar{X} := X_1, \cdots, X_n$*,* $R_0 = R[\bar{X}]$*,* $A_0 = \bar{X}R[\bar{X}]$*,* $G_0 = \exp(A_0)$ *and* $R_1 = R_0[G_0]$*, as seen in the construction of* $R[\bar{X}]^E$ *in Subsection 2.1.2. Then the torsion-free rank of* G_0 *is* $|A_0| = |R[\bar{X}]|$*.*

Let us say that a domain R has *field rank* α (denoted by $fr(R) = \alpha$) if its field of fractions $Frac(R)$ has a set of generators of cardinality α as a R-algebra and has no generating set of smaller cardinality (equivalently the field rank of R is the smallest cardinality of a set $S \subseteq R \setminus \{0\}$ such that the localization of R by S is a field). Note that R is a field iff $fr(R) = 0$.

One can show [31, Theorem 2] that if F is a field and G a group of torsion-free rank $\alpha \geq \omega$, then the group ring $F[G]$ is a Hilbert ring iff $|F| > \alpha$. Recall that a ring is said to be prime if the zero ideal $\{0\}$ is a prime ideal. When F is not especially a field, we have the following:

Fact 5.1.2 [31, Theorem 4] *Let G be a torsion free abelian group, of torsion-free rank α, and let R be a ring. Then the group ring $R[G]$ is a Hilbert ring iff the following conditions are satisfied:*

1. *All homomorphic images of R have cardinality strictly exceeding α.*
2. *For any prime homomorphic image A of R such that $fr(A) \leq \alpha$ and for any ring B such that $A \subseteq B \subseteq Frac(A)$, where $Frac(A)$ is the field of fractions of A, then $fr(B) \neq 1$.*

Facts 5.1.2 together with Example 5.1.1 show that neither

$R[\bar{X}]^E = R[\bar{X}][\exp(A_0 \oplus \cdots \oplus A_n \oplus \cdots)]$ nor $R_n = R[\bar{X}][\exp(A_0 \oplus \cdots \oplus A_{n-1})]$

are Hilbert rings.

5.2 Group rings and augmentation ideals

Let S_0 be a ring of characteristic 0, G a torsion free abelian group and consider the group ring $S_1 := S_0[G]$.

Let $\phi : S_1 \to S_0, \sum_i r_i g_i \mapsto \sum_i r_i \ (g_i \in G, r_i \in S_0)$

The kernel of ϕ is called the *augmentation ideal* of S_1. It is generated by elements of the form $g - 1$, $g \in G$. (write $\sum_i r_i g_i$ as $\sum_i r_i(g_i - 1) + \sum_i r_i$)

Then let $I \subseteq S_0$ an ideal and let $\phi_I = \pi \circ \phi$ be the composition of ϕ with the quotient map $\pi : S_0 \to S_0/I$.

Denote by I_1 the kernel of ϕ_I.

Lemma 5.2.1

- *The kernel I_1 of ϕ_I is an ideal of S_1 containing I.*
- $I_1 \cap S_0 = I$

- *I prime $\Rightarrow I_1$ prime*
- *I maximal $\Rightarrow I_1$ maximal*

Proof.

- Let $h \in S_1$ and $f \in I_1$, then $f.h \in I_1$: indeed, $\phi(f.h) = \phi(f).\phi(h)$. As $f \in I_1$, $\phi(f) \in I$ ideal, hence $\phi(f).\phi(h) \in I$. Consequently $\phi_I(f.h) = 0$, that is $f.h \in I_1$.
- Let $r \in S_0$ such that $\phi_I(r) = 0$. Then $\phi(r) = r + I = I$ hence $r \in I$.
- Let $f.h \in I_1$. Then $\phi(f).\phi(h) \in I$, therefore one of them belongs to I, as I is prime, hence $\phi_I(f) = 0$ or $\phi_I(h) = 0$.
- If I maximal, then S_0/I is a field. Let us show that S_1/I_1 is a field:
  - Consider $f = \sum_i r_i g_i \in S_1 \setminus I_1$.
  - Since $f \notin I_1$, $\phi(f) + I$ is invertible in S_0/I hence there is $s \in S_0$ s.t. $(\sum_i r_i + I).(s + I) = 1 + I$.
  - $(f + I_1).(s + I_1) = (\sum_i r_i(g_i - 1) + \sum_i r_i + I_1).(s + I_1) = (\sum_i r_i + I_1).(s + I_1) = 1 + I_1$

■

Some similar problems have been investigated by K.Manders [43].

Let $n \in \mathbb{N}$ and $I_n \subseteq R_n$ a proper ideal. Suppose we want to embed I_n in a proper ideal I_{n+1} of $R_{n+1} = R_n[\exp(A_n)]$ using the "augmentation ideal tool" so that $I_{n+1} \supseteq E(I_n) - 1$. As $I_n \subseteq R_{n-1} \oplus A_n$, we assume the following:

$$P \in I_n \cap R_{n-1} \Rightarrow E(P) - 1 \in I_n$$

and then "separate" I_n in two parts in order to construct I_{n+1} such that $I_{n+1} \supseteq E(I_n) - 1$.

As a divisible abelian group of the abelian group I_n, $I_n \cap R_{n-1}$ has a direct summand $\tilde{I}_n$ in I_n. First note that the projection of $\tilde{I}_n$ on A_n is injective:
let $u, v \in \tilde{I}_n$, and write $u = u_0 + u_1$, $v = v_0 + v_1$, with $u_0, v_0 \in R_{n-1}$ and

$u_1, v_1 \in A_n$. Suppose $u_1 = v_1$, then $u - v = u_0 - v_0 \in R_{n-1} \cap I_n \cap \tilde{I}_n = \{0\}$, consequently $u = v$.
Let $\tilde{A}_n$ be a direct summand of the projection $\tilde{I}_{n \restriction A_n}$ of $\tilde{I}_n$ in A_n.

Lemma 5.2.2 *Let $u \in R_n[\exp(A_n)]$. Then u can be rewritten in a unique way as*

$$\sum_{i=1}^{l} r_i E(u_i)$$

where $r_i \in R_n$, $u_i \in \tilde{I}_n \oplus \tilde{A}_n$, and for $1 \leq i, j \leq l$, $i \neq j \Rightarrow u_i \neq u_j$.

Proof. Let $u = \sum_{i=1}^{l} r_i.E(a_i) \in R_n[\exp(A_n)]$, where $r_i \in R_n$ and $a_i \in A_n$. Write a_i as

$$a_{i0} + a_{i1} \in \tilde{I}_{n \restriction A_n} \oplus \tilde{A}_n$$

Recall that $\tilde{I}_n \subseteq I_n \subseteq R_n = R_{n-1} \oplus A_n$. Because the projection of $\tilde{I}_n$ on A_n is injective as seen above, there is a unique $f_i \in \tilde{I}_n$ such that a_{i0} is the projection of f_i, that is there is $f_{i0} \in R_{n-1}$ such that $f_i = f_{i0} + a_{i0}$. Then we have

$$E(f_i) = E(f_{i0}).E(a_{i0}) \text{ and } E(a_i) = E(a_{i0}).E(a_{i1}) = E(-f_{i0}).E(f_i).E(a_{i1})$$

where $E(-f_{i0}) \in E(R_{n-1}) \subseteq R_n$ if $E(-f_{i0}) \neq 1$, $E(f_i) \in \tilde{I}_n$, $E(a_{i1}) \in \tilde{A}_n$.

Conversely, let $u = \sum_{i=1}^{l} r_i.E(u_i) \in R_n[E(\tilde{I}_n \oplus \tilde{A}_n)]$, where $r_i \in R_n$ and $u_i \in \tilde{I}_n \oplus \tilde{A}_n \subseteq R_{n-1} \oplus A_n$. Then, $E(u_i) = E(u_{i0}).\exp(a_i)$ for some $a_i \in A_n$ and $u_{i0} \in R_{n-1}$. Hence u belongs to $R_n[\exp(A_n)]$.

The expression is unique by injectivity of the projection of $\tilde{I}_n$ on A_n: if $f \neq g \in \tilde{I}_n$, their respective projections in A_n are different. ■

Lemma 5.2.3 *Let $I_n \subseteq R_n$ be a proper ideal such that $P \in I_n \cap R_{n-1} \Rightarrow E(P) - 1 \in I_n$. Let ϕ_{I_n} be the composition of the projection map $\pi : R_n \to R_n/I_n$ with the augmentation map $R_n[E(\tilde{I}_n \oplus \tilde{A}_n)] \to R_n$. Then* $\ker \phi_{I_n} \supseteq E(I_n) - 1$.

Proof. Let $f \in I_n$; write f as $f_0 + f_1$, with $f_0 \in I_n \cap I_{n-1}$ and $f_1 \in \tilde{I}_n$. Then

$$E(f) - 1 = (E(f_1) - 1).E(f_0) + (E(f_0) - 1)$$

By hypothesis, $E(f_0) - 1 \in I_n$, and by Lemma 5.2.1, $I_n \subseteq \ker \phi_{I_n}$. Furthermore, $E(f_1) - 1$ is in the kernel of the augmentation map, hence in $\ker \phi_{I_n}$. So finally $E(f) - 1 \in \ker \phi_{I_n}$. ■

Lemma 5.2.4 *Let $n \in \mathbb{N}$, $I_n \subseteq R_n$ be a proper (resp. prime, maximal) ideal with the property $(\dagger_n)$. Then I_n embeds in a proper (resp. prime, maximal) ideal I_{n+1} of R_{n+1} with properties $(\dagger_{n+1})$ and $(\dagger\dagger_{n+1})$.*

Proof. By Lemmas 5.2.1 and 5.2.3, setting $I_{n+1} := \ker \phi_{I_n}$. ■

Proposition 5.2.5 *Let $n_0 \in \mathbb{N}$ and $M \subseteq R_{n_0}$ be a proper (resp. prime, maximal) ideal with property $(\dagger_{n_0})$. Then it embeds into an E-ideal $M^E \subseteq R[\bar{X}]^E$ which is a proper (resp. maximal) ideal.*

Proof. Let $M_0 := M$. Lemma 5.2.4 allows to construct a proper (resp. prime, maximal) ideal M_1 of R_{n_0+1} containing $E(M_0) - 1$ and satisfying $(\dagger\dagger_{n_0+1})$, therefore we can reiterate the construction. Then let $M^E := \bigcup_n M_n$. It is an E-ideal by construction, and it is proper because for all $i \in \mathbb{N}$, $M^E \cap R_{n_0+i} = M_i$. It is prime by construction if M is prime. Suppose towards a contradiction that M is maximal but M^E is not. Then it would be properly contained in a proper ideal N of $R[\bar{X}]^E$, hence for some $i \geq 0$ we would have $M_i \subsetneq N \cap R_{n_0+i}$, a contradiction by Lemma 5.2.4. ■

5.3 Rabinowitsch's trick

Recall that Rabinowitsch's trick corresponds to the introduction of an extra variable, in order to prove the algebraic strong Nullstellensatz from the weak one by considering, given $P_1(\bar{X}), \cdots, P_m(\bar{X})$ and another $Q(\bar{X})$ vanishing on all their common zeroes, the introduction of a new variable Y and

$P_1, \cdots, P_m, 1 - Y.Q$ which do not have any common zeroes.

Then one expresses by an equality that 1 belongs to $\langle P_1, \cdots, P_m, 1 - Y.Q \rangle$, the ideal generated by $P_1, \cdots, P_m$ and $1 - Y.Q$. After that, one substitutes Y by Q^{-1} in the equality, and clears denominators.

To mimick Rabinowitsch's trick, we need a "non-E" variable:

Let (K, R, E) be a partial E-field, and let

$$S_n := R_n \otimes_R K[Y]$$

$$S = \bigcup_n S_n \supseteq R[\bar{X}]^E$$

and for G an abelian group we denote by $S_n[G]$ the tensor product of the group ring $R_n[G]$ with $K[Y]$:

$$S_n[G] := R_n[G] \otimes_R K[Y]$$

For J_n an ideal of S_n such that $J_n \cap R_n = I_n$ an ideal of R_n, define ϕ_{J_n} to be the composition of the projection $\pi_n : S_n \to S_n/J_n$ with $\phi_n : S_n[G] := R_n[G] \otimes_R K[Y] \to S_n := R_n \otimes_R K[Y]$, $(\sum_i r_i g_i, P(Y)) \mapsto (\sum_i r_i, P(Y))$.

Lemma 5.3.1 *Let $J_n \subseteq S_n$ be a proper (resp. prime, maximal) ideal and let $I_n := J_n \cap R_n$. Suppose that J_n contains $E(I_n \cap R_{n-1}) - 1$ $(\dagger'_n)$. Then J_n embeds into a proper (resp. prime, maximal) ideal J_{n+1} of S_{n+1} which contains $E(I_n) - 1$ $(\dagger'_{n+1})$ and such that $J_{n+1} \cap R_n = I_n$ $(\dagger\dagger'_{n+1})$.*

Proof. We follow the lines of Lemma 5.2.4's proof. Let $I_{n-1} := I_n \cap R_{n-1}$, it has a direct summand $\tilde{I}_n$ in I_n. Let $\tilde{A}_n$ be a direct summand in A_n of the projection of $\tilde{I}_n$ on A_n.
Then $\ker \phi_{J_n}$ is an ideal of S_{n+1} containing $E(I_n) - 1$ and $\ker \phi_{J_n} \cap S_n = J_n$.
■

Corollary 5.3.2 *Let $n \geq 0$, and $M \subseteq S_n$ be a maximal (resp. prime) ideal. Suppose $E(M \cap R_{n-1}) - 1 \subseteq M$. Then M embeds into a maximal ideal M^E of S such that $M^E \cap R[\bar{X}]^E$ is an E-ideal and $M^E \cap S_n = M$.*

Proof. Let us iterate the construction of Lemma 5.3.1's proof, the same way we followed Lemma 5.2.4 in Proposition 5.2.5's proof. ■

5.4 Statements

Corollary 5.4.1 ***"Weak Nullstellensatz"*** *Let (R, E) be an E-ring and $P_1, \cdots, P_m \in R[\bar{X}]^E$. Let $n \in \mathbb{N}$ be chosen minimal such that $P_1, \cdots, P_m \in R_n$. Assume the ideal I generated by $P_1, \cdots, P_m$ is proper and that there is a maximal ideal M of R_n containing I and such that $M \supseteq E(M \cap R_{n-1}) - 1$ $(\dagger_n)$. Then $P_1, \cdots, P_m$ have a common zero in an E-field extending (R, E).*

Proof. By Proposition 5.2.5, M embeds into an E-ideal M^E of $R[\bar{X}]^E$ which is a maximal ideal; we take the quotient $R[\bar{X}]^E/M^E$; it is an E-field in which $\bar{X} + M^E \in V(I)$. ■

Proposition 5.4.2 ***"Strong Nullstellensatz"*** *Let (K, R, E) be a partial E-field, $P_1, \cdots, P_m, Q \in R[\bar{X}]^E$ and let $n \in \mathbb{N}$ be chosen minimal such that $P_1, \cdots, P_m, Q \in R_n$. Assume the ideal I generated by $P_1, \cdots, P_m$ in R_n is proper and that there is a maximal ideal M of R_n containing I and such that $M \supseteq E(M \cap R_{n-1}) - 1$ $(\dagger_n)$ and that Q vanishes at each common zero of $P_1, \cdots, P_m$ in any partial E-field containing (R, E). Then Q belongs to M, and a power of Q belongs to I.*

Proof. Consider the ideal J of S_n generated by M and $1 - Y.Q$. Note that $J \cap R_n = M$. Assume that J is proper, so one can embed it in a maximal ideal of S_n, and then in a maximal ideal J^E of S which is an E-ideal by Corollary 5.3.2. Since $J^E \cap R[\bar{X}]^E$ is a prime ideal, the quotient

$$R[\bar{X}]^E/(J^E \cap R[\bar{X}]^E)$$

is an E-domain containing (R, E) which embeds in the partial E-field

$$(S/J^E, R[\bar{X}]^E/(J^E \cap R[\bar{X}]^E), E)$$

and where $P_1, \cdots, P_m, Q$ would have a common zero, a contradiction. Consequently:

$$1 = \sum_i t_i(\bar{X}, Y).P_i(\bar{X}) + (1 - Y.Q).r(\bar{X}, Y)$$

with $t_i(\bar{X}, Y), r(\bar{X}, Y) \in S_n \cong R[\bar{X}][\exp(A_0 \oplus \cdots \oplus A_{n-1})] \otimes K[Y]$. Then let us substitute Y by Q^{-1} and multiply each $t_i(\bar{X}, Q^{-1})$ by a power d of Q big enough to obtain an element of S_n. We get:

$$Q^d = \sum_i t_i(\bar{X}, Q^{-1}).Q^d.P_i(\bar{X}) \in R_n$$

since $t_i(\bar{X}, Q^{-1}).Q^d \in R_n$. Therefore $Q^d \in I \subseteq M$, hence $Q \in M$ as M is maximal. ■

5.5 Real Nullstellensatz

We now want to see if we can have similar results in ordered E-fields.

Definition 5.5.1 [36, p.279] A ring $(R, +, ., -, 0, 1)$ is *formally real* if there exists an order $<$ on R such that $(R, +, ., -, 0, 1, <)$ is an ordered ring, that is $<$ is a total ordering such that for $a, b, c \in R$,

$$a < b \Rightarrow a + c < b + c \text{ and } 0 < a, 0 < b \Rightarrow 0 < ab$$

Fact 5.5.2 [36, Theorem 17.11] *A ring R is formally real iff -1 is not a sum of squares of R.*

Definition 5.5.3 Let R be a ring. An ideal I of R is said to be *real* if for any $u_1, \cdots, u_n \in R$ such that $\sum_{i=1}^{n} u_i^2 \in I$, then $u_i \in I$ for all $i = 1, \cdots, n$. The *real radical* of I is:

$$\mathcal{R}(I) := \{f \in R : f^{2m}+s \in I \text{ for some } m \in \mathbb{N} \text{ and } s \text{ a sum of squares of } R\}$$

Fact 5.5.4 [5, Lemmas 4.1.5, 4.1.6 and Proposition 4.1.7] *Let R be a ring, and I an ideal of R.*

- *Suppose I is prime. Then I is real iff the fraction field of R/I is formally real.*
- *The real radical of I is the smallest real ideal containing I.*
- *If I is real, then I is radical.*

Fact 5.5.5 [35, Lemma 1.2 & Remark 1.3] *Let R be a commutative formally real ring, and let Σ be the set of sums of squares in R. Let $I \subseteq R$ be an ideal, maximal with respect to the property that I is disjoint from $1 + \Sigma$. Then I is a prime real ideal.*

Let S_0 be a ring of characteristic 0, G a torsion free abelian group and consider the group ring $S_1 := S_0[G]$.

Let $\phi : S_1 \to S_0, \sum_i r_i g_i \mapsto \sum_i r_i \ (g_i \in G, r_i \in S_0)$

Then let $I \subseteq S_0$ an ideal and let ϕ_I be the composition of ϕ with the quotient map $\pi : S_0 \to S_0/I$.

Denote by I_1 the kernel of ϕ_I.

Lemma 5.5.6 *If I is prime and real then I_1 is prime and real.*

Proof. We have already shown in Lemma 5.2.1 that I prime implies I_1 prime. Suppose $\sum u_i^2 \in I_1$, where forall i, $u_i \in S_1$. Then $\phi(\sum u_i^2) = 0 + I = \sum \phi(u_i)^2$. Because I is real, forall i, $\phi(u_i) \in I$, hence $\phi_I(u_i) = 0$, that is $u_i \in I_1$, which shows that I_1 is real. ∎

Lemma 5.5.7 *Let $n \in \mathbb{N}$, $I_n \subseteq R_n$ be a prime real ideal with the property $(\dagger_n)$. Then I_n embeds in a prime real ideal I_{n+1} of R_{n+1} with properties $(\dagger_{n+1})$ and $(\dagger\dagger_{n+1})$.*

Proof. By Lemmas 5.2.4 and 5.5.6. ∎

Corollary 5.5.8 *Let $P_1,\cdots,P_m \in R[\bar{X}]^E$ and let $n \in \mathbb{N}$ be chosen minimal such that $P_1, \cdots, P_m \in R_n$. Let Σ be the set of sums of squares in R_n. Assume the ideal I generated by $P_1, \cdots, P_m$ is disjoint from the set $1 + \Sigma$ and let $M \subseteq R_n$ be an ideal that is maximal for the properties of containing I and being disjoint from $1 + \Sigma$. Suppose M can be chosen such that $M \supseteq E(M \cap R_{n-1}) - 1$ $(\dagger_n)$. Then $P_1, \cdots, P_m$ have a common zero in an ordered E-field extending (R, E).*

Proof. By 5.5.5, M is a prime real ideal. By Proposition 5.2.5 and Lemma 5.5.7, M embeds in a prime real ideal M^E of $R[\bar{X}]^E$ that is an E-ideal. The quotient $R[\bar{X}]^E/M^E$ is an orderable E-field where $P_1, \cdots, P_m$ have a common zero. ∎

Chapter 6

E-varieties and E-torsors

In this chapter we want to adapt the notions of variety, generic point, and torsor as a tool to extend E-derivations in the E-algebraic case. The notions of E-variety, generic point of an E-variety, regular variety and desingularisation of a variety have already been developped (see for example [25] in the setting of definably complete expansions of real closed fields–and Section 3.1), and we extend results on torsors from [51] to results on 'E-torsors'. Given a regular E-variety V defined over a partial E-field (K, R, E), we show how to construct a generic point of it, lying in some power of $K((\bar{t}))$, field of Laurent series over K, or in an elementary extension.
Assuming moreover that $Th(K)$ satisfies an implicit function theorem in Section 6.3, we construct a generic point of V in an elementary extension of (K, R, E).
We will work with the results of Chapter 4.

6.1 Generic points of regular E-varieties

In the algebraic case one can define a notion of generic point, and show a correspondence between generic points, prime ideals and irreducible varieties. In subsubsection 2.1.5.1 of the setting, we have seen that, although it is weaker, we still have a correspondance between prime ideals and irreducible varieties in the (non-Noetherian, non-Hilbertian) E-algebraic setting. Nevertheless, the definition of E-algebraicity involves inequalities as well as equalities, and other coordinates, consequently we cannot use a definition of generic point of a variety based on an ideal or E-ideal of E-polynomials vanishing at it. We focus on the notion of regular E-variety instead of irreducible E-variety, as it enables us to construct generic points using Hensel's Lemma.

Let $(K, R, E) \subseteq (L, R', E)$ be partial E-fields, $n, m \in \mathbb{N} \setminus \{0\}$, $n := |\bar{X}|$, $P = (P_1, \cdots, P_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$.

Definition 6.1.1 Let us denote by $V^{reg}(P) \subseteq L^n$ the set of points $\bar{a}$ of L such that there is a m-tuple $\bar{y}$ for which, after possible renumerotation of the columns, $JP_{\bar{a}}$ has a non-singular submatrix $J_{(0,\bar{y})}P_{\bar{a}}$, that is to say $\det J_{(0,\bar{y})}P_{\bar{a}} \neq 0$. If $V(P) = V^{reg}(P)$, $V(P)$ is called a *regular* variety, and the system $P(\bar{X}) = \bar{0}$ a *regular* system, while the elements of $V(P)$ are called *regular zeros.*

Recall that $\bar{a} \in V^{reg}(P)$ iff $\nabla P_1(\bar{a}), \cdots, \nabla P_m(\bar{a})$ are linearly independent.

If $A = V(P)$, we denote $A^{reg} := V^{reg}(P)$.

Remark 6.1.2 *If $m = n$, and we are given a squared $m \times m$ system $P(\bar{X}) = \bar{0}$ that is regular, that is a Hovanskii system, then the coordinates of elements of $V^{reg}(P_1, \cdots, P_m)$ are E-algebraic over the set of coefficients of $P_1, \cdots, P_m$.*

Remark 6.1.3 *We implicitly consider varieties associated to a particular presentation. Indeed, $V(P_1, \cdots, P_m) = V(P_1^2, \cdots, P_m^2)$, but $V^{reg}(P_1^2, \cdots, P_m^2)$ is always empty even if $V^{reg}(P_1, \cdots, P_m)$ is not.*

Recall that there is a good notion of dimension associated to *ecl* by Fact 4.0.1–see Definition 4.0.2–.

Definition 6.1.4 Let $A \subseteq L^n$ be a set defined over K, and $\dim_K A := \sup\{ecl^L - dim_K\bar{a} \ : \ \bar{a} \in A\}$ be the dimension of A over K.
Let $V \subseteq L^n$ be an E-variety defined over K. If $\bar{a} \in V$ is such that $\dim_K V = ecl^L - \dim_K \bar{a}$, then $\bar{a}$ is called a *generic point* of V over K.

Remark 6.1.5 *Let $P = (P_1, \cdots, P_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $\bar{a} := a_1, \cdots, a_n \in V^{reg}(P) \subseteq L^n$. Then:*

- *There are m coordinates $a_{i_1}, \cdots, a_{i_m}$ of $\bar{a}$ that form a tuple solution of a $m \times m$ Hovanskii system defined over $K \cup \{a_{i_{m+1}}, \cdots, a_{i_n}\}$, where $a_{i_{m+1}}, \cdots, a_{i_n}$ are the $n-m$ other coordinates of $\bar{a}$.*

- *$ecl^L - \dim_K \bar{a} \leq n - m$. In particular, if $V(P) = V^{reg}(P)$, then V has dimension over K smaller than $n - m$.*

Proof. Let $\bar{a} \in V(P) = V^{reg}(P)$. By regularity $JP_{\bar{a}}$ admits a $m \times m$ non-zero minor. Without loss of generality suppose it is $\det J_{(\bar{0}_{n-m}, \bar{y}_m)} P_{\bar{a}}$, that is the determinant of the submatrix formed by the m right columns. Let $\bar{a}_{n-m} := a_1, \cdots, a_{n-m}$, and $\bar{a}' := a_{n-m+1}, \cdots, a_n$, $\bar{X}' := X_{n-m+1}, \cdots, X_n$. Then, for $i = 1, \cdots, m$, let

$$P'_i(\bar{X}') := P_i(\bar{a}_{n-m}, \bar{X}')$$

This shows $\bar{a}' \in V^{reg}(P'_1, \cdots, P'_m)$ and, in other words, that $\bar{a}'$ is a solution of the Hovanskii system formed by the P'_i's, with $\det JP'_{\bar{a}'} = \det J_{(\bar{0}_{n-m}, \bar{y}_m)} P_{\bar{a}} \neq 0$, that is $a_{n-m+1}, \cdots, a_n \in ecl^L(K(a_1, \cdots, a_{n-m}))$. Consequently $ecl^L - \dim_K \bar{a} \leq n - m$. ∎

From now on, let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$, and let $\mathcal{V}$ be a definable base of neighborhoods of 0 in K, such that $(K, R, E, \mathcal{V})$ is a topological $\mathcal{L}$-partial-E-field.

Proposition 6.1.6 *Let $Q = (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. Suppose that there is $\bar{a} \in V^{reg}(Q) \cap K^n$.*
Then there is a topological elementary $\mathcal{L}$-extension $(K_1, R_1, E, \mathcal{V}_1)$ of $(K, R, E, \mathcal{V})$ such that K_1 contains K and some elements $t_1, \cdots, t_{n-m}$ with:
for $i = 1, \cdots, n - m$, $t_i \sim_K 0$ and
there is $\bar{g} \in V^{reg}(Q) \subseteq K((t_1)) \cdots ((t_{n-m}))^n$ with $\bar{g} - \bar{a} \sim_K \bar{0}$
(and hence $\bar{g} - \bar{a} \sim_{\mathcal{V}_1(K)} \bar{0}$, for $\mathcal{V}_1(K) \models Comp\, K$), and
$ecl^{K((t_1)) \cdots ((t_{n-m}))} - dim_K(\bar{g}) = n - m$.

Furthermore, if $V(Q) = V^{reg}(Q)$, then $\bar{g}$ is a generic point of $V(Q)$.

Proof. By Corollary 4.1.4, let $(K_1, R_1, E, \mathcal{V}_1)$ be a topological elementary $\mathcal{L}$-extension of $(K, R, E, \mathcal{V})$ such that K_1 contains $K \cup \{t_1, \cdots, t_{n-m}\}$, where $t_1, \cdots, t_{n-m}$ are $ecl^{K((t_1)) \cdots ((t_{n-m}))}$-independent over K and such that each $t_i \sim_K 0$.

Without loss of generality suppose det $J_{(\bar{0}_{n-m},\bar{y}_m)}Q_{\bar{a}} \neq 0$.
Let $\bar{a}_{n-m} := a_1, \cdots, a_{n-m}$, $\bar{a}' := a_{n-m+1}, \cdots, a_n$, $\bar{X}' := X_{n-m+1}, \cdots, X_n$, and $Q'(\bar{X}') := Q(\bar{a}_{n-m}, \bar{X}')$. Then Q' defines a $(m \times m)$ Hovanskii system. Let $Q'^{\bar{t}}(\bar{X}') := Q(\bar{a}_{n-m} + \bar{t}_{n-m}, \bar{X}')$. It also defines a Hovanskii system and one gets $Q'^{\bar{t}}(\bar{a}') \sim_K \bar{0}$ and det $JQ'^{\bar{t}}_{\bar{a}'} \nsim_K 0$ by continuity.
By Hensel's Lemma 3.3.4 in $K[[t_1]] \cdots [[t_{n-m}]]$, one finds a regular zero $\bar{c}'$ of $Q'^{\bar{t}}$. Consequently $\bar{a}_{n-m} + \bar{t}_{n-m}, \bar{c}' \in V(Q)$ and has $ecl^{K((t_1))\cdots((t_{n-m}))}$-dimension $n - m$ over K. ∎

6.2 E-Torsors and extensions of E-derivations

Let $(K, R, E) \subseteq (L, R', E)$ be partial E-fields, $n \in \mathbb{N}$, $n := |\bar{X}|$.

Definition 6.2.1 *Let $V := V(Q) \subseteq L^n$ be an E-variety defined over K, and $I(V) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ be the ideal of E-polynomials with coefficients in K which vanish on V. Let*

$$\tau(V) := \{(\bar{a}, \bar{b}) : \bar{a} \in V \textit{ and } \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).b_i + P^D(\bar{a}) = 0 \, \textit{for all}\, P(\bar{X}) \in I(V)\}$$

the torsor of V.

Let $\bar{a} \in V \subseteq L^n$. The fibre of the torsor at $\bar{a}$ is set as:

$$\tau_{\bar{a}}(V) := \{\bar{b} \; : \; \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).b_i + P^D(\bar{a}) = 0 \, \text{for all}\, P(\bar{X}) \in I(V)\}$$

And the tangent space of V at $\bar{a}$ is:

$$T_{\bar{a}}(V) := \{\bar{b} \; : \; \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).b_i = 0 \, \text{for all}\, P(\bar{X}) \in I(V)\}$$

For $\bar{a} \in L^n$, recall that $I(\bar{a})$ denote the ideal of E-polynomials of $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ which vanish at $\bar{a}$. In what follows, let dim denote the linear dimension of $ecl^L(K(\bar{a}))$-vector spaces.

Lemma 6.2.2 *Let* $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. *Let* $V := V(Q)$. *Suppose that there is* $\bar{a} \in V^{reg}(Q) \subseteq L^n$ *such that* $ecl^L - \dim_K \bar{a} = n - m$. *Let*

$$T'_{\bar{a}}(V) := \{\bar{b} \ : \ \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{a}).b_i = 0 \, for \, all \, f \in I(\bar{a})\}$$

and

$$\ker JQ_{\bar{a}} := \{\bar{b} \ : \ \sum_{i=1}^{n} \frac{\partial Q_j}{\partial X_i}(\bar{a}).b_i = 0 \, for \, j = 1, \cdots, m\}$$

1. $T'_{\bar{a}}(V) \subseteq T_{\bar{a}}(V) \subseteq \ker JQ_{\bar{a}}$
2. $\dim T'_{\bar{a}}(V) \geq ecl^L - \dim_K \bar{a} = n - m$.
3. $\dim(\ker JQ_{\bar{a}}) = \dim T'_{\bar{a}}(V) = \dim T_{\bar{a}}(V) = ecl^L - \dim_K \bar{a} = n - m$.

Proof. Let $\bar{a}_{n-m} := a_1, \cdots, a_{n-m}$ and suppose without loss of generality that $a_1, \cdots, a_{n-m}$ are ecl^L-independent over K. Let $L_0 := K(\bar{a}, E(\bar{a})) \subseteq L_1 := ecl^L(K(\bar{a}_{n-m}))$.

1. Let $J := \langle Q_1, \cdots, Q_m \rangle_E$, the E-ideal of $K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ generated by $Q_1, \cdots, Q_m$.
   We have $I(\bar{a}) \supseteq I(V(J)) \supseteq J$, hence
   $$T'_{\bar{a}}(V) \subseteq T_{\bar{a}}(V) \subseteq \ker JQ_{\bar{a}}$$
2. By Lemma 4.2.3, we have that the L_1-linear dimension of $EDer(L_1/K)$ is $ecl^L - dim_K(\bar{a}) = n - m$. We use the base $D_1, \cdots, D_{n-m}$ of Lemma 4.1.2 to construct a L_1-linear independent family of $T'_{\bar{a}}(V)$. For $i = 1, \cdots, n$, $j = 1, \cdots, n-m$, let $b_{ji} := D_j(a_i)$, and $\bar{b}_j := (b_{j1}, \cdots, b_{jn})$, so $\bar{b}_1 := D_1(\bar{a}), \cdots, \bar{b}_{n-m} := D_{n-m}(\bar{a})$. For $\alpha_1, \cdots, \alpha_{n-m} \in L_1$, we do have
   $$\sum_{i=1}^{n-m} \alpha_i \bar{b}_i = \bar{0} \Rightarrow \alpha_1 = 0, \cdots, \alpha_{n-m} = 0$$
   Thus it remains to verify that each $\bar{b}_j = D_j(\bar{a}) \in T'_{\bar{a}}(V)$: let $f \in I(\bar{a})$. Then in $EDer(L_1/K)$,
   $$D_j(f(\bar{a})) = 0 = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{a}) D_j(a_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{a}).b_{ji}$$

3. By regularity, $\nabla Q_1(\bar{a}), \cdots, \nabla Q_m(\bar{a})$ are L_1-linearly independent, so rank $JQ_{\bar{a}} = m$, hence dim(ker $JQ_{\bar{a}}) = n - m$.

   Then as $T'_{\bar{a}}(V) \subseteq T_{\bar{a}}(V) \subseteq \ker JQ_{\bar{a}}$ as L_1-vector spaces, and as the L_1-linear dimension of $T'_{\bar{a}}(V) \geq n - m$ and the L_1-linear dimension of $\ker JQ_{\bar{a}}$ is $n - m$, we have equality between our three vector spaces.

■

Let $m \in \mathbb{N}$, $P = (P_1, \cdots, P_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. Let $V = V(P)$, and let $\bar{a} \in V^{reg}(P) \subseteq L^n$ be a point of maximal ecl^L-dimension over K, $n - m$. Let

$$\tau_{\bar{a}}(P) := \{\bar{b} \ : \ \sum_{i=1}^{n} \frac{\partial P_j}{\partial X_i}(\bar{a}).b_i + P_j^D(\bar{a}) = 0 \, \text{for} \, j = 1, \cdots, m\}$$

By Lemma 6.2.2, $\tau_{\bar{a}}(P) = \tau_{\bar{a}}(V)$: indeed let $\bar{c} \in \tau_{\bar{a}}(P)$ and $\bar{c}' \in \tau_{\bar{a}}(V)$. Then $\bar{c} - \bar{c}' \in T_{\bar{a}}(V) = \ker JQ_{\bar{a}}$. Hence $\bar{c} \in \bar{c}' + T_{\bar{a}}(V) = \tau_{\bar{a}}(V) \subseteq \tau_{\bar{a}}(P)$.

Remark 6.2.3 *Let $m \in \mathbb{N}$, $P = (P_1, \cdots, P_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. Let $\bar{a} \in V^{reg}(P) \subseteq L^n$ be a point of maximal ecl^L-dimension over K, $n - m$. Without loss of generality suppose $a_1, \cdots, a_{n-m}$ are ecl^L-independent over K and that $a_{n-m+1}, \cdots, a_n \in ecl^L(K(a_1, \cdots, a_{n-m}))$. As seen with Remark 6.1.5, there is then a Hovanskii system expressing that a_{n-m+1}, $\cdots$, $a_n \in ecl^L(K(a_1, \cdots, a_{n-m}))$, without involving other elements of L. This kind of play the role, in the "$V^{reg}(P)$-context" of the minimal polynomial in the algebraic case, which allow us to extend to the E-algebraic case a theorem of S.Lang [37, Th.1 p.184] which is a main tool for extending derivations in the algebraic case:*

Lemma 6.2.4 *Let $(K, R, E) \subseteq (L, R', E)$ be partial E-fields, and let D be an E-derivation on K. Let $H = (H_1, \cdots, H_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. Suppose that there are $\bar{a} := a_1, \cdots, a_n$, $\bar{b} := b_1, \cdots, b_n \in L^n$ such that $\bar{a} \in V^{reg}(H) \subseteq L^n$ and $ecl^L - dim_K(\bar{a}) = n - m$, and*

$$\bar{b} \in \tau_{\bar{a}}(H)$$

Then there is a unique E-derivation D^* on $ecl^L(K(\bar{a}, E(\bar{a})))$ extending D and such that for $i = 1, \cdots, n$, $D^*(a_i) = b_i$.

Proof. Without loss of generality, let $d := n-m = ecl^L - dim_K(\bar{a})$, and assume $a_1, \cdots, a_d$ are ecl^L-independent. By Remark 2.1.21, as $a_1, \cdots, a_d$ are ecl^L-independent, $a_1, \cdots, a_d$ do not satisfy 'relations' over K: there is no $p \in \mathbb{N}$, $c_1, \cdots, c_p \in K$ and $P \in \mathbb{Z}[\bar{X}\bar{Y}]^E \setminus \{0\}$ such that $P(\bar{a}\bar{c}) = 0$. Define a mapping D^* on $K(\bar{a}_d, E(\bar{a}_d))$ by:

$$D^*(a_i) := b_i;\ D^*(E(a_i)) := E(a_i)D^*(a_i) \text{ for } i = 1, \cdots, d$$

$$D^*(P(\bar{a}_d)) := \sum_{i=1}^{d} \frac{\partial P}{\partial X_i}(\bar{a}_d).b_i + P^D(\bar{a}_d)$$

$$D^*((P \circ E)(\bar{a}_d)) := \sum_{i=1}^{d} \frac{\partial P}{\partial X_i}(\bar{a}_d).E(a_i)b_i + P^D(E(\bar{a}_d))$$

$$D^*(P(\bar{a}_d)/Q(\bar{a}_d)) := \frac{P(\bar{a}_d)D^*(Q(\bar{a}_d)) - Q(\bar{a}_d)D^*(P(\bar{a}_d))}{Q(\bar{a}_d)^2}$$

for all $P(\bar{X})$, $Q(\bar{X})$ in $K[\bar{X}]$ ($Q(\bar{a}_d) \neq 0$). The mapping D^* is a well-defined E-derivation. By construction, it is unique as a mapping satisfying $D^*(a_i) = b_i$ for $i = 1, \cdots, d$ on $K(\bar{a}_d, E(\bar{a}_d))$.

Then as in the proof of Proposition 4.2.1, by derivation of the $m \times m$ Hovanskii system H_H defined over $K(\bar{a}_d)$ by $H(\bar{a}_d, \bar{X}') = \bar{0}$ and $\det J_{(\bar{0},\bar{y}_m)} H_{\bar{a}_d\bar{X}'} \neq 0$, where $\bar{X}' := X_{d+1}, \cdots, X_n$, one obtains that for $i = d+1, \cdots, n$, D^* extends uniquely on a_i by $D^*(a_i) := b_i$: indeed, for $i = 1, \cdots, m$, let

$$H'_i(\bar{X}') := H_i(\bar{a}_d, \bar{X}')$$

Then, by derivation of H_H, we obtain, as was done in the proof of Proposition 4.2.1, a squared linear system $\mathcal{S}$ in the Da_j's:
for $i = 1, \cdots, m$,

$$\sum_{j=d+1}^{n} \frac{\partial H'_i}{\partial X_j}(\bar{a}').Da_j + H'^D_i(\bar{a}') = 0$$

the determinant of which is non-zero:

$$\det JH'_{\bar{a}'} = \det J_{(\bar{0}_d,\bar{y}_m)} H_{\bar{a}_d\bar{a}'} \neq 0$$

hence the system $\mathcal{S}$ admits a unique solution. Furthermore,

$$\sum_{j=d+1}^{n} \frac{\partial H_i'}{\partial X_j}(\bar{a}').Da_j + H_i'^D(\bar{a}') = \sum_{j=d+1}^{n} \frac{\partial H_i'}{\partial X_j}(\bar{a}').Da_j + \sum_{k=1}^{d} \frac{\partial H_i}{\partial X_k}(\bar{a}).b_k + H_i^D(\bar{a})$$

$$= \sum_{j=1}^{d} \frac{\partial H_i}{\partial X_j}(\bar{a}).b_j + \sum_{j=d+1}^{n} \frac{\partial H_i}{\partial X_j}(\bar{a}).Da_j + H_i^D(\bar{a})$$

Consequently $\bar{b}' := b_{d+1}, \cdots, b_n$ is a solution of $\mathcal{S}$, because $\bar{b} \in \tau_{\bar{a}}(V)$, and therefore it is the unique solution. ■

Although we will not use them later, we show how to adapt Lemmas 1.3 and 1.6 of [51].

In Lemmas 6.2.5, 6.2.6, and Corollary 6.2.7, we make the following assumptions:

Let $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, and $H := (H_1, \cdots, H_p) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, where $|\bar{Y}| = |\bar{X}|$.
Let $V := V(Q) \subseteq L^n$, $W := V(H) \subseteq L^{2n}$.
Let $\bar{a}, \bar{b}$ be n-tuples from L such that $\bar{a}\bar{b} \in W^{reg} \cap \tau(V)$, $\bar{a} \in V^{reg}$, $ecl^L - dim_K(\bar{a}\bar{b}) = 2n - p$, and $ecl^L - dim_K(\bar{a}) = n - m$.

Lemma 6.2.5 *Let π be the projection map from $\tau_{\bar{a}\bar{b}}(W)$ onto the first n coordinates.*
Then π is onto $\tau_{\bar{a}}(V)$.

Proof. We first want to show that $\pi(\tau_{\bar{a}\bar{b}}(W)) \subseteq \tau_{\bar{a}}(V)$.
Notice that $\pi(\tau_{\bar{a}\bar{b}}(W)) \subseteq \tau_{\bar{a}}(V)$.
Then, the corresponding projection on tangent spaces maps $T_{\bar{a}\bar{b}}(W)$ onto $T_{\bar{a}}(V)$, because by Lemma 6.2.2, as $ecl^L(K(\bar{a}))$-vector spaces, $\dim T_{\bar{a}\bar{b}}(W) = ecl^L - dim_K \bar{a}\bar{b}$, and $\dim T_{\bar{a}}(V) = ecl^L - dim_K \bar{a}$, hence the rank of the projection is maximal, so it is onto.
Consequently, as all the maps and actions commute, one obtains that π maps $\tau_{\bar{a}\bar{b}}(W)$ onto $\tau_{\bar{a}}(V)$. ■

Corollary 1.7 of [51], page 111, stays true in the E-algebraic case–the statements are the same, replacing varieties by E-varieties and torsors by torsors involving E-polynomials–

Lemma 6.2.6 *There is $\bar{c} \subseteq ecl^L(K(\bar{a}\bar{b}))$ such that $\bar{b}\bar{c} \in \tau_{\bar{a}\bar{b}}(W)$.*

Proof. Word to word the proof from [51], that goes through in the E-algebraic case too. By Lemma 6.2.5, $\{\bar{w} \in L^n \ : \ \bar{b}\bar{w} \in \tau_{\bar{a}\bar{b}}(W)\}$ is nonempty. But this set is defined as a finite set of linear equations over $ecl^L(K(\bar{a}\bar{b}))$, so has a solution in $ecl^L(K(\bar{a}\bar{b}))$. ■

Corollary 6.2.7 *An E-derivation on K uniquely extends on $ecl^L(K(\bar{a}\bar{b}))$ in such a way that $\bar{b} = D(\bar{a})$.*

Proof. By Lemmas 6.2.6 and 6.2.4 ■

6.3 Hypothesis $\mathcal{I}m$ and regular E-varieties

In this section, considering $(K, R, E, \mathcal{V}) \subseteq (L, R', E, \mathcal{W})$ topological partial E-fields satisfying Hypothesis $\mathcal{I}m$, we state a few more results. We will not use these results later on, except Proposition 6.3.4.

Recall that if $\bar{a}$ is a generic point of an algebraic variety V, and $\bar{b} \in V$, then $I(\bar{a}) = I(V) \subseteq I(\bar{b})$.

Let $(K, R, E) \subseteq (L, R', E)$ be partial E-fields. Let $\bar{X} := X_1, \cdots, X_n$, $n \in \mathbb{N} \setminus \{0\}$.

Note that by Lemma 3.1.11, if a topological partial E-field $(K, R, E, \mathcal{V})$ satisfies Hypothesis $\mathcal{I}m$ and 'Lack of flat functions' (Fact 3.1.10) then

whenever $G = (g_1, \cdots, g_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ and $\bar{x}_0 \in V^{reg}(G)$, then

- either there exists $h \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ such that $\bar{x}_0 \in V^{reg}(G, h)$,
- or for all $h \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, if $h(\bar{x}_0) = 0$, then h vanishes on a neighborhood, for the induced topology on $V^{reg}(G)$, of $\bar{x}_0$ in $V^{reg}(G)$.

If $V(Q) = V^{reg}(Q)$, this gives us a local counterpart of the classical algebraic result on ideals of generic points of algebraic varieties recalled above:

Corollary 6.3.1 *Let* $(K, R, E, \mathcal{V}) \subseteq (L, R', E, \mathcal{W})$ *be topological partial* E*-fields satisfying Hypothesis* $\mathcal{I}m$ *and 'Lack of flat functions' (Fact 3.1.10). Let* $\bar{a} \in V^{reg}(Q) \subseteq L^n$*,* $Q = (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ *such that* $ecl^L - dim_K(\bar{a}) = n - m$*.*
Then there is $O \in \mathcal{W}(K)$ *such that* $I(\bar{a}) \subseteq I(\bar{a} + O \cap V^{reg}(Q))$*, where* $I(.) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$*.*

Although we will not use the following result, we show how to reduce problems involving E-varieties to problems dealing with regular E-varieties:

Lemma 6.3.2 *Let* $(K, R, E, \mathcal{V})$ *be a topological partial* E*-field satisfying Hypothesis* $\mathcal{I}m$ *and 'Lack of flat functions'(Fact 3.1.10), and let* $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$*,* $\bar{a} \in V(P) \subseteq K^{|\bar{X}|}$*.*

Then there exists $Q = (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ *and a definable open subset* S *(for the induced topology on* $V^{reg}(Q)$*) of* $V^{reg}(Q) \subseteq K^{|\bar{X}|}$ *such that* $\bar{a} \in S \subseteq V(P) \cap V^{reg}(Q) \subseteq K^{|\bar{X}|}$*.*

Proof. Almost word to word the proof from [61, instance of Theorem 32 P.195].
By Lemma 2.1.22, there exists H such that $\bar{a} \in V^{reg}(H)$. Let $k_0(\bar{a}) := min\{k | \exists H := (H_1, \cdots, H_{|\bar{a}|-k}) \in K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E, \bar{a} \in V^{reg}(H)\}$, and let $Q = (Q_1, \cdots, Q_m)$ such that $m = |\bar{a}| - k_0(\bar{a})$ and $\bar{a} \in V^{reg}(Q)$.

As we have chosen $V^{reg}(Q)$ with $|Q|$ maximal, Lemma 3.1.11–as noticed above– gives us that every function h such that $h(\bar{a}) = 0$ also satisfies $h \equiv 0$ on an open neighborhood S of $\bar{a}$ within $V^{reg}(Q)$. Consequently, as $P(\bar{a}) = 0$, P vanishes on some S, so there is a set S, open for the induced topology on $V^{reg}(Q)$, such that $\bar{a} \in S \subseteq V(P) \cap V^{reg}(Q)$. ■

Note that proof of Lemma 6.3.2 constructs $Q = (Q_1, \cdots, Q_m)$ with m maximal –as length of sequence of E-polynomials vanishing on some point such that their gradients are linearly independent at this point–.

Remark 6.3.3 *If we had supposed in the hypotheses of Proposition 6.1.6, that* $Th(K)$ *also satisfies Hypothesis* $\mathcal{I}m$*, we could have alternatively constructed a generic point using the implicit function theorem instead of Hensel's Lemma:*

Proposition 6.3.4 *Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$, and let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field such that $Th(K)$ satisfies Hypothesis $\mathcal{I}m$ and $\mathcal{D}$. Let $Q = (Q_1, \cdots, Q_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$. Suppose that there is $\bar{a} \in V^{reg}(Q) \cap K^n$.*
Then there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$ such that there is $\bar{a}_1 \in V^{reg}(Q) \subseteq L^n$ with $\bar{a}_1 - \bar{a} \sim_{\mathcal{W}(K)} \bar{0}$, for $\mathcal{W}(K) \models Comp\, K$, and $ecl^L - dim_K(\bar{a}_1) = n - p$.

Furthermore, if $V(Q) = V^{reg}(Q)$, then $\bar{a}_1$ is a generic point of $V(Q)$.

Proof. Without loss of generality suppose $\{k_1, \cdots, k_p\} = \{n - p + 1, \cdots, n\}$. Let $\bar{a}_{n-p} := a_1, \cdots, a_{n-p}$ and $\bar{a}' := a_{n-p+1}, \cdots, a_n$. By Hypothesis $\mathcal{I}m$, there are $O \subseteq K^{n-p}$, $O' \subseteq K^p$ open sets with $\bar{a} \in O \times O'$, and a continuous function $g : O \to O'$ such that $\bar{a}' = g(\bar{a}_{n-p})$ and such that for all $\bar{x}_{n-p} := x_1, \cdots, x_{n-p} \subseteq O$ and $\bar{x}' := x_{n-p+1}, \cdots, x_n \subseteq O'$, $\bar{x}' = g(\bar{x}_{n-p})$ iff $Q(\bar{x}_{n-p}\bar{x}') = \bar{0}$.

By Fact A.0.8, there is a topological elementary $\mathcal{L}$-extension $(K_1, R_1, E, \mathcal{V}_1)$ of $(K, R, E, \mathcal{V})$ that contains $K \cup \{t_1\}$, where t_1 is ecl^{K_1}-independent over K, and for all $V \in \mathcal{V}_1(K)$, $t_1 \in V$, where $\mathcal{V}_1(K)$ satisfies $Comp(K)$, hence $t_1 \sim_{\mathcal{V}_1(K)} 0$.

By repeating use of Fact A.0.8, for $i = 2, \cdots, n-p$, there is a topological elementary $\mathcal{L}$-extension $(K_i, R_i, E, \mathcal{V}_i)$ of $(K_{i-1}, R_{i-1}, E, \mathcal{V}_{i-1})$, and thus of $(K, R, E, \mathcal{V})$ that contains $K_{i-1} \cup \{t_i\} \supseteq K \cup \{t_1, \cdots, t_i\}$, where t_i is ecl^{K_i}-independent over K_{i-1}, and $t_i \sim_{\mathcal{V}_i(K_{i-1})} 0$, where $\mathcal{V}_i(K_{i-1})$ satisfies $Comp(K_{i-1})$. Moreover by Fact 2.2.5 there is $\tilde{\mathcal{V}}_i(K) \subseteq \mathcal{V}_i(K_{i-1})$ satisfying $Comp(K)$, hence $t_i \sim_{\tilde{\mathcal{V}}_i(K)} 0$.

Let $(L, R', E, \mathcal{W}) := (K_{n-p}, R_{n-p}, E, \mathcal{V}_{n-p})$.
Then $t_1, \cdots, t_{n-p}$ are ecl^L-independent over K, and for $i = 1, \cdots, n-p$, $t_i \sim_{\mathcal{W}(K)} 0$, where $\mathcal{W}(K)$ satisfies $Comp(K)$.

Let $\bar{t}_{n-p} := t_1, \cdots, t_{n-p}$. We have that $\bar{a}_1 := \bar{a}_{n-p} + \bar{t}_{n-p}, g(\bar{a}_{n-p} + \bar{t}_{n-p})$ is a regular zero of Q of ecl^L-dimension $n - p$ over K. ■

Chapter 7

A differential lifting scheme

In this chapter, in order to generalize results of [23], we consider first-order theories T of expansions of topological partial E-fields that are model-complete and T_D of their expansion to differential topological partial E-fields. In an attempt of selecting the models that are existentially closed in the class of models of T_D, we give a geometric scheme satisfied by this subclass.
We first state an hypothesis $(I)_E$ on the class $\mathcal{C}$ of models of T, which, like Hypothesis (I) in [23], implies on elements K of $\mathcal{C}$ that there is an extension of K in $\mathcal{C}$ that contains the field of Laurent series $K((\bar{t}))$. Then if T is model-complete, this implies that K, as a field, is large, that is existentially closed in $K((\bar{t}))$.

We then state a differential lifting scheme $(DL)_E$, in the spirit of other differential lifting schemes ([51], [48], [23]); it reduces an E-differential problem to an E-algebraic one, then 'lifts' the existence of solutions. We show that if $\tilde{K} \models T_D$ has a reduct K such that $Th(K)$ satisfies either Hypothesis $(I)_E$ or Hypothesis $\mathcal{I}m$, then $\tilde{K}$ has an extension in the class $\tilde{\mathcal{C}}$ of models of T_D, that satisfies a 'pre-scheme' $(DL)_E$.

As a corollary we get that for $T := T_{\mathbb{R},\exp}$ and for $T := T_{\mathcal{O}_p,E_p}$, the models of T_D that are existentially closed in T_D satisfy $(DL)_E$.

Finally we construct exponential derivations that satisfy scheme $(DL)_E$ on $(\mathbb{R}, \exp)$, $(\mathbb{R}((t))^{LE}, \exp)$ and on $(\mathbb{C}, \exp)$.

7.1 Hypothesis $(I)_E$

Let $(K, R, E, \mathcal{V})$ be a topological partial E-field; $n, m \in \mathbb{N} \setminus \{0\}$.
Let $K((\bar{t})) := K((t_1)) \cdots ((t_n))$. Consider the structure $(K((\bar{t})), R[[\bar{t}]], \mathcal{W}_n)$, where $\mathcal{W}_n$ is as defined in Section 3.3.1. Because $\mathcal{V}$ is not discrete then $\mathcal{W}_n(K)$ satisfies $Comp(K)$, which induces an equivalence relation $\sim_{\mathcal{W}_n(K)}$ on $K((\bar{t}))$ and so on $K((\bar{t}))^m$ with in particular:

$$\bar{t} \sim_{\mathcal{W}_n(K)} \bar{0}$$

This endows $K((\bar{t}))$ with the structure of a topological field. By Remark 2.3.2, $R[[\bar{t}]]$ can be endowed with an exponential E making it an E-ring extending (R, E).

Then consider $(K((\bar{t})), |.|)$, where $|.|$ is the canonical ultrametric absolute value as set in Section 3.3.1 – for which the topology on K is trivial. It has valuation ring $K[[\bar{t}]]$. Recall that for $k, l \in K((\bar{t}))$, we have $k \sim_K l$ iff $|k-l| > 0$ that is $k - l \in \mathbf{m}(K[[\bar{t}]]) = \bar{t}.K[[\bar{t}]]$, so then in $K((\bar{t}))^m$, $\bar{t} \sim_K \bar{0}$.

Recall that by Remark 3.3.1, for an element $\bar{a} \in K((\bar{t}))^m$, $\bar{a} \sim_K \bar{0}$ iff $\bar{a} \sim_{\mathcal{W}_n(K)} \bar{0}$. This will allow us to work in $(K((\bar{t})), |.|)$ when we need a Hensel's lemma, and to be able to construct topological extensions of $(K, R, E, \mathcal{V})$ in which $\bar{t}$ is in all neighborhoods of 0:

Let $\bar{X} := X_1, \cdots, X_m$.

Definition 7.1.1 Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$.
An inductive class $\mathcal{C}$ of topological partial E-fields satisfies *Hypothesis* $(I)_E$ if for every element $(K, R, E, \mathcal{V})$ of $\mathcal{C}$, the following conditions are verified:

Considering $(K((\bar{t})), R[[\bar{t}]], E, \mathcal{W}_n)$ as a topological $\mathcal{L}$-extension of $(K, R, E, \mathcal{V})$ and given G a tuple of E-polynomials in $K[[\bar{t}]][\bar{X}] \otimes_{R[[\bar{t}]]} (R[[\bar{t}]])[\bar{X}]^E$;

if there is $\bar{a} \subseteq K[[\bar{t}]]$ such that $G(\bar{a}) \sim_K \bar{0}$ and $\det JG_{\bar{a}} \nsim_K 0$ in $(K[[\bar{t}]], |.|)$, then there is a topological $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K((\bar{t})), R[[\bar{t}]], E, \mathcal{W}_n)$ such that:

1. $(L, R', E, \mathcal{W})$ is a topological elementary $\mathcal{L}$-extension of $(K, R, E, \mathcal{V})$ (thus belongs to $\mathcal{C}$).

2. There is a subset $\mathcal{W}(K)$ of $\mathcal{W}$ which satisfies $Comp(K)$ and with $\bar{t} \sim_{\mathcal{W}(K)} \bar{0}$.

3. There is $\bar{b} \in L$ such that $G(\bar{b}) = \bar{0}$, $\det JG_{\bar{b}} \nsim_{\mathcal{W}(K)} 0$ and $\bar{a} \sim_{\mathcal{W}(K)} \bar{b}$.

Remark 7.1.2 *Note that if we want to encompass cases like*

$$(K, R, E, \mathcal{V}) = (\mathbb{C}_p, \mathcal{O}_p, E_p, |.|_p)$$

where $\mathcal{O}_p$ is both the domain of definition of E_p and the valuation ring of $\mathbb{C}_p$ for $|.|_p$, we need to consider $R \oplus \bar{t}.K[[\bar{t}]]$ instead of $R[[\bar{t}]]$. Note that E is well-defined on $R \oplus \bar{t}.K[[\bar{t}]]$ by Remark 2.3.2, and that $\mathcal{O}_p \oplus \bar{t}.\mathbb{C}_p[[\bar{t}]]$ is the valuation ring of $\mathbb{C}_p[[\bar{t}]]$ for $\mathcal{W}_n$ in that case. Furthermore the results using Hensel's Lemma 3.3.4 go through, as the proofs stays the same, replacing $R[[\bar{t}]]$ by $R \oplus \bar{t}.K[[\bar{t}]]$.

7.1.1 Example

The languages, theories and structures we refer to in this paragraph are detailed in Section 2.4.

Proposition 7.1.3 *The class of models of the $\mathcal{L}_{or,E}$-theory $T_{\mathbb{R},\exp}$ that can be endowed with a $\mathcal{L}_{an}$-structure satisfies $(I)_E$.*

Proof. Let $(K, E, <) \models T_{\mathbb{R},\exp}$, such that K can be endowed with a $\mathcal{L}_{an}$-structure, and let $\mathcal{V}$ be a base of neighborhoods of 0 in K for the order topology. Consider both structures $(K((\bar{t})), E, \mathcal{W}_n)$, and $(K((\bar{t})), E, |.|)$. By Lemma 3.3.2, $(K[[\bar{t}]], |.|)$ is complete; hence given G a regular system with coefficients in $K[[\bar{t}]]$ such that $G(\bar{a}) \sim_K \bar{0}$ and $\det JG_{\bar{a}} \nsim_K 0$ for an $\bar{a} \subseteq K[[\bar{t}]]$, we find $\bar{b} \in K[[\bar{t}]]$ such that $H(\bar{b}) = \bar{0}$, $\bar{b} \sim_K \bar{a}$, and $\det JH_{\bar{b}} \nsim_K 0$. Then $\bar{b} \sim_{\mathcal{W}(K)} \bar{a}$ and $(K((\bar{t})), E, \mathcal{W}_n) \subseteq (K((\bar{t}))^{LE}, E, \mathcal{T}_n)$, where $\mathcal{T}_n$ is as recalled in Subsection 2.3.3.

Then note that if $\phi(x, \bar{k})$ is a formula of $\mathcal{L}_{or,E}$ with parameters $\bar{k} \in K$– and which has a solution in $(K((\bar{t}))^{LE}, E, \mathcal{T}_n)$–, the formula ϕ can be expressed as a formula of $\mathcal{L}_{an,E}$ with parameters in K, hence by Fact 2.4.11,

$K \preccurlyeq K((\bar{t}))^{LE}$ as $\mathcal{L}_{an,E}$-structures and consequently $(K((\bar{t}))^{LE}, E, \mathcal{T}_n) \models T_{\mathbb{R},\exp}$. ∎

7.2 Generic points

Let $\mathcal{L} \supseteq \mathcal{L}_{rings} \cup \{E\}$.

Proposition 7.2.1 *Let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field, $m, n \in \mathbb{N} \setminus \{0\}$, and $Q = (Q_1, \cdots, Q_m) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, where $|\bar{X}| = n$. Suppose that there is $\bar{a} \in V^{reg}(Q) \cap K^n$ and that $Th(K)$ satisfies Hypotheses $(I)_E$ and $\mathcal{D}$.*

Then there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$, that contains $K((\bar{t})) := K((t_1)) \cdots ((t_{n-m}))$ and such that there is $\bar{g} \in V^{reg}(Q) \cap L^n$ such that $ecl^L - dim_K \bar{g} = n - m$ and $\bar{g} \sim_{\mathcal{W}(K)} \bar{a}$.

Furthermore if $V(Q) = V^{reg}(Q)$ then $\bar{g}$ is a generic point of $V(Q)$.

Proof. By Proposition 6.1.6, there is an elementary $\mathcal{L}$-extension $(K_1, R_1, E, \mathcal{V}_1)$ of $(K, R, E, \mathcal{V})$ that contains K and some elements $t_1, \cdots, t_{n-m}$ such that for $i = 1, \cdots, n-m$, $t_i \sim_K 0$ and there is $\bar{g} \in V^{reg}(Q) \subseteq K((t_1)) \cdots ((t_{n-m}))^n$ with $\bar{g} - \bar{a} \sim_K \bar{0}$, and $ecl^{K((t_1))\cdots((t_{n-m}))} - dim_K(\bar{g}) = n - m$. By Hypothesis $(I)_E$, there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$ such that $L \supseteq K((t_1)) \cdots ((t_{n-m}))$ and $\bar{g} \sim_{\mathcal{W}(K)} \bar{a}$. To see that $ecl^L - dim_K \bar{g} = ecl^{K((t_1))\cdots((t_{n-m}))} - dim_K \bar{g} = n - m$, consider the $n - m$ $K((t_1)) \cdots ((t_{n-m}))$-linearly independent E-derivations D_i constructed in the proof of Corollary 4.1.4. By Corollary 4.3.16, each D_i extends to an E-derivation on L, and by construction, the extensions of the D_i's are L-linearly independent, consequently by Fact 4.0.1, $ecl^L - dim_K \bar{g} = n - m$. ∎

Definition 7.2.2 Let $n, p \in \mathbb{N} \setminus \{0\}$, $n = |\bar{X}|$, and let (K, R, E) be a partial E field. Let $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, and let $\bar{a} \in V^{reg}(P) \cap K^n$.
Let $\{k_1, \cdots, k_p\} \subseteq \{1, \cdots, n\}$ be the set of the indexes of the columns involved in a non-zero subminor of $JP_{\bar{a}}$. We will denote by $Ind_{\bar{k}_p}(JP_{\bar{a}}) := \{1, \cdots, n\} \setminus \{k_1, \cdots, k_p\}$ the set of the indexes of the other columns.

Definition 7.2.3 Let $m, n, p \in \mathbb{N}$, $m, n \neq 0$, $n = |\bar{X}| = |\bar{Y}|$, and let (K, R, E) be a partial E-field, $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $A = V(Q)$, $B = V(P)$. Suppose there is a tuple $\bar{a}\bar{b} \in K^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$.
We will say that $\bar{a}\bar{b} \in A^{reg}$ *virtually projects generically on* B if we have the following condition on A, B:

- $n - p > 0$ and there is a non-zero subminor of $JQ_{\bar{a}\bar{b}}$ involving a set of columns indexed by $\{j_1, \cdots, j_m\} \subseteq \{n+1, \cdots, 2n\}$.

Example 7.2.4 *Let $H = (H_1, \cdots, H_n) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$ defining a Hovanskii system H_H.*
Let $P = (P_1, \cdots, P_{n-1})$, where for $i = 1, \cdots, n-1$, $P_i = H_i$, and let $Q = (Q_1, Q_2) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, where $Q_1(\bar{X}, \bar{Y}) = H_n(\bar{X}) + \sum_{i=1}^{n-1} E(Y_i) + c$, and $Q_2(\bar{X}, \bar{Y}) = H_n(\bar{X}) + E(Y_n) + d$, $c, d \in R$. Suppose $\bar{a}\bar{b} \in V(Q)$ and $\bar{a} \in V(P)$.
Then $\nabla P_i(\bar{a})$ are linearly independent, and the subminor of $JQ_{\bar{a}\bar{b}}$ indexed by the two last columns has determinant $E(b_{n-1})E(b_n) \neq 0$.

Lemma 7.2.5 *Let $m, n, p \in \mathbb{N}$, $m, n \neq 0$, $n = |\bar{X}| = |\bar{Y}|$, and let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field, $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$.*
Suppose $Th(K)$ satisfies Hypotheses $(I)_E$, and $\mathcal{D}$.
Let $\bar{a}\bar{b} \in V^{reg}(Q) \cap K^{2n}$, such that $\bar{a} \in V^{reg}(P)$.
Suppose that $\bar{a}\bar{b}$ virtually projects generically on B.

Then there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$ and $\bar{a}_1\bar{b}_1 \in V^{reg}(Q) \subseteq L^{2n}$ such that $\bar{a}_1\bar{b}_1 \sim_{\mathcal{W}(K)} \bar{a}\bar{b}$, and such that $\bar{a}_1 \in V^{reg}(P)$ has ecl^L-dimension $n - p$ over K.

Furthermore if $V^{reg}(P) = V(P)$, then $\bar{a}_1$ is a generic point of $V(P)$.

Proof. First, notice that there is $\bar{k}_p$, $Ind_{\bar{k}_p}(JP_{\bar{a}}) \subseteq Ind_{\bar{j}_m}(JQ_{\bar{a}\bar{b}})$. By Proposition 7.2.1 there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$, that contains $K((\bar{t})) := K((t_1))\cdots((t_{n-p}))$ and such that there is $\bar{a}_1 \in V^{reg}(P) \cap L^n$ such that $ecl^L - dim_K \bar{a}_1 = n - p$ and $\bar{a}_1 \sim_{\mathcal{W}(K)} \bar{a}$.

Then supppose without loss of generality that $\{j_1, \cdots, j_m\} = \{2n -$

$m+1, \cdots, 2n\}$, and let $\bar{b}_{n-m} := b_1, \cdots, b_{n-m}$ if $n-m \geq 1$, and let $\bar{Y}' := Y_{n-m+1}, \cdots, Y_n$, $\bar{b}' := b_{n-m+1}, \cdots, b_n$. Let

$$Q'(\bar{Y}') := Q(\bar{a}, \bar{b}_{n-m}, \bar{Y}')$$

$$Q'^t(\bar{Y}') := Q(\bar{a}_1, \bar{b}_{n-m}, \bar{Y}')$$

We have that $J_{(\bar{0},\bar{y}_m)}Q'_{\bar{b}'} = J_{(\bar{0},\bar{y}_m)}Q_{\bar{b}'}$, hence $\det J_{(\bar{0},\bar{y}_m)}Q'_{\bar{b}'} = \det J_{(\bar{0},\bar{y}_m)}Q_{\bar{b}'} \neq 0$. Note that $\bar{a}_1 \in K[[t_1]]\cdots[[t_{n-p}]]$ as seen in the proof of Proposition 6.1.6, hence Q'^t has coefficients in $K[[t_1]]\cdots[[t_{n-p}]]$. By continuity $\det J_{(\bar{0},\bar{y}_m)}Q'^t_{\bar{b}'} \neq 0$ and $Q'^t(\bar{b}') \sim_K \bar{0}$. Consequently by Hensel's Lemma Proposition 3.3.4, there is $\bar{c}' \in L^m$ such that $Q'^t(\bar{c}') = \bar{0}$. Letting $\bar{b}_1 := \bar{b}_{n-m}, \bar{c}'$, we have found $\bar{a}_1\bar{b}_1 \in V^{reg}(Q) \subseteq L^{2n}$ such that $\bar{a}_1\bar{b}_1 \sim_{\mathcal{W}(K)} \bar{a}\bar{b}$, and such that $\bar{a}_1 \in V^{reg}(P)$ has ecl^L-dimension $n-p$ over K. ■

Lemma 7.2.6 *Let $m, n, p \in \mathbb{N}$, $m, n \neq 0$, $n = |\bar{X}| = |\bar{Y}|$, and let $(K, R, E, \mathcal{V})$ be a topological $\mathcal{L}$-partial-E-field, $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$.*
Suppose $Th(K)$ satisfies Hypotheses $\mathcal{I}m$, and $\mathcal{D}$.
Let $\bar{a}\bar{b} \in V^{reg}(Q) \cap K^{2n}$, such that $\bar{a} \in V^{reg}(P)$.
Suppose that $\bar{a}\bar{b}$ virtually projects generically on B.

Then there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$ and $\bar{a}_1\bar{b}_1 \in V^{reg}(Q) \subseteq L^{2n}$ such that $\bar{a}_1\bar{b}_1 \sim_{\mathcal{W}(K)} \bar{a}\bar{b}$, and such that $\bar{a}_1 \in V^{reg}(P)$ has ecl^L-dimension $n-p$ over K.

Furthermore if $V^{reg}(P) = V(P)$, then $\bar{a}_1$ is a generic point of $V(P)$.

Proof. The proof follows the same lines as the proof of Lemma 7.2.5 above, but we use Proposition 6.3.4 instead of Proposition 6.1.6, and then instead of using Hensel's Lemma we use Hypothesis $\mathcal{I}m$.

First, notice that there is $\bar{k}_p$, $Ind_{\bar{k}_p}(JP_{\bar{a}}) \subseteq Ind_{\bar{j}_m}(JQ_{\bar{a}\bar{b}})$ and supppose without loss of generality that $\{j_1, \cdots, j_m\} = \{2n-m+1, \cdots, 2n\}$.
Let $\bar{b}_{n-m} := b_1, \cdots, b_{n-m}$ if $n-m \geq 1$, and let $\bar{Y}_{n-m} := Y_1, \cdots, Y_{n-m}$, $\bar{Y}' := Y_{n-m+1}, \cdots, Y_n$, $\bar{b}' := b_{n-m+1}, \cdots, b_n$.
By Hypothesis $\mathcal{I}m$, there are $O \subseteq K^{2n-m}$, $O' \subseteq K^m$ open sets with $\bar{a}\bar{b} \in O \times O'$, and a continuous function $g : O \to O'$ such that $\bar{b}' = g(\bar{a}\bar{b}_{n-m})$ and such that for all $\bar{X}\bar{Y}_{n-m} \subseteq O$ and $\bar{Y}' \subseteq O'$, $\bar{Y}' = g(\bar{X}\bar{Y}_{n-m})$ iff

$Q(\bar{X}\bar{Y}_{n-m}\bar{Y}') = \bar{0}$.
By Proposition 6.3.4, there is a topological elementary $\mathcal{L}$-extension $(L, R', E, \mathcal{W})$ of $(K, R, E, \mathcal{V})$ such that there is $\bar{a}_1 \in V^{reg}(P) \subseteq L^n$ with $\bar{a}_1 - \bar{a} \sim_{\mathcal{W}(K)} \bar{0}$, for $\mathcal{W}(K) \models Comp\, K$, and $ecl^L - dim_K(\bar{a}_1) = n - p$.
Let $\bar{b}_1 := \bar{b}_{n-m} g(\bar{a}_1 \bar{b}_{n-m})$.
Then $\bar{a}_1 \bar{b}_1 \in V^{reg}(Q) \subseteq L^{2n}$ is such that $\bar{a}_1 \bar{b}_1 \sim_{\mathcal{W}(K)} \bar{a}\bar{b}$, and such that $\bar{a}_1 \in V^{reg}(P)$ has ecl^L-dimension $n - p$ over K. ∎

7.3 Scheme $(DL)_E$

Definition 7.3.1 Let $(K, R, E, D, \mathcal{V})$ be a differential topological partial E-field. We say that it satisfies the scheme $(DL)_E$ if:

for any $U \in \mathcal{V}$,
for any finitely generated E-varieties $A \subseteq K^{2n}$, $B \subseteq K^n$ defined over K such that $A^{reg} \subseteq \tau(B)$, if there is a tuple $(\bar{a}, \bar{c}) \in K^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$, that virtually projects generically on B, then

there is $\bar{b} \in K^n$ such that $(\bar{b}, D\bar{b}) \in A^{reg}$ and

$$(\bar{a}, \bar{c}) - (\bar{b}, D\bar{b}) \in U^{2n}$$

Remark 7.3.2 *If $(K, R, E, D, \mathcal{V})$ satisfies $(DL)_E$, then for any $U \in \mathcal{V}$ and $a \in K$, there is $b \in K$, $Db \in a + U$.*

Theorem 7.3.3 *Let $(K, R, E, D, \mathcal{V})$ be a topological differential partial E-field. Suppose that the theory $Th(K)$ of the $\mathcal{L}$-reduct $(K, R, E, \mathcal{V})$ satisfies Hypotheses $\mathcal{D}$ and either $(I)_E$ or $\mathcal{I}m$.*
Let $m, n, p \in \mathbb{N}$, $m, n \neq 0$, $n = |\bar{X}| = |\bar{Y}|$, and let $Q := (Q_1, \cdots, Q_m) \subseteq K[\bar{X}\bar{Y}] \otimes_{R[\bar{X}\bar{Y}]} R[\bar{X}\bar{Y}]^E$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}] \otimes_{R[\bar{X}]} R[\bar{X}]^E$, $A = V(Q)$, $B = V(P)$.

Suppose that $A^{reg} \subseteq \tau(B)$, and that there is $(\bar{a}, \bar{c}) \in K^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$, that virtually projects generically on B.
Let $U \in \mathcal{V}$.

Then there is a $\mathcal{L} \cup \{D\}$-extension $(L, R', E, D, \mathcal{W}) \supseteq (K, R, E, D, \mathcal{V})$ such that $(L, R', E, \mathcal{W})$ is a topological elementary $\mathcal{L}$-extension of

$(K,R,E,\mathcal{V})$, and there are $\bar{b} \in L^n$, and $W \in \mathcal{W}(K)$ with $W \cap K = U$, $(\bar{b}, D(\bar{b})) \in A^{reg}$ and

$$(\bar{a},\bar{c}) - (\bar{b}, D(\bar{b})) \in W^{2n}$$

Proof. By Lemma 7.2.5 or 7.2.6 there is a topological elementary $\mathcal{L}$-extension $(L,R',E,\mathcal{W})$ of $(K,R,E,\mathcal{V})$ such that there is $\bar{a}'\bar{c}' \in A^{reg} \subseteq L^{2n}$ such that $\bar{a}' \in B^{reg}$ has ecl^L-dimension $n-p$ over K, and $\bar{a}'\bar{c}' \sim_{\mathcal{W}(K)} \bar{a}\bar{c}$, for $\mathcal{W}(K) \models Comp(K)$ hence $\bar{a}'\bar{c}' \in \bar{a}\bar{c} + W^{2n}$, for W such that $W \cap K = U$.

By Lemma 6.2.4 applied to $\bar{a}'\bar{c}'$, D extends uniquely to an E-derivation on $ecl^L(K(\bar{a}'\bar{c}'))$ in such a way that $\bar{c}' := D\bar{a}'$. Therefore we let $(\bar{b}, D(\bar{b})) := (\bar{a}', \bar{c}')$. Then, D extends to L by Corollary 4.3.16. ■

Remark 7.3.4 *Note that if we were able to construct $\bar{a}_1\bar{b}_1$ to be of maximal ecl^L-dimension $2n-m$ over K in Lemmas 7.2.5 and 7.2.6, then in the proof above of Theorem 7.3.3 we could also have used Corollary 6.2.7 instead of Lemma 6.2.4.*

7.3.1 Existentially closed differential expansions

Let $\mathcal{L} = \mathcal{L}_{rings} \cup \{E\} \cup \{R_i, i \in I\}$, where R_i, $i \in I$ are relations symbols, a first-order language, let T be a $\mathcal{L}$-theory of topological partial E-fields satisfying Hypotheses $\mathcal{D}$.
Suppose that either models of T satisfy $\mathcal{I}m$, or that κ-saturated models of T satisfy $(I)_E$ for some κ, and let $(K,R,E,\mathcal{V}) \models T$.

Then set $\mathcal{L}_D := \mathcal{L} \cup \{D\}$, and let T_D be the $\mathcal{L}_D$-theory

$$T \cup \{D \text{ is an } E\text{-derivation}\}$$

Theorem 7.3.5 *Under the above setting and hypotheses, the models of the $\mathcal{L}_D$-theory T_D that are existentially closed in T_D satisfy $(DL)_E$ and have a $\mathcal{L}$-reduct that is existentially closed in the class of models of T.*

Proof. Let $(L_0, R_0, E, D, \mathcal{V}_0) \models T_D$ be existentially closed in T_D.
Let $(L_1, R_1, E, \mathcal{V}_1) \models T$ be a $\mathcal{L}$-extension of $(L_0, R_0, E, \mathcal{V}_0)$ (by Hypothesis $\mathcal{D}$ and Fact 2.2.6, it is a topological $\mathcal{L}$-extension). By Corollary 4.3.16, D extends to an E-derivation on L_1. By construction,

$$(L_1, R_1, E, D, \mathcal{V}_1) \models T_D$$

hence we get that $(L_0, R_0, E, \mathcal{V}_0) \preccurlyeq (L_1, R_1, E, \mathcal{V}_1)$ as $\mathcal{L}$-structures because $(L_0, R_0, E, D, \mathcal{V}_0) \preccurlyeq (L_1, R_1, E, D, \mathcal{V}_1)$ as $\mathcal{L}_D$-structures and $\mathcal{L} \subseteq \mathcal{L}_D$.

Now let $U \in \mathcal{V}_0$, and A, B be finitely generated E-varieties defined over L_0 such that $A^{reg} \subseteq \tau(B)$, and there is $(\bar{a}, \bar{c}) \in L_0^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$, that virtually projects generically on B. By Fact A.0.8, $(L_0, R_0, E, D, \mathcal{V}_0)$ can be embedded in a κ-saturated elementary extension $(L_0', R_0', E', D', \mathcal{V}_0')$. Let $U' \in \mathcal{V}_0'$ such that $U' \cap L_0 = U$. By Theorem 7.3.3, there is $(L_2, R_2, E, D, \mathcal{V}_2) \models T_D$ a $\mathcal{L}_D$-extension of $(L_0', R_0', E', D', \mathcal{V}_0')$, and thus of $(L_0, R_0, E, D, \mathcal{V}_0)$ such that there is $(\bar{a}'', \bar{c}'') \in A \cap L_2^{2n}$ and $W \in \mathcal{V}_2(L_0')$ with $W \cap L_0 = U$ and $(\bar{a}'', \bar{c}'') - (\bar{a}, \bar{c}) \in W^{2n}$ and $\bar{c}'' = D\bar{a}''$. By existential closedness of $(L_0, R_0, E, D, \mathcal{V}_0)$ in $(L_2, R_2, E, D, \mathcal{V}_2)$, there is $(\bar{a}', \bar{c}') \in L_0^{2n} \cap A$ such that $(\bar{a}', \bar{c}') - (\bar{a}, \bar{c}) \in U^{2n}$, and $\bar{c}' = D\bar{a}'$ which shows $(L_0, R_0, E, D, \mathcal{V}_0) \models (DL)_E$. ■

The following $\mathcal{L}$-theories T are model-complete by Facts 2.4.8 and 2.4.4, and Hypotheses $\mathcal{D}$, $\mathcal{I}m$ are satisfied by models of T. Furthermore, the $\aleph_1$-saturated models of $T_{\mathbb{R},\exp}$ can be endowed with a $\mathcal{L}_{an}$-structure, hence satisfy $(I)_E$.

Corollary 7.3.6 *Let $\mathcal{L} := \mathcal{L}_{or,E}$, and $T := T_{\mathbb{R},\exp}$. Then the models of the $\mathcal{L}_D$-theory T_D that are existentially closed in T_D satisfy $(DL)_E$.*

Corollary 7.3.7 *Let $\mathcal{L} := \mathcal{L}_{|,E}$, and T the $\mathcal{L}$-theory $Th(\mathbb{C}_p, \mathcal{O}_p, E_p)$. Then the models of the $\mathcal{L}_D$-theory T_D that are existentially closed in T_D satisfy $(DL)_E$.*

7.3.2 Examples of structures satisfying $(DL)_E$

In his PhD thesis [10] (2.5.2 p.32), Q.Brouette endows $\mathbb{R}$ with a derivation making it a model of $CODF$. He constructs by induction on $\alpha \in 2^{\aleph_0}$

a chain of subfields K_α of $\mathbb{R}$ that contains $\mathbb{Q}$, and such that the transcendence degree of $\mathbb{R}$ over K_α is $2^{\aleph_0}$. The iteration step is an adaptation of a lemma of C.Michaux's PhD thesis ([47], Chapter 2, Lemma 2.3.4) used to show that the theory $CODF$ has countable archimedean models, and it is itself an adaptation of a lemma of M.Singer ([62], p.85) used to show the existence of a model of $CODF$ containing a given model of ODF.

In this section we extend Q. Brouette's proof to show that $\mathbb{R}$ and $\mathbb{R}((t))^{LE}$, and $\mathbb{C}$ can be endowed with an E-derivation making them models of scheme $(DL)_E$. We replace the iteration step by Theorem 7.3.9.

First note that $ecl^{\mathbb{R}}(\mathbb{Q})$ and $ecl^{\mathbb{C}}(\mathbb{Q})$ both have cardinality $\aleph_0$, as the Hovanskii systems with coefficients in $\mathbb{Q}$ can be enumerated and because their solutions are isolated zeros. Moreover, in $\mathbb{R}$, a given Hovanskii system can only have a finite number of solutions because $\mathbb{R}$ is o-minimal hence a definable set is a finite union of intervals and points so a countable definable set is finite.
Then notice that a Hovanskii system in $\mathbb{R}((t))^{LE}$ has also countably many solutions:
indeed let $H_H(\bar{X}, \bar{a})$ corresponds to equations $h_1(\bar{X}, \bar{a}) = 0, \cdots, h_n(\bar{X}, \bar{a}) = 0$, together with inequation $\det J_{(\bar{x},\bar{0})} H_{\bar{X},\bar{a}} \neq 0$, where $H = (h_1, \cdots, h_n) \subseteq \mathbb{Z}[\bar{X}\bar{Y}]^E$, $n = |\bar{X}|$, $p = |\bar{a}|$, $\bar{a} \subseteq \mathbb{R}((t))^{LE}$. Let

$$\theta(\bar{x}\bar{y}) \equiv H(\bar{x}\bar{y}) = \bar{0} \wedge \det J_{(\bar{x},\bar{0})} H_{\bar{x},\bar{y}} \neq 0$$

By Fact 2.4.9, there is N such that

$$\exists^{\leq N} \bar{x} \forall \bar{a}\ \theta(\bar{x}, \bar{a})$$

is true in $\mathbb{R}$. By Fact 2.4.11, this is also true in $\mathbb{R}((t))^{LE}$. Consequently $ecl^{\mathbb{R}((t))^{LE}}(\mathbb{Q})$ also has cardinality $\aleph_0$. Notice that $\mathbb{R}((t))^{LE}$ has cardinality $2^{\aleph_0}$ [18, Corollary p.13]

Now let $\mathbb{L} := \mathbb{R}$ or $\mathbb{C}$ (resp. $\mathbb{R}((t))^{LE}$), and $\mathcal{W}$ be a base of neighborhoods of 0 in $\mathbb{L}$ for the topology of the absolute value $|.|$. Let $K \subseteq \mathbb{L}$ be such that $\aleph_0 = |K| < |\mathbb{L}|$.

Remark 7.3.8 $|K| = |ecl^{\mathbb{L}}(K)| < |\mathbb{L}|$

Indeed, as shown above for $\mathbb{Q} \subseteq \mathbb{L}$, the number of Hovanskii systems with parameters in K has also cardinality $\aleph_0$, consequently $|ecl^{\mathbb{L}}(K)| = \aleph_0$.

Like in Quentin Brouette's proof we consider for all $k \in \mathbb{N}$ the set

$$T_k := \{r \in \mathbb{L} : \frac{1}{k+2} < |r| < \frac{1}{k+1}\}$$

We let

$$B := \{b_\lambda : \lambda \in 2^{\aleph_0}\}$$

be an ecl-transcendence basis of $\mathbb{L}$ over $ecl^{\mathbb{L}}(K)$ such that for all $k \in \mathbb{N}$, $|B \cap T_k| = 2^{\aleph_0}$.

Suppose $(K, \exp, \mathcal{V})$ is a topological $\mathcal{L}$-subfield of $(\mathbb{L}, \exp, \mathcal{W})$, for $\mathcal{L} = \mathcal{L}_{rings} \cup \{E\}$ if $\mathbb{L} = \mathbb{C}$, and $\mathcal{L} = \mathcal{L}_{or,E}$ otherwise. Let D be an E-derivation on K. Assume that the E-algebraic transcendence degree of $\mathbb{L}$ over K is strictly greater than the cardinality of K. We want to construct a topological $\mathcal{L}$-extension of $(K, \exp, \mathcal{V})$ that is a topological $\mathcal{L}$-subfield of $(\mathbb{L}, \exp, \mathcal{W})$, equiped with an E-derivation extending D and satisfying a kind of 'pre-scheme' $(DL)_E$. We will then iterate this operation, thanks to the fact that $|ecl^{\mathbb{L}}(K)| = |K| < |\mathbb{L}|$, and take the transfinite union of all these differential topological E-fields. We begin by modifying slightly Theorem 7.3.3:

Theorem 7.3.9 *Given $n, m, p \in \mathbb{N}, n, m \neq 0$, $|\bar{X}| = |\bar{Y}| = n$, $Q = (Q_1, \cdots, Q_m) \subseteq K[\bar{X}\bar{Y}]^E$, $P = (P_1, \cdots, P_p) \subseteq K[\bar{X}]^E$ such that $A = V(Q) \subseteq \mathbb{L}^{2n}$, $B = V(P) \subseteq \mathbb{L}^n$, and $A^{reg} \subseteq \tau(B)$;*

let $U \in \mathcal{W}$ and suppose that there is $(\bar{a}, \bar{b}) \in K^{2n} \cap A^{reg}$, with $\bar{a} \in B^{reg}$, that virtually projects generically on B,

Then there is a neighborhood $W \in \mathcal{W}$, such that for any elements $t_1, \cdots, t_i$, $s_1, \cdots, s_j$ in W that are E-algebraic independent over K, where $i = n-p$, $j = n-m$, there is $(\bar{c}, \bar{d}) \in A$ such that $ecl^{\mathbb{L}}(K(\bar{c}, \bar{d})) = ecl^{\mathbb{L}}(K(\bar{t}_i, \bar{s}_j))$ and

$$(\bar{a}, \bar{b}) - (\bar{c}, \bar{d}) \in U^{2n}$$

and such that it is possible to extend uniquely D to $ecl^{\mathbb{L}}(K(\bar{c}, \bar{d}))$ by letting $D\bar{c} := \bar{d}$.

Proof. There is $\bar{k}_p$, $Ind_{\bar{k}_p}(JP_{\bar{a}}) \subseteq Ind_{\bar{j}_m}(JQ_{\bar{a}\bar{b}})$.

By Lemma 3.2.2, there is t^* and a neighborhood $O \in \mathcal{W}$, such that for any elements $t_1, \cdots, t_i$, in O that are E-algebraic independent, letting

$$\bar{a}_0 := (a_1 + t_1, \cdots, a_d + t_d, a_{d+1}, \cdots, a_n)$$

then there is a regular zero $\bar{a}^*$ of P in $ecl^{\mathbb{L}}(K(t_1, \cdots, t_d)) \cap B(\bar{a}_0, t^*) \subseteq \bar{a}_0 + (U/2)^n$.

Then we reapply Lemma 3.2.2, to find t'^* and a neighborhood $O' \in \mathcal{W}$, such that for any elements $s_1, \cdots, s_j$ in O' that are E-algebraic independent, letting

$$(\bar{a}^*, \bar{b}_0) := (a_1^*, \cdots, a_n^*, b_1 + s_1, \cdots, b_j + s_j, b_{j+1}, \cdots, b_n)$$

then there is a regular zero $\bar{a}^*\bar{b}^*$ of Q in

$$ecl^{\mathbb{L}}(K(t_1, \cdots, t_d, s_1, \cdots, s_j)) \cap B(\bar{a}^*, \bar{b}_0, t'^*) \subseteq \bar{a}_0\bar{b}_0 + U^{2n}$$

By Lemma 6.2.4 applied to $\bar{a}^*\bar{b}^*$, D extends uniquely to an E-derivation on $ecl^L(K(\bar{a}^*\bar{b}^*))$ in such a way that $\bar{b}^* := D\bar{a}^*$. Therefore we let $(\bar{c}, d) := (\bar{a}^*\bar{b}^*)$. Then, D extends to L by Corollary 4.3.16. ■

Theorem 7.3.10 *Let* $(\mathbb{R}, \exp)$ *(resp.* $(\mathbb{R}((t))^{LE}, \exp)$*,* $(\mathbb{C}, \exp)$*) be the field of real numbers (resp. of logarithmic-exponential series, resp. of complex numbers) equipped with exponentiation. There exists a non-trivial E-derivation D on* $\mathbb{R}$ *(resp.* $\mathbb{R}((t))^{LE}$*,* $\mathbb{C}$*) such that*

$$(\mathbb{R}, +, ., -, ^{-1}, \exp, 0, 1, <, D) \quad resp. \quad (\mathbb{R}((t))^{LE}, +, ., -, ^{-1}, \exp, 0, 1, <, D)$$

is an ordered differential E-field satisfying scheme $(DL)_E$*. (resp.*

$$(\mathbb{C}, +, ., -, ^{-1}, \exp, 0, 1, |.|, D)$$

where $|.|$ *is the absolute value or module, a topological differential E-field satisfying scheme* $(DL)_E$*.*

Proof. Notice that $ecl^{\mathbb{R}}(\mathbb{Q})$, $ecl^{\mathbb{R}((t))^{LE}}(\mathbb{Q})$ are real closed while $ecl^{\mathbb{C}}(\mathbb{Q})$ is algebraically closed. Let $\mathbb{L} := \mathbb{R}$, $\mathbb{C}$ or $\mathbb{R}((t))^{LE}$. Let us begin with

$ecl^{\mathbb{L}}(\mathbb{Q})$, equipped with a trivial derivation D. Recall that we consider we consider for all $k \in \mathbb{N}$ the set

$$T_k := \{r \in \mathbb{L} : \frac{1}{k+2} < |r| < \frac{1}{k+1}\}$$

Let

$$B := \{b_\lambda : \lambda \in 2^{\aleph_0}\}$$

be an ecl-transcendence basis of $\mathbb{L}$ over $ecl^{\mathbb{L}}(\mathbb{Q})$ such that for all $k \in \mathbb{N}$, $|B \cap T_k| = 2^{\aleph_0}$.

Let us now construct a chain of subfields F_α of $\mathbb{L}$ by induction on $\alpha \in 2^{\aleph_0}$.

- Let $\alpha \in 2^{\aleph_0}$. If $b_\alpha \in F_\alpha$, let $F_{\alpha+1} := F_\alpha$. Otherwise let

  $$F_{\alpha,0} := ecl^{\mathbb{L}}(F_\alpha(b_\alpha))$$

  and $Db_\alpha = 0$. Notice that it would be possible to let $Db_\alpha = 1$.) Recall that D extends to $ecl^{\mathbb{L}}(F_\alpha(b_\alpha))$ by Proposition 4.2.1. Then we enumerate all systems generating E-varieties A_δ, B_δ, $\delta \in |F_\alpha|$, defined on $F_{\alpha,0}$ such that $A_\delta^{reg} \subseteq \tau(B_\delta)$ and such that:
  there is a point in A_δ^{reg} the coordinates of which stand in $F_{\alpha,0}$, which projects on a point in B_δ^{reg}, and which virtually projects generically on B_δ.

  - Let $\delta \in |F_\alpha|$. Pick $\bar{t}_d, \bar{s}_r \in (B \cap T_k) \setminus F_\alpha$ for k big enough. Elements $\bar{t}_d, \bar{s}_r$ can be chosen ecl-independent because $|(B \cap T_k) \setminus F_\alpha| = 2^{\aleph_0}$. By Theorem 7.3.9, we get a point $(\bar{b}_\delta, D\bar{b}_\delta) \subseteq ecl^{\mathbb{L}}(F_{\alpha,\delta}(\bar{t}_d, \bar{s}_r))$, close enough to our starting point. Let

    $$F_{\alpha,\delta+1} := ecl^{\mathbb{L}}(F_{\alpha,\delta}(\bar{b}_\delta, D\bar{b}_\delta))$$

    Then let

    $$F_{\alpha+1} := ecl^{\mathbb{L}}(\bigcup_{\delta \in |F_\alpha|} F_{\alpha,\delta})$$

  - If $\delta \in |F_\alpha|$ is a limit ordinal, let $F_{\alpha,\delta} := \bigcup_{\gamma \in \delta} F_{\alpha,\gamma}$

- If $\alpha \in 2^{\aleph_0}$ is a limit ordinal, let $F_\alpha := \bigcup_{\beta \in \alpha} F_\beta$. By construction, F_α is real closed (resp. algebraically closed).

At the end, let $F := \bigcup_{\alpha \in 2^{\aleph_0}} F_\alpha$. If $\mathbb{L} = \mathbb{R}$ or $\mathbb{R}((t))^{LE}$, F is real closed and as $B \subseteq F \subseteq \mathbb{L}$, $F = \mathbb{L}$. By construction F is a model of scheme $(DL)_E$.
(resp. if $\mathbb{L} = \mathbb{C}$, F is algebraically closed and as $B \subseteq F \subseteq \mathbb{C}$, $F = \mathbb{C}$. By construction F is a model of scheme $(DL)_E$.)

■

Appendix A

Model theory

In this appendix we recall a few model-theoretic notions, namely elementary extensions, saturated models, existentially closed models, model-complete theories, quantifier elimination.

Let $\mathcal{L}$ be a language. We use a distinct notation for a $\mathcal{L}$-structure and its underlying domain in this appendix but not elsewhere.

Definition A.0.1 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called an *elementary embedding* if, whenever $a_1, \ldots, a_n \in M$, and $\phi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, then

$$\mathcal{M} \vDash \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \vDash \phi(j(a_1), \ldots, j(a_n))$$

If $\mathcal{M}$ is a substructure of $\mathcal{N}$ we then say that it is an *elementary substructure*, or that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$, and write $\mathcal{M} \preccurlyeq \mathcal{N}$.

Definition A.0.2 A $\mathcal{L}$-substructure $\mathcal{M}$ of a $\mathcal{L}$-structure $\mathcal{N}$ is said to be *existentially closed* in $\mathcal{N}$ if for every quantifier free $\mathcal{L}$-formula $\phi(x_1, \cdots, x_m, y_1, \cdots, y_n)$ and all elements $b_1, \cdots, b_n$ of $\mathcal{M}$ such that

$$\mathcal{N} \models \phi(x_1, \cdots, x_m, b_1, \cdots, b_n)$$

$$\text{then} \quad \mathcal{M} \models \phi(x_1, \cdots, x_m, b_1, \cdots, b_n)$$

A model $\mathcal{M}$ of a given $\mathcal{L}$-theory T is called existentially closed in T if it is existentially closed in every $\mathcal{L}$-extension $\mathcal{N}$ that is itself a model of T.

Definition A.0.3 A $\mathcal{L}$-theory T is said to be *model-complete* if it satisfies one of the following equivalent conditions:

- every embedding of models is elementary
- all models of T are existentially closed in T
- every (first-order) formula is equivalent to a universal formula
- every (first-order) formula is equivalent to an existential formula

Definition A.0.4 A $\mathcal{L}$-theory T is said to admit *quantifier elimination* if for every $\mathcal{L}$-formula ϕ there is a quantifier free $\mathcal{L}$-formula ψ such that

$$T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$$

A $\mathcal{L}$-theory T that admit quantifier elimination is model-complete.

Definition A.0.5 A class $\mathcal{C}$ of models of a $\mathcal{L}$-theory T is said to be *inductive* if it is closed by union of chains.

Fact A.0.6 [11, 4.3.13 p.208]**(Frayne's Theorem)** *Let $\mathcal{A}$, $\mathcal{B}$ be two $\mathcal{L}$-structures such that $\mathcal{A}$ is elementarily equivalent to $\mathcal{B}$. Then there is a set I and an ultrafilter $\mathcal{U}$ on I such that $\mathcal{B}$ embeds elementarily in the ultrapower $\Pi_{\mathcal{U}}\mathcal{A}$.*

We now recall the model-theoretic notion of saturation. Let T be a satisfiable $\mathcal{L}$-theory, and let $\mathcal{L}_{\overline{x}} := \mathcal{L} \cup \{x_1, \ldots, x_n\}$; $x_1, \ldots, x_n$ being new constant symbols, denoted like variables by commodity. Let $S_n(T)$ be the set of complete $\mathcal{L}_{\overline{x}}$-theories containing T. An element of $S_n(T)$ is called a *complete n-type*. A set of $\mathcal{L}$-formulas with n free variables is a *n-type* if it can be completed in a complete n-type.

Let $\mathcal{M} \vDash T$ and $A \subseteq M$. Given $\mathcal{L}_A$ obtained from $\mathcal{L}$ by adding one new constant symbol for each element of A, we let $Th_A(\mathcal{M})$ denote the theory of $\mathcal{M}$ in the language $\mathcal{L}_A$; and $S_n^{\mathcal{M}}(A) := S_n(Th_A(\mathcal{M}))$.

Let $\overline{c} \in M^n$. The set of $\mathcal{L}_A$-formulas $\phi(v_1, \ldots, v_n)$ for which $\mathcal{M} \vDash \phi(\overline{c})$ is denoted $tp_A^{\mathcal{M}}(\overline{c})$ or $p_A^{\mathcal{M}}(\overline{c})$. For every type $p(\overline{x}) \in S_n(T)$, there exists $\mathcal{M} \vDash T$ and $\overline{a} \in M^n$ such that $tp^{\mathcal{M}}(\overline{a})$ is equivalent to $p(\overline{x})$.

Definition A.0.7 Let κ be an infinite cardinal, $\mathcal{L}$ a language such that $|\mathcal{L}| < \kappa$, T a satisfiable $\mathcal{L}$-theory, and $\mathcal{M} \vDash T$.
We say that $\mathcal{M}$ is *κ-saturated* if, whenever $A \subseteq M$, $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.
We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Fact A.0.8 [46, Theorem 4.3.12] *Let $\kappa > \aleph_0$ be a cardinal and $\mathcal{M}$ be a $\mathcal{L}$-structure. Then $\mathcal{M}$ admits a κ^+-saturated elementary extension.*

Appendix B

$\aleph_1$-saturation of $\mathbb{R}((G_n))$ and $\mathbb{R}((G_{m,n}))$

By ordered set, we mean a totally ordered set.
If $(S, <)$ is an ordered set and $A, B \subseteq S$, let $A < B$ if for all $a \in A$ and for all $b \in B$, $a < b$.

Definition B.0.1 Let $\alpha \in O_n$. The ordered set $(S, <)$ is called an η_α-*set*, if whenever $A, B \subseteq S$ with $A < B$ and $|A \cup B| < \aleph_\alpha$, then there is $s \in S$ such that $A < \{s\} < B$.

Remark B.0.2 *Let $(S_1, <_1)$ and $(S_2, <_2)$ be η_α-sets. Then $S := S_1 \overleftarrow{\times} S_2$ equipped with the (anti)lexicographic order $<$, is an η_α-set.*

Recall that an abelian group G is *divisible* if for all n in $\mathbb{N} \setminus \{0\}$, $\forall g \in G$, $\exists x \in G, nx = g$.

We now want to introduce the characterization of $\aleph_\alpha$-saturated divisible ordered abelian groups using η_α-sets from [32]. For this, we briefly recall a classical way to endow such a group with a valuation (we follow here [13, paragraph 2.2, p.2]):

Definition B.0.3 [1, p.62] Let G be an abelian (additively written) group. A *valuation* on G is a function $v : G \to S \cup \{\infty\}$ where S is an ordered set, such that for all $x, y \in G$ the following conditions are satisfied:

- $v(x) = \infty$ iff $x = 0$

- $v(-x) = v(x)$
- $v(x + y) > \min\{v(x), v(y)\}$

(G, S, v), where S is an ordered set, and $v : G \to S \cup \{\infty\}$ is a surjective valuation, is called a *valued group*. The ordered set S is called *the value set* of (G, S, v).

The open balls $B_a(s) := \{x \in G : v(x - a) > s\}$ form a basis for a topology that endow G with the structure of a Hausdorff topological group.

Definition B.0.4 [1, p.64] Let (G, S, v) be a valued abelian group. Let $\lambda \in On$ a limit ordinal and $(a_\rho)_{\rho<\lambda}$ be a sequence in G, and $a \in G$. Then $(a_\rho)_{\rho<\lambda}$ is said to *pseudoconverge* to a, if $v(a - a_\rho)$ is eventually strictly increasing, that is, for some index ρ_0 we have

$$v(a - a_\rho) < v(a - a_\sigma) \text{ whenever } \sigma > \rho > \rho_0$$

We also say in that case that a is a *pseudolimit* of $(a_\rho)_{\rho<\lambda}$.
A *pseudo-Cauchy sequence* in (G, S, v), is a sequence $(a_\rho)_{\rho<\lambda}$ in G such that for some index ρ_0 we have

$$\tau > \sigma > \rho > \rho_0 \to v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$$

Let $(G, +, 0, <)$ be a divisible ordered abelian group. For $x \in G$, set $|x| := \max\{x, -x\}$. For non-zero $x, y \in G$, let $x \approx y$ if there exists $n \in \mathbb{N}$, $n|x| \geq |y|$ and $n|y| \geq |x|$. This is an equivalence relation; let $\tilde{x}$ denote the equivalence class of x, and let

$$\Gamma := \{\tilde{x} : x \in G \setminus \{0\}\}$$

the set of equivalence classes of non-zero elements of G. It can be ordered by:

$$x <_\Gamma y \text{ iff for all } n \in \mathbb{N}, n|x| < |y|$$

Then one can define a valuation $v : G \to \Gamma \cup \{\infty\}$, $0 \mapsto \infty$ and $0 \neq x \mapsto \tilde{x}$. For $\tilde{x} \in \Gamma$, the *archimedean component* of $\tilde{x}$ is the maximal archimedean subgroup $A_{\tilde{x}}$ of G (that is $A_{\tilde{x}}$ is an ordered subgroup of G such that for every $a_1, a_2 \in A_{\tilde{x}} \setminus \{0\}$, there is $n \in \mathbb{N}$ with $a_1 \leq na_2$) containing $\tilde{x}$. For each $\tilde{x}$, $A_{\tilde{x}}$ is isomorphic to an ordered subgroup of $(\mathbb{R}, +, 0, <)$.

Fact B.0.5 [32, Th.C, p.10] *Let $\alpha \in On$ and let $G \neq 0$ be a divisible ordered abelian group. Then G is $\aleph_\alpha$-saturated (in the language of ordered groups) iff its value set is an η_α-set, and all its archimedean components are isomorphic to $\mathbb{R}$, and every pseudo-Cauchy sequence indexed by $\lambda < \aleph_\alpha$ has a pseudolimit in G.*

Let G be an ordered abelian group. Recall that the field of Hahn series $\mathbb{R}((G))$ can be ordered by, for $s = \sum c_g g$:

$$s > 0 \text{ iff } s \neq 0 \text{ and } Lc(s) > 0$$

Fact B.0.6 [34, Consequence of Theorem 6.2] *Let $\alpha \in On$, and let G be an $\aleph_\alpha$-saturated divisible ordered abelian group. Then the field of Hahn series $\mathbb{R}((G))$ is an $\aleph_\alpha$-saturated real closed field.*

Let $n, m \in \mathbb{N}$, and G_n, $G_{m,n}$, θ_n, θ_m as defined in Subsection2.3.2. Then G_n is divisible, and $G_{m,n}$ is divisible too, as θ_n, θ_m are isomorphisms and G^E is a union of divisible groups. As $G_{m,n} := \theta_m^{-1}(G_n)$ and θ_m is an order-preserving isomorphism, to show $\aleph_1$-saturation of G_n and $G_{m,n}$ as ordered groups, we will show $\aleph_1$-saturation of G_n:

Lemma B.0.7 *For $n \in \mathbb{N}$, $(G_n, <)$, where $G_0 = x^{\mathbb{R}}$ and for $n \in \mathbb{N} \setminus \{0\}$, $G_n = x^{\mathbb{R}} \overleftarrow{\times} x^{A_0} \overleftarrow{\times} \cdots \overleftarrow{\times} x^{A_n}$ is an η_1-group; all its archimedean components are isomorphic to $\mathbb{R}$; and pseudo-Cauchy sequences indexed by $\aleph_0$ do pseudoconverge in G_n.*

Proof. By recurrence on $n \in \mathbb{N}$. In order to use only additive notations when dealing with archimedean classes, we use the fact that $G \mapsto x^G$ is an order preserving isomorphism. We use notations from Subsection 2.2.3.

- <u>$n = 0$</u>: $(\mathbb{R}, +, 0, <)$ has a unique archimedean class $\tilde{1}$ isomorphic to $\mathbb{R}$, in particular its value set $\Gamma := \{\tilde{x} : x \in \mathbb{R} \setminus \{0\}\} = \{\tilde{1}\}$ is trivially an η_1-set. It is complete as a metric space with $|.|$ so $\aleph_0$-indexed pseudo-Cauchy sequences do pseudoconverge. Hence as we have an order-preserving isomorphism $\mathbb{R} \mapsto x^{\mathbb{R}}$, its multiplicative copy $(x^{\mathbb{R}}, ., <)$ also has a unique archimedean class and a η_1-value set and pseudo-Cauchy sequences indexed by $\aleph_0$ do pseudoconverge in $x^{\mathbb{R}}$.

- $\underline{n=1}$: Consider $(A_0 = \{s \in \mathbb{R}((x^{\mathbb{R}})) : Supp\, s > 1\}, +, 0, <)$. Two elements $s_1, s_2 \in A_0$ are in the same archimedean class–satisfy that it exists $n \in \mathbb{N}$, $n|s_1| > |s_2|$ and $n|s_2| > |s_1|$ (where $|.|$ is not the canonical Hahn series field valuation $Lm(.)$ but the absolute value defined in Subsection 2.2.3 on an additive ordered group G by $|g| := \max\{g, -g\}$ for $g \in G$)– iff $Lm(s_2) = Lm(s_1)$: Indeed, for $s \in A_0$, $|s| = s$ and s_1 and s_2 are in the same archimedean class iff $Lc(ns_2 - s_1) > 0$ and $Lc(ns_1 - s_2) > 0$ iff $Lm(s_2) = Lm(s_1)$. Hence the value set of A_0 is $\Gamma_0 := \{x^r : r \in \mathbb{R}^{>0}\}$ which is an η_1-set by the properties of $\mathbb{R}$; and the set of archimedean components A_0^r of A_0 is
$$\left\{A_0^r = \{s \in A_0 : Lm(s) = x^r\} : r \in \mathbb{R}^{>0}\right\}$$
Each component contains a copy of $\mathbb{R}$, as $]0, r[$ is isomorphic to $\mathbb{R}$ and for $c \in]0, r[$, $x^c + x^r \in A_0^r$; and by Remark 2.3.9, each component has cardinality $\leq 2^{\aleph_0}$.
Now let (a_ρ) be an $\aleph_0$-indexed pseudo-Cauchy sequence in A_0. By Fact 2.3.4, $\mathbb{R}((x^{\mathbb{R}}))$ is spherically complete so (a_ρ) admits a pseudolimit a in $\mathbb{R}((x^{\mathbb{R}}))$. By definition of pseudoconvergence, $(v(a - a_\rho))_\rho$ is strictly increasing, which implies that $a \in A_0$.

  Hence we got the result for the additive ordered group A_0 and thus for its multiplicative copy x^{A_0}.

- Let $n \in \mathbb{N} \setminus \{0\}$ and consider the ordered additive group
$$(A_{n+1} = \{s \in \mathbb{R}((G_n)) : Supp\, s > 1\}, +, 0, <)$$
Similarly to the previous case, two elements $s_1, s_2 \in A_n$ are in the same archimedean class iff $Lm(s_1) = Lm(s_2)$. The value set of A_n is the set
$$\begin{aligned}\Gamma_{n+1} &:= \{x^a : a \in A_n^{>0}\} \\ &= \{x^{(r, a_0, \cdots, a_n)} : (r, a_0, \cdots, a_n) \in (x^{\mathbb{R}} \overleftarrow{\times} x^{A_0} \overleftarrow{\times} \cdots \overleftarrow{\times} x^{A_n})^{>0}\}\end{aligned}$$
which is an η_1-set by recurrence hypothesis and Remark B.0.2. The archimedean components of A_{n+1} are the
$$A_{n+1}^{(r, a_0, \cdots, a_n)} = \{s \in A_{n+1} : Lm(s) = x^{(r, a_0, \cdots, a_n)}\}$$

for $(r, a_0, \cdots, a_n) \in (x^{\mathbb{R}} \overleftarrow{\times} x^{A_0} \overleftarrow{\times} \cdots \overleftarrow{\times} x^{A_n})^{>0}$. The archimedan components are isomorphic to $\mathbb{R}$, and by the same arguments than above, $\aleph_0$-indexed pseudo-Cauchy sequences in A_n pseudoconverge in A_n. Then again we obtain the desired result for the multiplicative copy G_{n+1} from the results on A_{n+1}.

■

Corollary B.0.8 *For $n, m \in \mathbb{N}$, $(G_n, <)$ and $(G_{n,m}, <)$ are $\aleph_1$-saturated ordered groups and $(\mathbb{R}((G_n)), <)$, $(\mathbb{R}((G_{m,n})), <)$ are $\aleph_1$-saturated ordered fields.*

Proof. By Facts B.0.5 and B.0.6 and Lemma B.0.7. ■

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Additionnal comments

Here is a list of the main modifications made after the oral presentation. We thank the referees for their comments.

- We prove Theorem 7.3.5 assuming in the hypotheses on T that only κ-saturated models of T, for some κ, must satisfy $(I)_E$. This allows to consider $T_{\mathbb{R},\exp}$.
- We prove Theorem 7.3.5 assuming Hypothesis $\mathcal{I}m$ instead of $(I)_E$. For this, we have slightly modified Proposition 6.3.4, and have added Lemma 7.2.6, which is an alternative of Lemma 7.2.5, using Proposition 6.3.4 instead of Proposition 6.1.6, and Hypothesis $\mathcal{I}m$ instead of Hensel's Lemma 3.3.4 and $(I)_E$.
- We have added a remark under Definition 7.1.1, concerning the p-adic case, where the domain of definition of the exponential is also the valuation ring.