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Aspects of higher-spin symmetry in flat space

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CALIGULA

Où vas-tu, Hélicon ?

HÉLICON, *sur le seuil*

Te chercher la lune.

Albert Camus, *Caligula*.

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Contents

Chapter 1

Introduction

1.1 An invitation

Einstein's theory of general relativity and the standard model of particle physics constitute our best understanding of the physical laws of Nature so far. They describe respectively the classical evolution of space and time in the presence of matter, and the quantum interactions of matter with itself. While still not within our experimental grasp, a description of gravity in the *quantum* regime would allow for a more solid ground towards a unified description of the fundamental forces, and thus constitutes a challenge in theoretical physics. Unfortunately, applying the tools of quantum field theory to gravity, viewed as the classical field theory of a massless spin-two particle, the graviton, leads to technical, as well as conceptual problems. At two-loop level, general relativity displays divergences [1] which can only be regularised in the ultra-violet regime by the addition of an infinite amount of counter-terms, thereby losing predictive power. This result tells us that general relativity should be considered, at best, as a remarkably accurate effective field theory, whose high-energy behaviour is still unknown.

A candidate, high-energy-finite theory of quantum gravity is provided by string theory, which identifies elementary particles with the vibrating modes of a fundamental string, including the graviton itself. As observed by Veneziano [2], this change of paradigm has the astonishing feature of softening the divergences of gravitational scattering. This can be partly explained by the inclusion of an infinite tower of excitations in the spectrum of the theory with increasing mass, dictated by the famous formula

$$M^2 = \frac{4}{\alpha'}(N-1)\,, \tag{1.1.1}$$

where M^2 is the mass-squared of the state at excitation level N, and α' is a constant with dimension of length squared, parameterising the inverse string tension. The integer N also represents the maximal spin of the state in question, so that this process can be understood as the exchange of massive higher-spin states. These states are organised in so-called Regge trajectories, which are straight lines in the spin/mass-squared plane, whose slopes are given by $\frac{4}{\alpha'}$, the string tension.

As the center-of-mass energy rises or, equivalently, as $\alpha' \to \infty$, the masses of the higher-spin states become smaller and smaller, which means that these higher-spin excitations become increasingly more relevant in the effective description of string theory. In the limit of ultra-high energy, higher-spin excitations would become massless and one should in theory take all of them into account. The full treatment of string theory in the ultra-high energy regime is far from being an easy task, but some observations can nevertheless be made.

Firstly, Gross and Mende [3] proved a series of relations among the scattering amplitudes of string theory for excitations of different spins, in the high-energy, or low-tension, limit $\alpha' \to \infty$, pointing at the existence of a hidden underlying symmetry, see also [4, 5]. This led to the conjecture that string theory could be the broken phase of a higher-spin theory [6], whose gauge symmetry is restored in the ultra-high energy limit $\alpha' = \infty$ (for a review, see, e.g., [7]). Although the appearance of 'extra' symmetries, broken by a dimensionful constant is a perfectly reasonable statement, which is encountered e.g. in Proca or massive Fierz-Pauli theories where the mass term breaks gauge invariance, or in massive scalar field theory where the mass term breaks conformal invariance, the situation of string theory is more delicate due to the yet unclear status of the putative theory at $\alpha' = \infty$. More on the status of string theory in the tensionless limit can be found in [8, 9, 10, 11, 12].

Secondly, it was found in [13], see also [14, 15, 16, 17], that the bosonic part of the leading Regge trajectory in the limit $\alpha' \to \infty$ of string field theory is composed only of symmetric fields propagating representations of spins s, $s-2$, $s-4$, ... down to zero or one, and which are called triplets due to their particular description using a triple of gauge fields of rank s, $s-1$ and $s-2$. This constitutes a first step in the understanding of string field theory in the tensionless limit. Still, a careful treatment of all excitations, including the ones with mixed symmetry, is required to know what such gauge theory could look like, and its link, if any, with higher-spin theory, see, for instance, [18, 19, 7]. For the moment, the tensionless limit is only well-understood in AdS_3, see for instance [20], while other approaches to tensionless strings include, e.g., [21, 22, 23].

The question that arises at this stage is if the key to the understanding of the renormalisation mechanism at play in string theory could be contained in the exchange of higher-spin particles, or if this feature will only be achieved by describing Nature in terms of strings. If the first is true then one should be in position to propose a modification of general relativity based on the inclusion of *massless* higher-spin particles, that is finite in the high-energy regime and amenable to being described at the quantum level. This approach has the added benefit of not needing to introduce strings, and therefore avoiding all undesirable features that are necessary for the consistency of string theory, to wit the existence of extended objects, the necessity of increasing the number of space-time dimensions or the background-dependence of the formulation. The class of theories proposing such a modification fall under the name of 'higher-spin gravity', and are the subject of this thesis.

The story of higher-spin particles begins much earlier than the inception of string theory itself, with Majorana's proposal [24] of a Lorentz-covariant, relativistic equation of motion for particles of arbitrary integer spin, which was refined by Dirac a few years later [25] to the case of an irreducible spectrum and in the language of spinors. Eventually, Wigner's correspondence between elementary particles and the classification of the Unitary Irreducible Representations (UIR) of the Poincaré group of isometries of Minkowski space-time [26] provided a concrete group-theoretic framework to characterise particles of arbitrary spin. In order to find UIR of the Poincaré group, Wigner's classification instructs us to start by fixing the action of the translation generators on a given state. In the case of massless representations, this is equivalent to fixing a space-time vector with zero norm, and the 'little group' of residual transformations preserving this vector is (after quotienting out unphysical translations in two-dimensional Euclidean space, see, e.g., [27]) $SO(2)$, the Euclidean rotation group in two dimensions. The irreducible representations of the latter, using the isomorphism $SO(2) \simeq U(1)$, are characterised by an integer representing the winding mode. Therefore, massless representations of the four-dimensional Poincaré group are characterised by a single number, which means that only one type of higher-spin fields propagate, that we will call symmetric because they correspond to symmetric representations of the little group. Note however that this does not preclude the description of these symmetric higher-spin degrees of freedom by Lorentz tensors of $SO(1,3)$ of a more complicated, mixed symmetry type, since there can be multiple ways in which one can embed degrees of freedom into a Lorentz-covariant space-time tensor, see e.g., the off-shell dualisation procedure of [28].

At around the same time as Wigner's discovery, the Lagrangian formulation of Fierz and Pauli [29] describing the propagation of free spin-2 and spin-3/2 particles, the latter also found by Rarita and Schwinger [30], was proposed. It was already considered to be generalisable to any spin, though a full proposal was still lacking at this stage. Some years later, the Bargmann-Wigner equations[1] [32] were considered for this role, but eventually abandoned due to their problematic coupling to electromagnetism. Although some progress was made in [33] on the construction of Lagrangians for particles with spin $s \leq 4$, the general spin-s case would only be

[1]Although the Bargmann-Wigner equations are first-order equations formulated in $D = 4$, they were generalised in [31] where it was proven that they are equivalent to higher-derivative equations in terms of symmetric fields.

completed much later thanks to the work of Fronsdal [34]. The way that the Fronsdal equations were brought about is actually by considering the limit of vanishing mass of *massive* higher-spin field equations built in [35, 36], and showing that at the massless point, the theory acquires a novel gauge symmetry.

Fronsdal's equations are two-derivative, and use a completely symmetric tensor field with s indices $\varphi_{\mu_1\cdots\mu_s}$ which is doubly traceless and transforms with a traceless gauge parameter $\xi_{\mu_1\cdots\mu_{s-1}}$

$$\varphi_{\mu_1\cdots\mu_{s-4}\nu\rho}{}^{\nu\rho} = 0\,, \quad \delta\varphi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\,\xi_{\mu_2\cdots\mu_s)}\,, \quad \xi_{\mu_1\cdots\mu_{s-3}\nu}{}^{\nu} = 0\,, \tag{1.1.2}$$

where indices are raised and lowered thanks to the Minkowski metric $\eta_{\mu\nu}$ with signature $(-,+,+,+)$, and where our conventions for symmetrisation are displayed in section 1.3. This gauge symmetry is Abelian since the successive action of two gauge transformations automatically vanishes, and one can reconstruct the result of Fronsdal by starting from a two-derivative wave equation on the field $\varphi_{\mu_1\cdots\mu_s}$ and reconstructing the rest of the terms by imposing gauge invariance [37, 38]. This theory closely resembles the linearised regime of gravity, save for the trace constraints which start to appear at spin three, and admits a Lagrangian description.

In $D \geq 5$ space-time dimensions, the previous picture is qualitatively the same, although the classification of the UIR of the Poincaré group somewhat complicates, due to the representations of the 'little group' $SO(D-2)$ being richer. Symmetric representations exist and can still be described by Fronsdal's formulation, but there are also mixed-symmetry representations when $D \geq 6$, whose first covariant description is due to Labastida [39]. The latter are important in string theory, since mixed-symmetry fields always arise in the spectrum of higher excitations of the string, as can already be seen at the first excited level of the closed bosonic string which contains, in addition to the usual dilaton and graviton fields, a massless anti-symmetric field called the Kalb-Ramond two-form. Given the links between higher-spin theory and string theory, it is an interesting problem to try to formulate an interacting gauge theory of higher-spin gravity in flat space-time including fields of arbitrary mixed symmetry. However, from the vantage point of higher-spin gravity, being the *minimal* modification of general relativity potentially able to be described at a quantum scale, one is led to adopt a simplifying stance and work only with symmetric fields.

Although the description of free massless symmetric higher-spin fields poses little obstacle, problems start to emerge at the level of interactions. It was already noticed in [40] that a spin-three field cannot give rise to a consistent self-coupling[2] cubic term, as opposed to the case e.g. of the spin-two self-coupling in Einstein theory. This problem could potentially be solved by adding other higher-spin fields to the theory, but it was observed that this problem will always arise for any higher-spin

[2]In general, self-coupling terms for fields of odd spin are problematic, even for spin one. A way out consists in introducing colour factors, like in Yang-Mills theory.

theory containing a finite set of fields and in dimension at least four, and can have a chance to be solved only if *all spins* are introduced (one can eventually reduce this infinity by half by working with fields of even spin only). This makes the task of finding a consistent theory much harder, since one has to deal with all spins at once.[3]

Another unconventional feature of higher-spin interactions is related to the coupling of higher-spin fields to gravity. Indeed, it was already noticed by Weinberg [41] that the low-energy limit of scattering amplitudes involving a 'soft' particle (i.e. on-shell with vanishing momentum) with spin $s > 2$ yields, under the hypothesis of minimal coupling to gravity, conservation laws that have no non-trivial solution unless the corresponding coupling constant is zero. Along the same lines, it was found still for $s > 2$ that any cubic two-derivative vertex linear in the graviton and quadratic in the spin-s field will never be gauge-invariant [42], leading to the conclusion that in flat space-time, one has to forego the usual notion of minimal coupling to gravity. Further theorems constraining the form of higher-spin interactions, most of them using arguments on the form of the S-matrix, were presented for instance in [43, 44].

Higher-spin interactions preserving a deformation of the free gauge symmetry at the cubic level were eventually constructed in [40] and fully classified in [45, 46, 47, 48, 49, 50, 51, 52]. Crucially, it was remarked that some of the previous peculiarities disappear when the background space-time is not Minkowski but (Anti-) de Sitter. As an example, the presence of the cosmological constant, which has the dimension of an inverse length squared, allows for the existence a two-derivative term in the cubic coupling of higher-spin fields with gravity [53, 54] (see also, e.g. [55] or the review [56]), thereby recovering a notion of minimal coupling. The other usual no-go arguments revolving around the S-matrix are also not directly applicable since the latter is only properly defined in flat space-time.[4]

Eventually, the free theory of Fronsdal and the problem of constructing higher-spin interactions and checking their consistency at higher orders in perturbation theory was reformulated in algebraic terms [57, 58] as the existence of a non-Abelian algebra encoding higher-spin symmetry and satisfying some conditions, dictated by the need to reproduce the free theory. This approach uses a 'frame-like' field ${e_\mu}^{a_1\cdots a_{s-1}}$, which is a space-time one-form with frame indices a_i (to be contracted with the corresponding generator $M_{a_1\cdots a_{s-1}}$ of a putative higher-spin algebra) which

[3]Although one can build theories up to cubic order with only a finite number of fields, the necessity of closing of the algebra of gauge transformations at cubic level, or the consistency of interactions at quartic level, forces us to consider the full spectrum. This problem of an infinite spectrum is distinct from the potentially infinite number of interactions with increasing derivatives in a perturbative expansion, which is also present in the case of higher-spin gravity.

[4]The S-matrix measures the scattering amplitudes in an asymptotically far region of Minkowski space, and has to be replaced in Anti de Sitter by correlation functions of the boundary theory, while no direct equivalent seems to exist in de Sitter space-time.

are symmetrised and traceless. The Fronsdal field $\varphi_{\mu_1\cdots\mu_s}$ can then be recovered as the completely symmetric projection of $e_\mu{}^{a_1\cdots a_{s-1}}$, including both frame and form indices. This frame-like field verifies a first-order equation of motion, which is similar to the torsion constraint in Cartan's formulation. Additional fields, or 'spin connections' in the language of Cartan, $\omega_\mu{}^{a_1\cdots a_{s-1},b_1}$, $\omega_\mu{}^{a_1\cdots a_{s-1},b_1b_2}$, etc. need to be introduced, which are of mixed symmetry type and are identified with the successive derivatives of the frame-like field upon imposing structure equations.

In fact, this whole procedure can be thought of as a higher-spin extension of linearised general relativity *à la* Cartan, with torsion equation (here and in the following, we will use $\stackrel{!}{=}$ to denote when we impose an equation of motion)

$$\partial_{[\mu}\, e_{\nu]}{}^a + \eta_{b[\mu}\, \omega_{\nu]}{}^{ab} \stackrel{!}{=} 0\,, \tag{1.1.3}$$

where $e_\mu{}^a$ is the frame, or *vielbein* field gauging the space-time translation generator P_a, and $\omega_\mu{}^{ab}$ is the spin-connection field gauging Lorentz transformations J_{ab}. We chose as a background the Minkowski space-time with vielbein $\delta_\mu{}^a$, and brackets denote an anti-symmetrisation. The equation of motion for the propagation of the spin-two excitation encoded in the symmetric part of $e_\mu{}^a$ can be written as

$$\partial_{[\mu}\, \omega_{\nu]}{}^{ab} \stackrel{!}{=} \eta_{\mu c}\, \eta_{\nu d}\, C^{ac,bd}\,, \tag{1.1.4}$$

where $C^{ab,cd}$ has the symmetries of the Weyl tensor in general relativity, i.e. symmetric in the first two and the last two indices, symmetric under the exchange of the two groups of indices and completely traceless. The previous equations can be shown to be equivalent to the Fierz-Pauli equations of motion of linearised gravity, since the unique non-zero component of the Riemann tensor upon imposing Einstein's equations is the Weyl tensor. The identification of $C^{ab,cd}$ with the Weyl tensor is then completed by the integrability condition of the previous equation, that imposes Bianchi identities on $C^{ab,cd}$.

A unique candidate higher-spin algebra in AdS_4 which reproduces the equations [58] was identified in [59], constructed using higher products of the isometry generators of AdS_4 in an oscillator realisation [60]. The equations of motion up to first order in curvature bringing about its gauging were constructed in [61]. Although this algebra exists and is unique in $(\mathrm{A})\mathrm{dS}_4$, it was shown that, starting from the Poincaré algebra, no such symmetry algebra can be constructed under the same set of assumptions, which resounded as another no-go theorem for higher-spin theories in flat space-time. The four-dimensional construction was then generalised to higher dimensions[5] in [63, 64, 65, 66] and was an important milestone in the quest for the interacting theory of higher-spin gravity of [67], which is reviewed, e.g. in [68]. Its

[5] It was proven, much later after the original construction of Fradkin and Vasiliev, that this algebra is the unique one that satisfies the Jacobi identity (which is a necessary requirement if one hopes to push the deformation procedure to the next, quartic, order) and reproduces known cubic interactions [62] in any dimensions $D \geq 4$, save for a one-parameter deformation in $D = 5$.

main ingredient is undoubtedly the higher-spin symmetry algebra in AdS_D space-time, that we shall dub $\mathfrak{hs}_D$ in the following, which is in more modern language given by the quotient of the universal enveloping algebra of space-time isometries $\mathfrak{so}(2, D-1)$ by a certain two-sided ideal $\mathcal{I}$ that we shall detail in section 2.2

$$\mathfrak{hs}_D := \frac{\mathcal{U}(\mathfrak{so}(2, D-1))}{\langle \mathcal{I} \rangle} . \tag{1.1.5}$$

In this thesis, we will see how this construction, and the steps that led to it, can nevertheless be repeated in the case of flat space-time. More specifically, we will construct a flat higher-spin algebra, that we shall call $\mathfrak{ihs}_D$, from the quotient of the universal enveloping algebra of the isometries of Minkowski space by a two-sided ideal $\mathcal{I}^\flat$ whose definition will be given in eqs. (2.3.16). The algebra can also be recovered from a İnönü-Wigner contraction of the algebra $\mathfrak{hs}_D$. The ideal will be identified as the unique one allowing to reproduce the same spectrum of generators as in the AdS case, and the gauging of this algebra at the linearised level will give rise to equations that describe the correct free dynamics. A peculiar feature of this algebra is the factoring out of the product of higher-translation generators $P_{a_1} \cdots P_{a_{s-1}}$, that one would naively associate to the frame-like field ${e_\mu}^{a_1 \cdots a_{s-1}}$. Indeed, we will see that the fundamental gauge field of our construction will not be the frame-like field, but rather one of the usually auxiliary 'spin-connections', ${\omega_\mu}^{a_1 \cdots a_{s-1}, b_1 \cdots b_{s-2}}$. In turn, this will suggest why the only gauge-invariant vertices coupling higher-spin to gravity in flat space are necessarily higher-derivative.

In the construction of $\mathfrak{hs}_D$, the set of relations in $\mathcal{I}$ that one has to quotient plays an important role. This ideal was identified in [65, 69, 70] as the annihilator of a particular representation of the conformal algebra, the singleton, and as a consequence the full higher-spin algebra is isomorphic to the associative algebra of higher differential symmetries of this module. The latter is associated to the on-shell non-trivial isometries of a Klein-Gordon equation in one dimension less and constitutes, alongside other pieces of evidence [71], a first element in the characterisation of the holographic dual of higher-spin theory in AdS, which we will attempt to summarise.

A free, massless scalar field ϕ in $(D-1)$ dimensions has higher symmetries associated to differential operators of higher arbitrary order, which provide a differential realisation of the algebra $\mathfrak{hs}_D$, according to the argument of Eastwood [65]. The free scalar theory also displays conserved higher-spin currents of spins $s = 1, 2, \dots, \infty$ [72], schematically given by

$$J_{\mu_1 \cdots \mu_s} = \bar{\phi} \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_s} \phi - \text{ traces} , \tag{1.1.6}$$

such that $J_{\mu_1 \cdots \mu_s}$ are traceless and transverse when the field ϕ satisfies the massless Klein-Gordon equation. These currents couple to massless bulk fields reaching the conformal boundary of AdS space-time through the usual $\int \mathrm{d}^{D-1}x \; J_{\mu_1 \cdots \mu_s} \tilde{\varphi}^{\mu_1 \cdots \mu_s}$ term, where $\tilde{\varphi}^{\mu_1 \cdots \mu_s}$ has to be understood as the boundary value of a Fronsdal field in the bulk of AdS space-time. These kinematical considerations, backed by the AdS/CFT correspondence[6] of Maldacena [73], has led to the idea that gauge higher-spin symmetry in the bulk should correspond to rigid higher-spin symmetry on the boundary. Following the work of Klebanov, Polyakov [74], Sezgin and Sundell [75], the holographic dual of the theory of interacting massless higher-spin fields in AdS_4 of [76] was identified to be the large-N limit of the critical three-dimensional $O(N)$ vector model, where N massless scalars interact via a quartic term with fine-tuned coupling constant. Maldacena and Zhiboedov [77] then classified conformal field theories with exact higher-spin symmetry in dimensions three and greater, and found that higher-spin symmetry can unambiguously fix the form of all correlators of the theory to be those of a free theory. This result is somehow the AdS/CFT equivalent of the theorems constraining the form of the S-matrix in flat space.

Although not explicitly forbidden by these theorems, the existence of a higher-spin symmetry algebra in flat space, and its holographic reformulation are much less understood. This would be interesting for a number of reasons. First of all, it would allow to bridge the gap between higher-spin gravity in AdS and in flat-space.[7] Second, this would ever so slightly bring us closer to a description of string theory in the tensionless regime, which is formulated around the Minkowski background. Lastly, instances of a holographic correspondence including gravity in asymptotically flat space-time are scarce and the subject of an active investigation [87, 88, 89, 90, 91, 92, 93, 94, 95]. The relative simplicity of the holographic dual of AdS higher-spin gravity constitutes an ideal playground to develop such a correspondence in flat space.[8] Let us spend the rest of this introduction elaborating on this last point.

In asymptotically flat space-times, the symmetries of general relativity are enlarged when approaching null infinity. The Poincaré group of transformations enhances to the BMS group of Bondi, van der Burg, Metzner and Sachs [97, 98], which can be seen as the set of large diffeomorphisms preserving a certain class of solutions of Einstein's equations, characterised by boundary conditions, up to terms that are sub-leading in an asymptotic expansion. The new generators of symmetry are called 'super-translations', as they asymptotically perform a shift in the retarded time by an arbitrary function of the remaining coordinates, the angles $\mathbf{x}$ parameterising the two-dimensional 'celestial' sphere. This symmetry group can

[6]The original formulation of the AdS/CFT correspondence postulated the equivalence between type-IIB super-string theory on $AdS_5 \times S^5$ and a dual $\mathcal{N} = 4$ super-conformal Yang-Mill theory on four-dimensional Minkowski space, the boundary of AdS_5 .

[7]Interacting higher-spin theories in flat space are known to exist only in lower dimensions [78, 79, 80, 81], while tentative definitions of a higher-spin algebra in flat space were considered in [82, 83]. Another promising route is to formulate the theory *on the light-cone*, where additional types of interactions exist, and a quantum theory was explicitly constructed [45, 84, 85, 86].

[8]One can also wonder about the status of higher-spin symmetry and holography in other types of situations, such as near the horizon of a black hole [96].

be enlarged even more, by including transformations that modify the geometry of the sphere at infinity, and called 'super-rotations', leading to the *extended* [89] or *generalised* [99] BMS group depending on the allowed set of transformations. The fact that symmetries on the boundary are greatly enhanced compared to rigid symmetries in the bulk is a sign that flat-space holography, if such a theory exists, may not work in the same way as it does in AdS.

This enhancement of symmetries is also observed in other field theories, such as Maxwell or Yang-Mills [100], and even free spin-s fields [101, 102, 103]. In the latter case, under the most permissive boundary conditions, asymptotic symmetry generators are characterised by s arbitrary symmetric and traceless tensors, depending only on the coordinates on the celestial sphere

$$T(\mathbf{x})\,, \quad \rho^i(\mathbf{x})\,, \quad \dots\,, \quad K^{i_1\cdots i_{s-1}}(\mathbf{x})\,. \tag{1.1.7}$$

A promising route to flat-space holography can be found in the framework of *Carrollian* physics [94]. Its inception can be traced back to the work of Duval, Gibbons and Horvathy [104], who showed that the group of isometries of a Carrollian[9] manifold, that is a smooth manifold equipped with a degenerate metric and a nowhere-vanishing vector field (see appendix A), is isomorphic to the BMS group. The null manifold in question is identified with (past or future) null infinity $\mathscr{I}^\pm$, whose metric is degenerate in the direction of (advanced or retarded) time. The holographic dual of a gravitational theory in the bulk of asymptotically Minkowski space-time would then be encoded in a Carrollian conformal field theory. This idea already finds a concrete realisation in the case of the fluid-gravity correspondence [92, 105, 106], where the speed of light of the boundary fluid is directly proportional to the cosmological constant in the bulk.

We will argue in this thesis that the correspondence between the generators of the higher-spin algebra $\mathfrak{ihs}_D$ in the bulk and the higher differential symmetries of a certain free field living at the $(D-1)$-dimensional boundary of space-time, of which the ideal $\mathcal{I}^\flat$ is the annihilator, also holds. Moreover, the full set of symmetries is actually much bigger, and includes generators that can be assimilated with the asymptotic symmetry generators described in eq. (1.1.7) and which form a subalgebra. More specifically, we will consider two candidate theories of free Carrollian scalar fields, namely the 'electric' (or 'time-like')

$$\int \mathrm{d}u\, \mathrm{d}^d\mathbf{x}\, \sqrt{\gamma}\, \left(\bar{\phi}\, \partial_u{}^2 \phi\right)\,, \tag{1.1.8}$$

[9]The name *Carrollian* was inspired by Lewis Carroll's book *Through the looking glass*, where the protagonist Alice is puzzled by the Queen's remark 'it takes all the running you can do, to keep in the same place'. In Carrollian space-time, the speed of light is sent to zero, creating this sense of 'static motion'.

and the 'magnetic' (or 'space-like')

$$\int \mathrm{d}u\, \mathrm{d}^d\mathbf{x}\, \sqrt{\gamma}\, \left(\bar{\pi}\partial_u\phi + \pi\partial_u\bar{\phi} + \bar{\phi}\hat{\nabla}^2\phi\right)\,, \tag{1.1.9}$$

where $d = D - 2$ is the dimension of the celestial sphere, u is the null (retarded or advanced) time, $\hat{\nabla}^2 := \nabla^2 - \frac{(d-1)^2}{4}$ and ∇^2 is the Laplace-Beltrami operator on the d-dimensional round sphere with metric γ.

Both theories realise the higher-spin symmetry algebra $\mathfrak{ihs}_{d+2}$, and we will argue that the first is adapted for the holographic description of higher-spin symmetry in AdS-Carroll$_{d+2}$ space-time, while the latter is a candidate starting point for a holographic theory of higher-spin gravity in Minkowski$_{d+2}$ space-time.

In conclusion, the algebraic approach towards the construction of an interacting theory of higher-spin gravity in AdS has lead to the identification of a unique candidate algebra underlying higher-spin symmetry, which admits a co-dimension one realisation in terms of differential operators of a relativistic conformal field theory. A contraction of this algebra (also unique under certain considerations) admits a co-dimension one realisation in terms of differential operators of a Carrollian conformal field theory, and is shown to reproduce free equations of motion that propagate the correct number of degrees of freedom at the linear level. Any non-linear deformation of our free equations of motion will therefore provide a candidate gravitational dual of the simplest Carrollian field theory, thus fitting within the urgent quest for concrete dual pairs in flat-space holography, that is currently mainly driven by symmetry considerations.

1.2 Structure of this thesis

In this thesis, we bridge the gap between the gauge description of higher-spin gravity in AdS and in flat space by proving the existence of a higher-spin algebra with a Poincaré sub-algebra in any dimensions at least three, and prove that its gauging at the linearised level brings about equations that are equivalent to the ones of Fronsdal. The main advantage of our construction relies in the non-Abelian character of the algebra, translating into the possible existence of interaction terms for fields of every spin, including gravity. The explicit construction of an interacting theory based on this algebra will be addressed in a future work. We then present the first steps in the elaboration of a holographic dual, relying on a Carrollian, i.e. ultra-relativistic, scalar field theory, by showing that this higher-spin algebra can be realised as a subset of the higher symmetries of this scalar field. Extra symmetries are also uncovered, playing the role of putative asymptotic higher-spin symmetries. This thesis is divided as follows.

In chapter 2, the construction of non-Abelian symmetry algebras for higher-spin fields in Minkowski space-time is discussed in details. We begin with some reminders of the global symmetries of free higher-spin fields in the formulation of Fronsdal [34] presented in section 2.1, while the construction of the higher-spin algebra in AdS space-time is reviewed in section 2.2. We then present the İnönü-Wigner contraction leading to the flat-space higher-spin algebras in three and higher dimensions first presented in [107], and recover them from quotients of the universal enveloping algebras of space-time isometries in section 2.3. Lastly, section 2.4 is devoted to the study of the curvatures gauging the algebras found in the previous section, and we prove there how to reproduce the dynamics of Lopatin and Vasiliev [64], equivalent to the one of Fronsdal, from equations of motion, as presented in [108].

In chapter 3, we explore a potential holographic dual of that theory. Following the standard observation that higher-spin algebras are associative algebras of higher differential symmetries of free field theories [65] reviewed in section 3.1, we will focus on the realisation of the higher-spin algebra identified in the previous chapter as the algebra of symmetries of a Carrollian field theory, identified as the ultra-relativistic limit $c \to 0$ of a relativistic free scalar field. More precisely, we will study the two possible limits (electric and magnetic) of the free scalar field identified in the literature [109, 110, 111] in sections 3.2 and 3.3. The latter, magnetic theory, admits an extension as a field in Minkowski space which we show to be the direct flat limit of its AdS parent, the singleton [112, 71, 113].

Finally, chapter 4 closes this thesis with a summary of the results and a discussion of possible research directions.

The results of this thesis were obtained in collaboration with Xavier Bekaert, Nicolas Boulanger and Andrea Campoleoni, and were presented in [107, 114, 108]. The first reference also contains the construction of a different contraction of the higher-spin algebra $\mathfrak{hs}_D$ to a *Galilean* conformal algebra, not discussed in this thesis, as well as the construction of higher-spin algebras for theories with the same set of generators as the ones relevant for partially-massless higher-spin gravity which are briefly mentioned in section 2. This thesis also contains original material, which will be presented in [115]. Other original works not directly related to the subject of this thesis were presented in [116, 117] and lecture notes in [118].

1.3 Conventions

Throughout this thesis, the AdS radius will be denoted by R and the dimension of space-time by D . We will also use in chapter 3 the dimension of the celestial sphere $d = D - 2$. Our conventions for tensors will be as follow:

- A covariant and a contravariant index denoted by the same letter are meant to be contracted and thus summed, following the Einstein summation convention ;

- Repeated indices that are not contracted are symmetrised. For instance, $\varphi_\mu{}^\mu$ denotes a trace, but
$$\varphi_{\mu\mu} \quad \text{stands for} \quad \varphi_{\mu_1\mu_2}\,; \tag{1.3.1}$$

- A group of symmetrised indices will be replaced by a single index, with the multiplicity indicated with brackets. As an example
$$\varphi_{\mu(s)} \quad \text{stands for} \quad \varphi_{\mu_1\cdots\mu_s}\,, \tag{1.3.2}$$
where $\varphi_{\mu_1\cdots\mu_s}$ is symmetric under the exchange of any two indices ;

- Divergences of symmetric tensors will simply be denoted by a dot, e.g.
$$\nabla\cdot\varphi_{\mu(s-1)} \quad \text{stands for} \quad \nabla^\nu\,\varphi_{\nu\mu_2\cdots\mu_s}\,; \tag{1.3.3}$$

- The trace of a symmetric tensor with respect to a (non-degenerate) metric $g_{\mu\nu}$ will be denoted by a prime, e.g
$$\varphi'{}_{\mu(s-2)} \quad \text{stands for} \quad g^{\nu\rho}\,\varphi_{\nu\rho\mu_3\cdots\mu_s}\,; \tag{1.3.4}$$

- Symmetrisation (and anti-symmetrisation) are performed with unit weight, meaning that they are projections. For instance
$$\partial_\mu\,\xi_{\mu(s-1)} \quad \text{stands for} \quad \partial_{(\mu_1}\,\xi_{\mu_2\cdots\mu_s)} = \frac{1}{s}\sum_{k=1}^{s}\partial_{\mu_k}\,\xi_{\mu_1\cdots\widehat{\mu}_k\cdots\mu_s}\,; \tag{1.3.5}$$

- Unless otherwise specified, any tensor $T_{a(s_1),b(s_2),\cdots,d(s_n)}$ is in an irreducible representation of the Lorentz group characterised by the corresponding Young tableau of $SO(D)$, represented by the symbol $\mathbb{Y}_D(s_1,\,s_2,\,\ldots,\,s_n)$. We are working in the symmetric convention, so repeated indices are symmetrised
$$\mathbb{Y}_D(s_1,\,s_2,\,\ldots,\,s_n) \;\sim\; \begin{array}{|c|}\hline \quad\quad\quad s_1 \quad\quad\quad \\ \hline \quad\quad s_2 \quad\quad \\ \hline \quad \cdots \quad \\ \hline s_n \\ \hline\end{array} \tag{1.3.6}$$
where $n \leq \left\lfloor\frac{D-2}{2}\right\rfloor$ and $s_1 \geq s_2 \geq \cdots \geq s_n > 0$. Such tensors are completely irreducible and traceless, meaning that
$$\begin{aligned} T_{a(s_1),ab(s_2-1),\cdots,d(s_n)} &= 0\,,\\ T_{a(s_1),b(s_2),b\cdots,d(s_n)} &= 0\,,\\ &\cdots\\ T_{a(s_1),\cdots,c(s_{n-1}),cd(s_n-1)} &= 0\,, \end{aligned} \tag{1.3.7}$$
and $\eta^{aa}T_{a(s_1),b(s_2),\cdots,d(s_n)} = 0$ (the vanishing of the trace in the first row and the irreducibility conditions imply the vanishing of all traces) .

Chapter 2

Higher-spin symmetry in Minkowski space

We start this chapter with a review of the global symmetries of massless higher-spin fields, putting a particular emphasis on the case of Minkowski space-time. Even though it has already been discussed at length in various reviews, see for example [119, 120, 121, 122, 123, 118], we will emphasise some key features that will allow us to comment on the construction of an interacting theory.

The starting point is the covariant description of the propagation of free massless higher-spin particles on a flat (or constantly curved) background, which we review in section 2.1. Fronsdal's formulation of the dynamics [34, 124], which is a set of second order equations of motion for a field $\varphi_{\mu(s)}$ which is a completely symmetric and doubly-traceless tensor, fits this role while keeping the analogy with lower-spin theories, such as electromagnetism or (linearised) general relativity. This 'Fronsdal' field enjoys a gauge invariance $\delta\varphi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)}$ under a completely symmetric and traceless tensor $\xi_{\mu(s-1)}$ which is to the Fronsdal field what linearised diffeomorphisms are to the linearised graviton. Upon a partial gauge fixing, Fronsdal's equations resolve into a Fierz system, which can be shown to describe unitary irreducible representations of the Lorentz group. This formulation can be thought of as a direct extension of linearised gravity to higher spins and is often called 'metric-like' by analogy with linearised Einstein theory and is reviewed in section 2.1.1. By contrast, the 'frame-like' approach which we will come to in section 2.1.2 uses a generalisation of the vielbein and is more akin to an extension of Cartan's formulation of general relativity.

The metric-like formulation is accompanied with a Lagrangian formulation, and constitutes a natural basis upon which one can try to build interactions. Although many examples of interaction vertices of higher-spin fields in the Fronsdal approach are known [45, 49], gravitational interactions have been pointed out long ago to be problematic [125, 126, 40]. This result was somehow to be anticipated because of

Weinberg's low-energy theorem [41], the Weinberg-Witten theorem [127] or its generalisation [44], all stating in a way that higher-spin fields cannot couple *minimally* to gravity, that is with a two-derivative cubic vertex, unlike lower-spin theories such as Yang-Mills or the graviton itself.

This obstruction did not prevent the explicit construction of higher-derivative vertices in Minkowski space-time in the light-cone formulation [45, 128, 49, 50], in a covariant manner [129] or using the BV-BRST approach [130, 55, 131]. While not completely ruled out, an interacting theory in Minkowski space-time seemed increasingly exotic. Fortunately, a way out emerged in the 80's, corresponding to a shift of paradigm from flat to constantly curved background. It was realised in [59] that one can construct a two-derivative cubic vertex coupling higher-spin Fronsdal fields to gravity in (A)dS, which is necessarily accompanied by a tower of higher-derivative pieces.

A few years earlier, it was realised that the free dynamics could also be described in a similar way to the gauge (or Cartan's) formulation of gravity, not using Fronsdal fields but space-time one-forms taking value in a putative Lie algebra, understood as *connections* on a specific fibre bundle. This frame-like formulation of the dynamics started with the results of Vasiliev [57, 58] and was setting the stage for the construction of the interacting theory in (A)dS space-time by Fradkin, Vasiliev and collaborators [60, 54, 59, 64, 61, 76, 67, 66, 68]. Not only does the frame-like formulation reproduce the same linearised dynamics as the one of Fronsdal, treating the different connection one-forms in a more balanced way, but it also allows to reformulate the problem of finding an interacting theory into an algebraic one. This step will prove to be of crucial importance in our construction, so we will spend some time on it.

The main star of the construction of Fradkin and Vasiliev is the algebra of higher-spin symmetry $\mathfrak{hs}_D$, first constructed for $D = 4$ in [60] and generalised to higher dimensions in [132, 133, 65, 66], that we review in section 2.2. This algebra plays the same role as the (A)dS isometry algebra does for linearised general relativity: it encodes the isometries of the background, and one can recover perturbatively interacting equations of motion from its gauging [61, 76]. This algebra can be thought of as a *non-Abelian* completion of the algebra of global isometries of the free theory. Historically, the generators of this algebra were realised in $D = 4$ by products of space-time isometries [60], which leads to the modern picture of using a construction based on a Universal Enveloping Algebra (UEA). Specifically, $\mathfrak{hs}_4$ will be the UEA of $\mathfrak{so}(2,3)$ quotiented by an appropriate two-sided ideal, which is automatically factored out in the oscillator representation. The whole construction can be generalised to any dimensions. However, it was noticed in [60] that the algebra $\mathfrak{hs}_4$ does not seem to admit any *reasonable* flat contraction that one could use to repeat the same steps as in the (A)dS case, that is to impose equations on the curvatures of the fields gauging[1] this algebra, reproducing the free dynamics of Fronsdal upon linearisation around the Minkowski background.

A version of the higher-spin algebra $\mathfrak{hs}_D$ in $D = 3$ was also independently found as the algebra of area-preserving diffeomorphisms [134, 135] or from an oscillator construction [136]. They both represent an analytic continuation of $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ when N is sent to infinity. The $N = 3$ instance was analysed in [137, 138, 139, 140], where it was proven that the linearised equations of motion obtained from its gauging are equivalent to the three-dimensional Fronsdal theory. Moreover, one can build a fully-interacting theory using a Chern-Simons action, extending the usual construction for gravity [141, 142, 143, 144, 145] to an arbitrary (finite or infinite) number of higher-spin fields [134], and the asymptotic symmetry algebra of the non-linear theory falls into a class of algebras known as $\mathcal{W}$-algebras [146, 147], also studied in the context of string theory in [148, 149, 150].

A one-parameter family of deformation of the $\mathfrak{hs}_3$ algebra, that we shall call $\mathfrak{hs}_3[\lambda]$, was also found in [134], see also e.g. [150, 151, 152] and references therein. This three-dimensional higher-spin algebra has a particular status, since it is the only known case where a flat counterpart, that we shall denote as $\mathfrak{ihs}_3[\lambda]$ has been explicitly constructed [78, 79, 80], until recently [107].

The idea for this part of the thesis will be to build a higher-spin algebra extending the Poincaré algebra in any dimensions, following the UEA approach. The no-go results in the metric- and frame-like formulations somehow force us to think backwards: our main focus in section 2.3 will be to see if one can construct any algebra at all, while deferring a detailed analysis of the dynamics that can be described by its gauging to section 2.4.

Starting from the three-dimensional case, we show how to reproduce the flat-space higher-spin family of algebras $\mathfrak{ihs}_3[\lambda]$. We will show that it can be built from the same considerations that allows one to build the higher-spin algebra $\mathfrak{hs}_3[\lambda]$ from the UEA of the AdS_3 isometry algebra. This places both algebras on a similar footing and provides us with some key insights in order to generalise the construction to higher dimensions. We will then show how to build such a flat algebra in any dimensions, starting from an İnönü-Wigner contraction and then reconstructing the result from a UEA of the Poincaré algebra. We also provide arguments for the

[1]The word *gauging* is perhaps too vague to be used without an accompanying explanation. In the Cartan formulation of linearised gravity, gauge transformations of the vielbein and spin connection correspond to linearised diffeomorphisms and local Lorentz transformations respectively. They leave the curvatures of the fields built upon the Poincaré or (A)dS algebra invariant. In the non-linear setting, gauge transformations correspond only to local Lorentz transformations (interpreted as a change of frame), while linearised diffeomorphism invariance is promoted to full diffeomorphism invariance of the action (interpreted as a coordinate redefinition), and the curvatures transform covariantly under local Lorentz transformations. In general, it should always be clear that a non-linear gauge theory is the given of an algebra of *global* symmetries and its sub-algebra of *local* gauge transformations.

unicity of the UEA construction.

The resulting algebra, that we will call $\mathfrak{ihs}_D$, does not reproduce upon gauging the usual linearised curvatures employed in the unfolded formulation of the dynamics in the flat case [64], in accordance with the observation of [60]. Notwithstanding, we show that in spite of this mismatch, one can still define equations of motion that indeed describe the propagation of massless fields of arbitrary spin on a Minkowski background. This new formulation of the dynamics opens the way to a new paradigm for a putative interacting higher-spin theory in flat space-time, with the algebra $\mathfrak{ihs}_D$ at its centre.

2.1 Global symmetries of free massless higher-spin fields

2.1.1 Metric-like formulation

Our starting point is the Fronsdal theory in D-dimensional Minkowski space-time, which is the unique two-derivative equation of motion for completely symmetric, doubly-traceless fields $\varphi_{\mu(s)}$, where, as seen in section 1.3 of the introductory chapter, repeated indices denote a symmetrisation with weight one. It can be built as the unique gauge-invariant completion of the d'Alembertian operator acting on $\varphi_{\mu(s)}$, which is given by the Fronsdal tensor $F_{\mu(s)}[\varphi]$

$$F_{\mu(s)}[\varphi] := \partial^2 \varphi_{\mu(s)} - s\,\partial_\mu D_{\mu(s-1)}[\varphi]\,, \tag{2.1.1}$$

where we defined the spin-s De Donder tensor $D_{\mu(s-1)}[\varphi]$ as

$$D_{\mu(s-1)}[\varphi] := \partial\cdot\varphi_{\mu(s-1)} - \tfrac{s-1}{2}\partial_\mu\,\varphi'_{\mu(s-2)}\,. \tag{2.1.2}$$

The Fronsdal tensor is invariant under the gauge transformations

$$\delta\varphi_{\mu(s)} = \partial_\mu\xi_{\mu(s-1)}\,, \tag{2.1.3}$$

where the parameter $\xi_{\mu(s-1)}$ is traceless. The equations of motion are

$$F_{\mu(s)}[\varphi] \stackrel{!}{=} 0\,. \tag{2.1.4}$$

The AdS$_D$ version of Fronsdal theory is given by transforming the partial derivatives ∂ into the AdS background covariant derivatives ∇ and adding some linear terms to $F_{\mu(s)}[\varphi]$ so that it becomes

$$\nabla^2\varphi_{\mu(s)} - s\,\nabla_\mu D_{\mu(s-1)}[\varphi] - \tfrac{1}{R^2}\left[(R\,m_s)^2\,\varphi_{\mu(s)} + s(s-1)\,\bar{g}_{\mu\mu}\,\varphi'_{\mu(s-2)}\right]\,, \tag{2.1.5}$$

where R denotes the radius of AdS with metric $\bar{g}_{\mu\mu}$, the De Donder tensor takes the form $\nabla\cdot\varphi_{\mu(s-1)} - \frac{s-1}{2}\nabla_\mu\,\varphi'_{\mu(s-2)}$ and $(R\,m_s)^2 = (s-2)(s+D-3) - s$.

Going back to Minkowski background, we can follow the same steps that allow one to extract the physical degrees of freedom from the Fierz-Pauli equations of motion (going 'on-shell'), by asking the field $\varphi_{\mu(s)}$ be transverse and traceless, so that $D_{\mu(s-1)}[\varphi] = 0$, and check that the new equations of motion enjoy a residual gauge symmetry with the parameter $\xi_{\mu(s-1)}$ being now also transverse and harmonic

$$\partial^2 \varphi_{\mu(s)} = 0\,, \qquad \partial \cdot \varphi_{\mu(s-1)} = 0\,, \qquad {\varphi'}_{\mu(s-2)} = 0\,, \tag{2.1.6a}$$

$$\partial^2 \xi_{\mu(s-1)} = 0\,, \qquad \partial \cdot \xi_{\mu(s-2)} = 0\,, \qquad {\xi'}_{\mu(s-3)} = 0\,. \tag{2.1.6b}$$

This last system takes the form of a Fierz system and from there, one can use for instance a light-cone parameterisation to obtain the equations of motion in the little group $SO(D-2)$, and realise that they describe the propagation of a massless spin-s UIR of the Poincaré group.

One can find the *rigid* symmetries of the Fronsdal field by classifying the isometries preserving the vacuum solution $\varphi_{\mu(s)} = 0$

$$\partial_\mu \xi_{\mu(s-1)} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \xi_{\mu(s-1)} = \sum_{t=0}^{s-1} \Lambda_{\mu(s-1),\nu(t)}\, \underbrace{x^\nu \cdots x^\nu}_{t}\,, \tag{2.1.7}$$

where the constant tensors $\Lambda_{\mu(s-1),\nu(t)}$ span irreducible representations of the Lorentz group $SO(1, D-1)$, parameterised by two-row Young tableaux with symmetry $\mathbb{Y}_D(s-1, t)$. The difference of lengths between the two rows $(s-t-1)$ is often called the depth. The parameters $\Lambda_{\mu(s-1),\nu(t)}$ are often called the *reducibility parameters* of the theory, and there are as many in (A)dS space-time as there are in Minkowski space-time [153, 154].

Note that, even though we started with gauge parameters that we only required to be traceless (so that the double trace of $\partial_\mu \xi_{\mu(s-1)}$ has to vanish), we found that the parameters of rigid isometries are also transverse

$$\partial \cdot \xi_{\mu(s-2)} = \sum_{t=1}^{s-1} t\, \Lambda_{\mu(s-1),\mu\nu(t-1)}\, \underbrace{x^\nu \cdots x^\nu}_{t-1} = 0\,, \tag{2.1.8}$$

and harmonic

$$\partial^2 \xi_{\mu(s-1)} = \sum_{t=2}^{s-1} t(t-1)\, \Lambda_{\mu(s-1),\nu(t-2)\alpha}{}^{\alpha}\, \underbrace{x^\nu \cdots x^\nu}_{t-2} = 0\,, \tag{2.1.9}$$

because the parameters $\Lambda_{\mu(s-1),\nu(t)}$ are completely irreducible and traceless. Thus, these parameters act also as the reducibility parameters of the Fierz system (2.1.6). This observation signals that the reducibility parameters are somewhat insensitive to the formulation we started with. Indeed, we could have started with a partially gauged-fixed version of Fronsdal's formulation [155, 17, 156] and still get the same reducibility parameters.[2] The spectrum of isometries

$$\bigoplus_{t=0}^{s-1} \mathbb{Y}_D(s-1,t)\,, \tag{2.1.10}$$

will appear again in section 2.1.2 from different considerations.

For $s = 2$, the reducibility parameters are the Killing vectors of Minkowski space $\zeta = \zeta^\mu \partial_\mu$, taking the canonical form of the differential representation for the generators of translations parameterised by Λ_μ and Lorentz transformations parameterised by $\Lambda_{\mu,\nu}$

$$\zeta_\mu = \Lambda_\mu + \Lambda_{\mu,\nu}\, x^\nu\,. \tag{2.1.11}$$

One can extract from it the *algebra* of isometries, realised as the Lie bracket of vectors ζ_1 and ζ_2

$$(\mathcal{L}_{\zeta_1}\zeta_2)^\mu = [\zeta_1,\zeta_2]^\mu = \zeta_1^\nu \partial_\nu \zeta_2^\mu - \zeta_2^\nu \partial_\nu \zeta_1^\mu\,, \tag{2.1.12}$$

and can be seen to represent the Poincaré algebra.

In this representation, one can then have a look at the action of the Poincaré algebra on the spin-s gauge parameters, given by the Lie derivative of the tensors $\xi^{\mu(s-1)}$ with $s \geq 1$ along the vectors ζ

$$(\mathcal{L}_\zeta \xi)^{\mu(s-1)} = \zeta^\nu \partial_\nu \xi^{\mu(s-1)} - (s-1)\xi^{\nu\mu(s-2)}\partial_\nu \zeta^\mu\,. \tag{2.1.13}$$

This gives a tentative definition for the Lie bracket on an algebra whose spectrum, as a vector space, is composed of generators with Young symmetries $\mathbb{Y}_D(s-1,t)$. In order to have access to the full set of structure constants, however, one has to generalise the notion of a Lie bracket to include the action of Killing tensors with $s > 2$ on themselves. One possibility is given by the Schouten-Nijenhuis bracket [162, 163, 164], which was already studied in [125] in the context of higher-spin (see also [82, 107] for a discussion)

$$\begin{aligned}[\xi_1,\xi_2]_{\mathrm{S}}{}^{\mu(s_1+s_2-3)} := k(s_1,s_2)\,\big((s_1-1)\,{\xi_1}^{\lambda\mu(s_1-2)}\,\partial_\lambda {\xi_2}^{\mu(s_2-1)} \\ -(s_2-1)\,{\xi_2}^{\lambda\mu(s_2-2)}\,\partial_\lambda {\xi_1}^{\mu(s_1-1)}\big)\,,\end{aligned} \tag{2.1.14}$$

with $k(s_1,s_2) = \frac{(s_1+s_2-3)!}{(s_1-1)!(s_2-1)!}$.

The Schouten bracket has the advantage of preserving the property of being a Killing tensor, since

$$\begin{aligned}&k(s_1,s_2)^{-1}\,\partial^\mu\,[\xi_1,\xi_2]_{\mathrm{S}}{}^{\mu(s_1+s_2-3)} = \\ &\quad (s_1-1)\,\partial^\mu\,{\xi_1}^{\lambda\mu(s_1-2)}\partial_\lambda\,{\xi_2}^{\mu(s_2-1)} - (s_2-1)\,\partial^\mu\,{\xi_2}^{\lambda\mu(s_2-2)}\,\partial_\lambda\,{\xi_1}^{\mu(s_1-1)} \\ &\quad = -\partial^\lambda\,{\xi_1}^{\mu(s_1-1)}\,\partial_\lambda\,{\xi_2}^{\mu(s_2-1)} + \partial^\lambda\,{\xi_2}^{\mu(s_2-1)}\,\partial_\lambda\,{\xi_1}^{\mu(s_1-1)} = 0\,,\end{aligned} \tag{2.1.15}$$

[2] However, unconstrained and geometric theories [38, 157, 14, 158, 159, 160, 161] are placed on a different footing from the get-go, since they require *traceful* gauge parameters to start with.

where we used $(s_1 - 1)\, \partial^\mu\, {\xi_1}^{\lambda\mu(s_1-2)} + \partial^\lambda\, {\xi_1}^{\mu(s_1-1)} = s_1\, \partial^{(\mu}\, {\xi_1}^{\lambda\mu(s_1-2))} = 0\,.$

However, in general, the Schouten bracket of traceless Killing tensors is not traceless, nor can it be decomposed into a sum of traceless Killing tensors. The failure of the Schouten bracket to define a Lie algebra with the desired spectrum does not make it a good candidate for our purpose. However, it makes it a natural candidate in the case of *unconstrained* theories [157, 14, 159, 160, 161], whose reducibility parameters are given by *all* Killing tensors, without any constraint on their trace, see, e.g., [107] for the construction of such algebras, as well as candidate non-unitary theories in flat space exhibiting this spectrum of rigid symmetries.

Even though the previous analysis was performed purely in terms of Fronsdal fields, one can learn many things regarding the global isometries of the theory and the structure of a putative Lie algebra playing the role of higher-spin symmetry. Drawing a parallel with the Cartan formulation of general relativity, the gauge field associated with the generator of translations – the vielbein – carries with it the metric tensor, while the one associated with Lorentz transformations – the spin connection – can be expressed in terms of the former through a torsion equation. Non-linear general relativity can then be obtained by constructing objects invariant under gauge transformations given by local Lorentz transformations. This is the spirit of the frame-like formulation that we will present now.

2.1.2 Frame-like formulation

Following Cartan's formulation of general relativity, we can try to reformulate the free higher-spin theory using one-forms. This is known as the frame-like formulation of higher-spin gravity and was originally developed in [57, 58]. The idea is to realise the free higher-spin theory using a gauge potential ${e_\mu}^{a(s-1)}$ and an associated parameter of gauge transformations $\xi^{a(s-1)}$ such that

$$\delta {e_\mu}^{a(s-1)} = \partial_\mu \xi^{a(s-1)}\,, \tag{2.1.1}$$

which are both in an irreducible Lorentz representation in the fibre, i.e. it is symmetric and traceless. The field ${e_\mu}^{a(s-1)}$ is a generalisation of the vielbein for $s=2$. It is easy to see how to recover a Fronsdal field from ${e_\mu}^{a(s-1)}$ by converting frame indices into space-time indices using the background vielbein ${h_\mu}^a$, and completely symmetrising over the indices μ

$$\varphi_{\mu(s)} = {h_\mu}^a \cdots {h_\mu}^a\, e_{\mu\, a(s-1)}\,. \tag{2.1.2}$$

We will work in a coordinate system such that ${h_\mu}^a = {\delta_\mu}^a$ (and therefore the background spin connection is zero) so that the background Lorentz-covariant derivative is simply the exterior derivative $\mathrm{d} = \partial_\mu \mathrm{d}x^\mu$. The same argument can be repeated in an arbitrary coordinate system, upon replacing d by the nilpotent Lorentz-covariant derivative ∇.

The field ${e_\mu}^{a(s-1)}$ being traceless in the fibre indices, the field $\varphi_{\mu(s)}$ defined in eq. (2.1.2) is naturally doubly-traceless because any double trace would necessarily involve a trace in the fibre. Moreover, the gauge variation of $e^{a(s-1)}$ imposes

$$\delta\varphi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)}\,. \tag{2.1.3}$$

Pure-gauge components

There is however an extra component in ${e_\mu}^{a(s-1)}$ corresponding to the 'hook' projection $\mathbb{Y}_D(s-1,1)$ in the decomposition of the tensor product between a vector and a completely symmetric, traceless tensor of rank $(s-1)$, involving the anti-symmetrisation

$$h^{\mu b}\,{e_\mu}^{a(s-1)} - h^{\mu a}\,{e_\mu}^{ba(s-2)} - \text{traces}\,. \tag{2.1.4}$$

Note that we are still working in the manifestly symmetric convention, so that the a indices are symmetrised and the mixed-symmetry tensor defined in eq. (2.1.4) verifies the property that a complete symmetrisation of the indices gives zero.

In order for the frame-like field ${e_\mu}^{a(s-1)}$ not to contain extra degrees of freedom as compared to a Fronsdal field, it is necessary that this component be gauged away algebraically, i.e. we impose the gauge transformation

$$\delta e^{a(s-1)} = h_b\,\lambda^{a(s-1),b}\,, \tag{2.1.5}$$

where the gauge parameter $\lambda^{a(s-1),b}$ has the same symmetries as the tensor of eq. (2.1.4). Here and in the following, we omit the one-form index μ.

The gauge parameter $\lambda^{a(s-1),b}$ is associated with a new gauge field $\omega^{a(s-1),b}$ which is a space-time one-form with the same symmetries as eq. (2.1.4) in its frame indices. By analogy with the Cartan formulation of general relativity, this field is expressed in terms of derivatives of the generalised frame field through a torsion-like equation of motion

$$T^{a(s-1)} := \mathrm{d}e^{a(s-1)} + h_b \wedge \omega^{a(s-1),b}\,, \quad T^{a(s-1)} \stackrel{!}{=} 0\,, \tag{2.1.6}$$

where d is the exterior derivative and $T^{a(s-1)}$ is a space-time two-form which is invariant under the set of gauge transformations

$$\delta e^{a(s-1)} = \mathrm{d}\xi^{a(s-1)} + h_b\,\lambda^{a(s-1),b}\,, \quad \delta\omega^{a(s-1),b} = \mathrm{d}\lambda^{a(s-1),b}\,. \tag{2.1.7}$$

Extra fields

Contrary to the spin-two case, not all of the irreducible components of $\omega^{a(s-1),b}$ are fixed by eq. (2.1.6). One can see that the components that are not fixed have the symmetry of the Young tableau $\mathbb{Y}_D(s-1,2)$, and can be gauged away algebraically using a new gauge parameter

$$\delta\omega^{a(s-1),b} = h_c\,\lambda^{a(s-1),bc}\,. \tag{2.1.8}$$

The torsion constraint (2.1.6) is invariant under the transformations (2.1.8), so that the previous discussion is not affected. In turn, this new gauge parameter is associated to a one-form $\omega^{a(s-1),b(2)}$ and one imposes

$$T^{a(s-1),b} := \mathrm{d}\omega^{a(s-1),b} + h_c \wedge \omega^{a(s-1),bc}\,, \qquad T^{a(s-1),b} \stackrel{!}{=} 0\,. \tag{2.1.9}$$

The second term in the definition of $T^{a(s-1),b}$ manifestly has the same symmetries as the first one, since $\omega^{a(s-1),ab} = 0$. This time, the expressions that make up $T^{a(s-1),b}$ in eq. (2.1.9) can be decomposed into three categories:

- The components that are shared by both $\mathrm{d}\omega^{a(s-1),b}$ and $\omega^{a(s-1),b(2)}$, and allow to express $\omega^{a(s-1),b(2)}$ as the derivative of $\omega^{a(s-1),b}$ and therefore as the second derivative of $e^{a(s-1)}$, that are given by

$$\mathbb{Y}_D(s-1,2,1) \oplus \mathbb{Y}_D(s,2) \oplus \mathbb{Y}_D(s-1,1) \oplus \mathbb{Y}_D(s-2,2)\,; \tag{2.1.10}$$

- The components of $\mathrm{d}\omega^{a(s-1),b}$ that are absent from $\omega^{a(s-1),b(2)}$, and impose a first-order equation on $\omega^{a(s-1),b}$ and therefore a second-order equation on $e^{a(s-1)}$, that are given by

$$\begin{aligned}&\mathbb{Y}_D(s-1,1,1,1) \oplus \mathbb{Y}_D(s,1,1) \oplus \mathbb{Y}_D(s-1,1) \oplus \mathbb{Y}_D(s-2,1,1)\\ &\quad \oplus \mathbb{Y}_D(s) \oplus \mathbb{Y}_D(s-2)\,;\end{aligned} \tag{2.1.11}$$

- The component of $\omega^{a(s-1),b(2)}$ that is not in $\mathrm{d}\omega^{a(s-1),b}$, that is given by

$$\mathbb{Y}_D(s-1,3)\,. \tag{2.1.12}$$

The components of the second kind precisely contain the contributions of a symmetric tensor of rank s and $s-2$, and impose Fronsdal's equation on the field $\varphi_{\mu(s)}$, while the components of the third kind have to be gauged away algebraically for the same reason as before. This will involve a series of additional gauge parameters and one-forms, starting with the ones that have the symmetry of the Young diagram $\mathbb{Y}_D(s-1,3)$ in their frame indices.

Action principle

An action [58] reproducing eq. (2.1.6) as well as all the components of eq. (2.1.9) save for the ones that are given in eq. (2.1.12) can be written, which generalises the linearised Einstein-Cartan action

$$\int \mathrm{d}^D x \left(\mathrm{d}e^{a_1 a(s-2)} + \tfrac{1}{2} h_b \wedge \omega^{a_1 a(s-2),b}\right) \wedge \omega^{a_2}{}_{a(s-2)}{}^{,a_3} \wedge K_{a_1 a_2 a_3}\,, \tag{2.1.13}$$

where $K_{a_1 a_2 a_3} := h^{a_4} \wedge \cdots \wedge h^{a_D} \varepsilon_{a_1 \cdots a_D}$ is a background $(D-3)$-form. The extra components of $\omega^{a(s-1),b}$ given by the irreducible representation $\mathbb{Y}_D(s-1,2)$ are not present in this action, hence the origin of the gauge variation (2.1.8).

Complete set of fields and curvatures

The procedure of finding the extra fields necessary to gauge away all spurious components repeats recursively, by introducing the gauge parameters $\lambda^{a(s-1),b(t+1)}$, connections $\omega^{a(s-1),b(t+1)}$ and torsion constraints $T^{a(s-1),b(t)}$ defined as

$$T^{a(s-1),b(t)} := \mathrm{d}\omega^{a(s-1),b(t)} + h_c \wedge \omega^{a(s-1),b(t)c}\,, \quad T^{a(s-1),b(t)} \stackrel{!}{=} 0\,, \tag{2.1.14}$$

for $2 \leq t \leq s-2$, invariant under the transformations

$$\delta\omega^{a(s-1),b(t)} = \mathrm{d}\lambda^{a(s-1),b(t)} + h_c\,\lambda^{a(s-1),b(t)c}\,, \tag{2.1.15}$$

until the final spin-connection is eventually reached, corresponding to a field whose symmetry in the frame indices is encoded by the rectangular Young diagram $\mathbb{Y}_D(s-1,s-1)$. This yields the following complete set of one-forms

$$e^{a(s-1)}\,, \quad \omega^{a(s-1),b}\,, \quad \dots\,, \quad \omega^{a(s-1),b(s-2)}\,, \quad \omega^{a(s-1),b(s-1)}\,, \tag{2.1.16}$$

and gauge parameters

$$\xi^{a(s-1)}\,, \quad \lambda^{a(s-1),b}\,, \quad \dots\,, \quad \lambda^{a(s-1),b(s-2)}\,, \quad \lambda^{a(s-1),b(s-1)}\,. \tag{2.1.17}$$

Note the perfect matching with the spectrum identified in eq. (2.1.10). The last field is identified (on-shell) with $(s-1)$ derivatives of the Fronsdal field, and one can impose an equation encoding the vanishing of the trace of its curvature, or equivalently by projection onto its purely traceless two-row component

$$R^{a(s-1),b(s-1)} := \mathrm{d}\omega^{a(s-1),b(s-1)}\,, \quad R^{a(s-1),b(s-1)} \stackrel{!}{=} h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}\,, \tag{2.1.18}$$

where the zero-form $C^{a(s),b(s)}$ represents the spin-s Weyl tensor, is completely irreducible and gauge-invariant. This is similar to the rewriting of the linearised Einstein's equations by equating the curvature of the spin-connection (on-shell the Riemann tensor) to its Weyl part as explained near eq. (1.1.4).

For completeness, we recall the complete form of the linearised equations

$$T^{a(s-1),b(t)} = \mathrm{d}\omega^{a(s-1),b(t)} + h_c \wedge \omega^{a(s-1),b(t)c} \stackrel{!}{=} 0\,, \tag{2.1.19a}$$

$$R^{a(s-1),b(s-1)} = \mathrm{d}\omega^{a(s-1),b(s-1)} \stackrel{!}{=} h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}\,, \tag{2.1.19b}$$

which are a cornerstone of the procedure known as *unfolding* (for a review as well as applications to other systems see, e.g. [64, 165, 166, 167]). Note that integrability of the previous set of equations yields the Bianchi identities

$$\mathrm{d}T^{a(s-1),b(t)} = -h_c \wedge T^{a(s-1),b(t)c}\,, \tag{2.1.20}$$

for $t \in \{0, \dots, s-3\}$, where we used $\mathrm{d}^2 = 0$, $\mathrm{d}h^a = 0$ and $h_c \wedge h_d \wedge \omega^{a(s-1),b(t-1)cd} = 0$ due to the anti-symmetry of $h_c \wedge h_d$. For $t = s-2$, we get

$$\mathrm{d}T^{a(s-1),b(s-2)} = -h_c \wedge R^{a(s-1),b(s-2)c}\,. \tag{2.1.21}$$

The equations (2.1.19a) and (2.1.19b) are compatible with the Bianchi identities, since $h_d \wedge h_e \wedge h_c\, C^{a(s-1)d,b(s-2)ce} = 0$. The compatibility of the equation $R^{a(s-1),b(s-1)} = h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}$ has to be studied separately and extends into the 'zero-form' sector as

$$\mathrm{d}R^{a(s-1),b(s-1)} = 0 \quad \Longrightarrow \quad \mathrm{d}C^{a(s),b(s)} = h_c \wedge C^{a(s)c,b(s)}\,, \tag{2.1.22}$$

with $C^{a(s+1),b(s)}$ another Lorentz-irreducible zero-form.[3] The unfolding of the dynamics then continues in the zero-form sector, whose description goes beyond the scope of this thesis.

AdS space-time

In the presence of a non-zero cosmological constant, the previous discussion is slightly modified by the introduction of extra terms. Like in the case of linearised (A)dS gravity, these terms are necessary to ensure gauge invariance of the equations of motion and their integrability, due to the fact that the (A)dS background covariant derivative, defined as

$$\nabla := \mathrm{d} + \varpi\,, \quad \text{i.e.} \quad \nabla X^a{}_b = \mathrm{d}X^a{}_b + \varpi_c{}^a \wedge X^c{}_b - \varpi^c{}_b \wedge X^a{}_c\,, \tag{2.1.23}$$

for an arbitrary p-form $X^a{}_b$, does not commute with itself. The background AdS vielbein and spin connection ϖ takes the local expression through the definition

$$\nabla h_a = 0\,. \tag{2.1.24}$$

For instance

$$\nabla^2 e^{a(s-1)} = -\frac{s-1}{R^2}\, h^a \wedge h_b \wedge e^{a(s-2)b}\,, \tag{2.1.25}$$

with R the AdS radius. All in all, the definitions of eqs. (2.1.14) and eq. (2.1.18) become

$$T^{a(s-1),b(t)} := \nabla\omega^{a(s-1),b(t)} + h_c \wedge \omega^{a(s-1),b(t)c} + \tfrac{\beta_{s,t}}{R^2}\, h^{\{b} \wedge \omega^{a(s-1),b(t-1)\}}\,, \tag{2.1.26}$$

for $0 \leq t \leq s-1$, where $\omega^{a(s-1)}$ has to be understood as $e^{a(s-1)}$ and $T^{a(s-1),b(s-1)}$ has to be understood as $R^{a(s-1),b(s-1)}$. Eq. (2.1.26) involves a certain coefficient $\beta_{s,t}$ and the braces denote a Young projection (valid for $D \geq 4$) such that the new term has the same symmetries as the other ones

$$\begin{aligned}
&(s-t+1)\, h^{\{b} \wedge \omega^{a(s-1),b(t-1)\}} \\
&\quad := (s-t)\, h^b \wedge \omega^{a(s-1),b(t-1)} - (s-1)\, h^a \wedge \omega^{a(s-2)b,b(t-1)} \\
&\qquad - \tfrac{(s-1)(t-1)}{D+s+t-5}\, \eta^{ab}\, h_c \wedge \left(\tfrac{s-t-1}{t-1}\, \omega^{a(s-2)c,b(t-1)} - \tfrac{D+2s-6}{D+2t-6}\, \omega^{a(s-2)b,b(t-2)c}\right) \\
&\qquad + \tfrac{(s-1)(s-2)}{D+s+t-5}\, \eta^{aa}\, h_c \wedge \left(\omega^{a(s-3)bc,b(t-1)} - \tfrac{t-1}{D+2t-6}\omega^{a(s-3)b(2),b(t-2)c}\right) \\
&\qquad + \tfrac{(s-t)(t-1)}{D+2t-6}\, \eta^{bb} h_c \wedge \omega^{a(s-1),b(t-2)c}\,.
\end{aligned} \tag{2.1.27}$$

[3] As explained in e.g. [121], the component of $\mathrm{d}C^{a(s),b(s)}$ whose Young diagram has three rows is killed as a consequence of eq. (2.1.19b), which is also the case of its trace, so that the only surviving component is a Lorentz-irreducible tensor $C^{a(s+1),b(s)}$ with two rows of length $(s+1)$ and s.

This unpalatable expression is referred to in the literature as σ_+ , since it performs the necessary projection to add an index b in the second row. Similarly, the term $h_c \wedge \omega^{a(s-1),b(t)c}$ removes one index in the second row and is therefore called σ_- .

The coefficient $\beta_{s,t}$ is such that $\beta_{s,0} = 0$ and $\beta_{s,1} = s\,\frac{D+s-4}{D-2}$. The precise expression of $\beta_{s,t}$ for general t is rather tedious to obtain and not strictly necessary here so we shall refrain from giving it, but it is completely fixed by the requirement of gauge invariance in AdS and can be found, e.g. in [64] for $D = 4$.

Initial data for a gauge algebra

This is the full description of the linearised dynamics. We now want to reformulate the problem of adding interactions into an algebraic one. The first step is to identify the set of 'initial data' that a higher-spin algebra must satisfy in order to reproduce the free dynamics in the way that we explained above. More explicitly, we wish to interpret the previous torsions and curvatures (2.1.26) as the different components of the field strength of a linearised Yang-Mills connection taking value in a Lie algebra that is yet to be determined.

From the form of the field strengths (torsions and curvature), one can read off some structure constants of the putative higher-spin symmetry algebra. These correspond to the Lie brackets of the Poincaré or (A)dS sub-algebra spanned by J_{ab} and P_a with the higher-spin generators $M_{a(s-1),b(t)}$ with $s \geq 3$ and $0 \leq t \leq s-1$

$$\begin{aligned}
[J_{cd}, M_{a(s-1),b(t)}] &= (s-1)\left(\eta_{da}\, M_{ca(s-2),b(t)} - \eta_{ca}\, M_{da(s-2),b(t)}\right) \\
&\quad + t\left(\eta_{db}\, M_{a(s-1),cb(t-1)} - \eta_{cb}\, M_{a(s-1),db(t-1)}\right) , &\text{(2.1.28a)}\\
[P_c, M_{a(s-1),b(t)}] &= \eta_{c\{b}\, M_{a(s-1),b(t-1)\}} + \tfrac{\beta_{s,t+1}}{R^2}\, M_{a(s-1),b(t)c} , &\text{(2.1.28b)}
\end{aligned}$$

where the shift $\beta_{s,t} \to \beta_{s,t+1}$ in the last term between eq. (2.1.26) and eq. (2.1.28b) is a consequence of the transition from curvature to structure constant, and where braces also denote a Young projection similar to the one of eq. (2.1.27)

$$\begin{aligned}
&(s-t+1)\, \eta_{c\{b}\, M_{a(s-1),b(t-1)\}} \\
&\quad := (s-t)\, \eta_{cb}\, M_{a(s-1),b(t-1)} - (s-1)\, \eta_{ca}\, M_{a(s-2)b,b(t-1)} \\
&\quad - \tfrac{(s-1)(t-1)}{D+s+t-5}\, \eta_{ab}\left(\tfrac{s-t-1}{t-1}\, M_{a(s-2)c,b(t-1)} - \tfrac{D+2s-6}{D+2t-6}\, M_{a(s-2)b,b(t-2)c}\right) \qquad \text{(2.1.29)}\\
&\quad + \tfrac{(s-1)(s-2)}{D+s+t-5}\, \eta_{aa}\left(M_{a(s-3)bc,b(t-1)} - \tfrac{t-1}{D+2t-6} M_{a(s-3)b(2),b(t-2)c}\right) \\
&\quad + \tfrac{(s-t)(t-1)}{D+2t-6}\, \eta_{bb}\, M_{a(s-1),b(t-2)c} .
\end{aligned}$$

Note that this time, the limit $R \to \infty$ in the Lie brackets takes away the simplest term, while the only remaining term is the one involving a trace and a Young projection. There is a slight abuse of language when talking about the limit of a dimensionful parameter such as the AdS radius. The correct way to present things, which we will adopt in the following, is to define the AdS radius as $R = \epsilon^{-1}\hat{R}$ with a dimensionless parameter ϵ and to send ϵ to 0 while keeping $\hat{R}$ fixed. We will see that this distinction will become of capital importance when discussing the flat contraction of the AdS higher-spin algebra, since the contraction parameter ϵ will not appear with the AdS radius R in a pairwise manner.

The commutation relations (2.1.28) define in themselves a Lie algebra satisfying the Jacobi identity, albeit one that does not give rise to an interacting theory beyond the coupling of higher-spin fields to gravity. However, it is interesting to note that the Lie derivative of the traceless Killing tensors of AdS (or Minkowski space when $R \to \infty$) along the corresponding Killing vectors is a tensor representation of this Lie algebra. Although handy, this representation is not suited to our goal of constructing a non-Abelian algebra satisfying the initial conditions of (2.1.28), so we will bypass the representation given in terms of Killing tensors and Lie derivatives and work directly with the generators and the Lie bracket.

2.2 Higher-spin algebras for massless theories in (A)dS

In the metric-like formulation, one can introduce interactions perturbatively starting from the free theory. This is usually called Nœther procedure, and is reviewed for higher-spin fields e.g. in [168]. As an example, a self-interaction term for a spin-s field $\varphi_{\mu(s)}$ deforms the free equations of motion by terms quadratic in $\varphi_{\mu(s)}$

$$F_{\mu(s)}[\varphi] \stackrel{!}{=} g\,\mathcal{O}(\varphi^2)_{\mu(s)}\,, \tag{2.2.1}$$

and deforms the gauge transformations as well by terms linear in $\varphi_{\mu(s)}$ and $\xi_{\mu(s-1)}$

$$\delta\varphi_{\mu(s)} = \partial_\mu \xi_{\mu(s-1)} + g\,\mathcal{O}(\varphi, \xi)_{\mu(s)}\,. \tag{2.2.2}$$

Still on the example of the self-interacting spin-s field, performing two gauge variations and commuting the order, one remarks that the deformed gauge transformations can become non-Abelian, in the sense that the right-hand side of

$$[\delta_{\xi_1}, \delta_{\xi_2}]\varphi_{\mu(s)} = g\,\mathcal{O}(\delta_{\xi_2}\varphi, \xi_1)_{\mu(s)} - (1 \leftrightarrow 2)\,, \tag{2.2.3}$$

can be non-zero. In the case of spin-one, non-Abelian deformations can arise only if the fields take value in a matrix group like $SU(N)$. Given this group, one can reconstruct full Yang-Mills theory in an unequivocal way. In the case of spin-two self interactions, asking that the deformation of gauge transformations at cubic, quartic etc. levels close on the Poincaré or (A)dS algebra leads to a non-empty set of vertices. At each order, interactions are fully determined and the whole procedure re-sums to the Einstein-Hilbert theory of gravity [169, 170, 171, 172, 173]. The application of this philosophy to higher-spin fields can be broadly denominated as the ‘gauge’ approach to interacting higher-spin theories and is reviewed, e.g., in [174, 175, 69, 56, 121, 176, 123]. The classification of allowed cubic vertices and the associated gauge transformations in Minkowski space-time shows that interactions involving Fronsdal fields are higher-derivative[4] (for instance, $s-s-2$ vertices have $2s-2$ derivatives), and that some gauge transformations are indeed non-Abelian. All of this hints at the existence of a non-Abelian higher-spin algebra underlying the whole construction.[5]

In (A)dS space, we will recall the conditions under which the cubic gravitational vertex of Fradkin and Vasiliev defines a unique Lie algebra which satisfies the Jacobi identity. Moreover, this Lie algebra was shown to be associative (therefore it automatically satisfies the Jacobi identity), and which is unique except in $D=5$ [62] and up to deformations.

The hypotheses that we will work with are the following:

- the higher-spin algebra should have a spectrum which, as a vector space, is given by a set of generators whose Young symmetry was identified in eq. (2.1.10) and that we will denote by $M_{a(s-1),b(t)}$, representing the rigid isometries of massless fields of every spin ;
- the algebra should be a non-Abelian extension of the Poincaré (or (A)dS) algebra, meaning that gauge transformations are deformed at every spin ;
- the action of the Lorentz sub-algebra is the usual adjoint one: generators transform under the action of the Lorentz group as is specified by their tensorial representation, see eq. (2.1.28a) ;
- it should contain the isometries of the vacuum (i.e. Poincaré or (A)dS) as a sub-algebra .

To this set of hypotheses, we will add a couple more:

- the Lie bracket of translations with higher-spin generators in AdS is given by eq. (2.1.28b) ;
- there should be at least one Lie bracket $\left[M_{a(s-1),b(t_1)}, M_{c(s-1),d(t_2)}\right]$ such that the right-hand side contains a generator J_{ab} or P_a for every value of s .

[4] A complete classification of cubic vertices in light-cone was performed in [49] which exhibits $(2s-2)$, $(2s+2)$ and 2-derivative couplings with gravity. The latter class can not be seen within the Fronsdal formulation and only the first leads to a deformation of gauge transformations [130, 55]

[5] A Lagrangian formulation is not strictly necessary to construct an interacting theory, but if one wants to quantise it using the Feynman rules, it is a desirable feature. However, an underlying algebra is always necessary to have a consistent theory. Indeed, in order to build quartic interactions, the cubic vertices must satisfy a consistency condition which is that the structure constants of the algebra they define satisfy the Jacobi identity.

This first extra hypothesis seems reasonable to be able to reproduce the free dynamics as explained in section 2.1.2, while the second one is more of a technical nature. In mathematical terms, we want to make sure that we define a filtered algebra and not a graded one. From a filtered algebra, one can always perform a contraction that makes it graded, and inversely, from a graded algebra one may introduce deformations that will make it filtered. The physical interpretation is to insist on the fact that higher-spin fields have a *back-reaction* on the graviton.

In the original paper of Fradkin and Vasiliev [60], such an algebra was constructed in dS_4, in spinor language, by classifying all the structure constants that satisfy the Jacobi identity. This construction gives a unique result in $(A)dS_4$, but none in flat space. Moreover, it was remarked that all İnönü-Wigner contractions of the algebra $\mathfrak{hs}_4$ admitting a Poincaré sub-algebra violate the hypothesis on the form of the Lie brackets with P_a .

In [65, 67, 70], the construction in generic D was performed. In [82, 168], the problem of finding an algebra in the flat case was performed again using different techniques, and again no solution was found. In [62], it was proven that the $(A)dS_D$ algebra $\mathfrak{hs}_D$ is essentially unique in any $D = 4$ and $D \geq 6$, while the case $D = 5$ brings an extra parameter.

Before showing the explicit construction of the algebra *per se*, let us point out a certain number of properties that it satisfies:

- it verifies the usual rules for the addition of angular momenta

$$|s\rangle \otimes |s'\rangle = |s_{\max}\rangle \oplus |s_{\max} - 2\rangle \oplus \cdots \oplus |s_{\min} + 2\rangle \oplus |s_{\min}\rangle \,, \tag{2.2.4}$$

  with $s_{\max} = s+s'$ and $s_{\min} = |s-s'|$, where the spin of a higher-spin generator (i.e. the representation of the Lorentz group that it carries) is shifted by one with respect to the spin of the corresponding field. In practice, this means that the Lie brackets of generators of spin s with generators of spin s' can be decomposed into generators of spin $s_{\max} - 2$, $s_{\max} - 4$, $\dots$, $s_{\min} + 4$, $s_{\min} + 2$;

- it admits super-symmetric generalisations, see e.g. [177, 66, 178], which are non-Abelian algebras for the rigid symmetries of fermionic higher-spin fields described by the equations of motion of Fang and Fronsdal [179, 180] ;

- it is associative .

This last point is non-trivial. The fact that the algebra is associative is something extra that we did not ask for. It signals the existence of an extra structure whose origin can be traced back to the higher-spin algebra being an algebra of non-Abelian gauge transformations, which is always associative under composition of gauge transformations. Moreover, as we will see in section 3.1.2, this higher-spin algebra (like all known higher-spin algebras) can be realised as the algebra of higher symmetries of a differential equation, which is also always associative under composition of differential operators.

In the following, we will present the construction of the AdS higher-spin algebra $\mathfrak{hs}_D$ in arbitrary dimension D , and then focus on the case of $D = 3$ separately. Even though massless higher-spin fields in three space-time dimensions propagate no degrees of freedom, this case is interesting for multiple reasons. Firstly, the higher-spin algebra admitting a one-parameter extension $\mathfrak{hs}_3[\lambda]$ is substantially simpler than the case of generic dimension, because it contains only two types of generators at each spin. Secondly, it admits finite-dimensional truncations to the double copy of the well-known matrix algebras $\mathfrak{sl}(N, \mathbb{R})$, which makes their structure constants more tractable. Thirdly, it remains the best-understood case yet where a flat contraction of the algebra exists and successfully led to the construction of interacting, albeit topological, higher-spin theories in three-dimensional Minkowski space-time.

Let us also note that the method of employing the quotient of a UEA to construct interacting higher-spin algebra has also proven useful for other theories than the one of massless fields: algebras for partially-massless higher-spin theories [181] and theories with mixed-symmetry fields [182] can be defined along the same lines.

2.2.1 Any dimensions

The algebra of Fradkin and Vasiliev was built by exploiting the special isomorphism between the universal cover of the Lorentz group and $SL(2, \mathbb{C})$, allowing one to use spinorial language and picking a particular oscillator representation for the (A)dS$_4$ algebra. Indeed, the oscillator representation used in [60] provides a simple way of building higher-spin generators which reproduces automatically the desired spectrum. We will not reproduce its construction here, preferring instead to present the general case. In modern language, we are representing the *Universal Enveloping Algebra* of AdS$_4$ thanks to a particular module.

One can understand the idea of the UEA construction as follows: there is a one-to-one correspondence between Killing tensors on constantly curved manifolds and higher products of Killing vectors [183] (the conformal version of this statement can be found in [65]). In other words, Killing tensors solutions of eq. (2.1.7) can be expressed as products of Killing vectors with the appropriate Young projection.

Rather than picking a particular representation for the generators of the higher-spin algebra, this can be formulated in algebraic terms by defining the higher-spin algebra as a suitable quotient of the UEA of the algebra of space-time isometries. The UEA of a Lie algebra $\mathfrak{g}$, called here $\mathcal{U}(\mathfrak{g})$ is constructed by introducing an associative product $\star$ that will be omitted in the following, asking that

$$[a, b] = a \star b - b \star a \,, \tag{2.2.1}$$

for any $a, b \in \mathfrak{g}$. Since the $\star$-product of two elements in $\mathcal{U}(\mathfrak{g})$ can be written as a symmetrised product and a commutator, elements of $\mathcal{U}(\mathfrak{g})$ can be represented by symmetrised products using the anti-commutator

$$a \odot b = a \star b + b \star a \,, \tag{2.2.2}$$

which is also a consequence of the Poincaré-Birkhoff-Witt theorem, see, e.g., the review part in [182].

We will resort to ambient space in order to simplify this construction. From now on, we will also work in AdS_D as it will be more convenient than dS_D to talk about holography in the later parts, and the dS_D case can be recovered by a Wick rotation. One may view the bulk of AdS_D space-time with radius R as embedded in an ambient space of one dimension more $\mathbb{R}^{2,D-1}$

$$\mathrm{AdS}_D := \left\{ X^2 = -R^2 \,\middle|\, X \in \mathbb{R}^{2,D-1} \right\}, \tag{2.2.3}$$

where the metric η_{AB} on $\mathbb{R}^{2,D-1}$ is the flat one with signature $(-,+,\ldots,+,-)$, i.e.

$$X^2 = \eta_{AB} X^A X^B = -(X_0)^2 - (X_D)^2 + (X_1)^2 + \cdots + (X_{D-1})^2 \,. \tag{2.2.4}$$

The generators of the algebra of ambient space isometries preserving a particular AdS sub-manifold are given by (the imaginary unit times) the Lorentz transformations of $\mathbb{R}^{2,D-1}$

$$J_{AB} = X_A\, \partial_B - X_B\, \partial_A \,, \tag{2.2.5}$$

and verify the algebra

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} + \eta_{AD} J_{BC} \,. \tag{2.2.6}$$

The usual transvections and Lorentz transformations can be identified by choosing a time-like vector in $\mathbb{R}^{2,D-1}$, say $\frac{\partial}{\partial X^D}$, and write the decomposition of the generators with respect to this direction $J_{AB} = (J_{ab},\ J_{a\,D}) := (J_{ab},\ R\,P_a)$

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc} \,, \tag{2.2.7a}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \,, \tag{2.2.7b}$$
$$[P_a, P_b] = R^{-2} J_{ab} \,, \tag{2.2.7c}$$

where $a \in \{0,\, \ldots,\, D-1\}$. The advantage of this formulation is that the spectrum of higher-spin generators greatly simplifies when the latter are grouped in ambient space tensors. Indeed, using the branching rules

$$\mathbb{Y}_{D+1}(s-1, s-1) \quad \xrightarrow[D+1\to D]{} \quad \bigoplus_{t=0}^{s-1} \mathbb{Y}_D(s-1, t) \,, \tag{2.2.8}$$

we can see that the whole collection of Killing tensors (2.1.10) can be grouped together in a single generator $\mathcal{M}_{A(s-1),B(s-1)}$ with two-row rectangular Young projection. Focusing on the coset construction, we want to define a quotient of the

UEA of $\mathfrak{so}(2,D-1)$ such that only the generators $\mathcal{M}_{A(s-1),B(s-1)}$ are present. In particular, we need to get rid of generators whose Young symmetry are given by tableaux with more than two rows. This is done by identifying an *ideal* of the UEA, that is a subalgebra $\mathfrak{k}$ such that $[\mathfrak{k},\mathcal{U}(\mathfrak{so}(2,D-1))]\subset\mathfrak{k}$. In the following, we will focus on the the quadratic sector of $\mathcal{U}(\mathfrak{so}(2,D-1))$ and identify those quadratic combinations that should not appear in the higher-spin algebra. The requirement of compatibility and the property of being an ideal will uniquely fix the whole construction.

The ambient space construction allows to define the higher-spin algebra $\mathfrak{hs}_D$ in a compact way by quotienting the UEA of $\mathcal{U}(\mathfrak{so}(2,D-1))$ by a two-sided ideal spanned by two quadratic elements

$$\mathfrak{hs}_D := \frac{\mathcal{U}(\mathfrak{so}(2,D-1))}{\langle \mathcal{I}_{[ABCD]}\oplus\mathcal{I}_{(AB)}\rangle}\,, \tag{2.2.9}$$

where $\langle\cdot\rangle$ stands for the left- and right- multiplication by $\mathcal{U}(\mathfrak{so}(2,D-1))$ using the associative product. The two-sided ideal is generated by $\mathcal{I}_{[ABCD]}$ and $\mathcal{I}_{(AB)}$, which take the following expressions

$$\mathcal{I}_{[ABCD]} := J_{[AB}\odot J_{CD]}\,, \tag{2.2.10a}$$
$$\mathcal{I}_{(AB)} := J^{C}{}_{(A}\odot J_{B)C} - \tfrac{4}{D+1}\,\eta_{AB}\,C_2\,, \tag{2.2.10b}$$

where the first element is completely anti-symmetric and the second one is symmetric and traceless. We defined the quadratic Casimir of $\mathfrak{so}(2,D-1)$ as

$$C_2 := \tfrac{1}{4}J_{AB}\odot J^{BA} = \tfrac{1}{2}J_{AB}J^{BA}\,. \tag{2.2.11}$$

The only other independent quadratic combination is

$$\begin{aligned}\mathcal{K}_{AB,CD} = J_{C(A}\odot J_{B)D} &- \tfrac{1}{D-1}\big(\eta_{AB}\,\mathcal{I}_{CD}+\eta_{CD}\,\mathcal{I}_{AB}-2\,\eta_{C(A}\,\mathcal{I}_{B)D}\big)\\ &- \tfrac{4}{D(D+1)}\big(\eta_{AB}\,\eta_{CD}-\eta_{C(A}\,\eta_{B)D}\big)\,C_2\,,\end{aligned} \tag{2.2.12}$$

which is completely traceless and irreducible. This latter is a good candidate to be the generator of rigid isometries of a spin-three field, described in section 2.1.2, owing to the branching rule $\mathbb{Y}_{D+1}(2,2)\longrightarrow\mathbb{Y}_D(2,2)\oplus\mathbb{Y}_D(2,1)\oplus\mathbb{Y}_D(2)$. This pattern is reproduced for higher-order expressions, so that the only remaining generators carry a rectangular two-row irreducible representation of the conformal group.

At first sight, one could think that the generator C_2 is still unconstrained, but the requirement that $\mathcal{I}_{[ABCD]}$ and $\mathcal{I}_{(AB)}$ define an ideal in $\mathcal{U}(\mathfrak{so}(2,D-1))$ forces it to assume a fixed expression, which can be seen by the identity

$$\frac{3}{4}\mathcal{I}_{[ABCD]}J^{CD}+\frac{1}{2}\mathcal{I}_{C[A}J_{B]}{}^{C}+\frac{D-1}{D+1}\left(C_2+\frac{(D+1)(D-3)}{4}\,\mathrm{id}\right)J_{AB}\equiv 0\,, \tag{2.2.13}$$

where id represents the identity generator. Factoring out $\mathcal{I}_{[ABCD]}$ and $\mathcal{I}_{(AB)}$ forces one to factor out $\left(C_2+\frac{(D+1)(D-3)}{4}\,\mathrm{id}\right)J_{AB}$ as well. If we do not want to factor out

J_{AB} and therefore trivialise the whole algebra, we are forced to adopt the following expression for the quadratic Casimir

$$C_2 = -\frac{(D+1)(D-3)}{4}\,\mathrm{id}\,. \tag{2.2.14}$$

The eigenvalue of C_2 matches the one of a unitary irreducible representation of the conformal algebra called the *singleton*, that we will discuss in section 3. Retrospectively, Schur's lemma confirms that the quadratic Casimir C_2 must be proportional to the identity generator when its UEA is evaluated on an irreducible module. At every order in the symmetrised product of $\mathfrak{so}(2, D-1)$ generators, one can show that we obtain more elements which are all factored out except for a generator with a two-row rectangular Young symmetry, indicating that the correct spectrum has been reached.

Up to now, we have only showed that eq. (2.2.14) is the value of the Casimir which is compatible with the quotient of the elements $\mathcal{I}_{[ABCD]}$ and $\mathcal{I}_{(AB)}$. When moving higher and higher in the tensor algebra, there will be other consistency requirements coming from the compatibility of the quotient with higher products appearing at each order, e.g. $\mathcal{I}_{[ABCD]}\,\mathcal{I}^{[CDEF]}\ldots$, establishing non-trivial relations between all higher-order Casimir operators C_2 , C_4 , etc. It is remarkable that all these relations are satisfied at the 'singleton point' defined by eq. (2.2.14), see, e.g. [182].

When splitting the generators into Lorentz-irreducible pieces, we obtain an algebra which displays the desired spectrum and the Lorentz-covariance property, and one can read off the structure constants of the commutators $[P_a, M_{b(s-1),c(t)}]$, for instance by a meticulous application of the Leibniz rule, and see that it verifies eq. (2.1.28b) as well.

It was shown [62] that $\mathfrak{hs}_D$ is the only admissible algebra in $D=4$ and $D \geq 6$ reproducing known interactions. In $D=5$, a one-parameter family of deformations exists [182, 184], due to the exceptional isomorphism $\mathfrak{so}(2,4) \simeq \mathfrak{su}(2,2)$ and the fact that the UEA of the $\mathfrak{sl}(N,\mathbb{R})$ family of algebras admits a one-parameter deformation [185].[6]

For the same reason, in $D=3$ a one-parameter family, called $\mathfrak{hs}_3[\lambda]$, also exists due to the isomorphism $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which we will now review. The algebras $\mathfrak{hs}_3[\lambda]$ are infinite-dimensional generalisations of the $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ algebras, where the real parameter λ interpolates between integral values of N . These algebras can be described by applying the same construction as in any D , with a slight relaxation on the factoring out of $\mathcal{I}_{ABCD}$ [134, 150].

[6]Remember that $\mathfrak{sl}(4,\mathbb{R})$ and $\mathfrak{su}(2,2)$ are two real forms of $\mathfrak{sl}(4,\mathbb{C})$.

2.2.2 Three dimensions

The isometry algebra $\mathfrak{so}(2,2)$ is isomorphic to two copies of $\mathfrak{so}(2,1)$, which are in turn isomorphic to two copies of $\mathfrak{sl}(2,\mathbb{R})$. To see this isomorphism in action, let us start with the generators J_{AB} of $\mathfrak{so}(2,2)$, with $A \in \{0,1,2,3\}$. Let us pose

$$m_{ab} := J_{ab}\,, \quad p_a := J_{a3}\,, \tag{2.2.1}$$

where $a,b \in \{0,1,2\}$ and m_{ab} can be dualised into

$$j^a := \tfrac{1}{2}\,\varepsilon^{abc}m_{bc} \quad \Longleftrightarrow \quad m_{ab} = -\varepsilon_{abc}j^c\,, \tag{2.2.2}$$

using the Levi-Civita tensor. For the latter we adopt the convention $\varepsilon^{012} = 1$ and use the metric $\eta = (-,+,+)$ to raise or lower indices. In this basis, the commutation relations read

$$[j_a, j_b] = \varepsilon_{abc}j^c\,, \quad [j_a, p_b] = \varepsilon_{abc}p^c\,, \quad [p_a, p_b] = \varepsilon_{abc}j^c\,. \tag{2.2.3}$$

One can rearrange the components of p_a and j_a into the generators

$$P_m := (p_0 - p_1,\, p_2,\, p_0 + p_1)\,, \quad L_m := (j_0 - j_1,\, j_2,\, j_0 + j_1)\,, \tag{2.2.4}$$

where $m \in \{-1,0,1\}$. This gives the commutation relations of two intertwined copies of $\mathfrak{sl}(2,\mathbb{R})$

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}\,,\\
[L_m, P_n] &= (m-n)P_{m+n}\,,\\
[P_m, P_n] &= (m-n)L_{m+n}\,.
\end{aligned} \tag{2.2.5}$$

One can eventually identify two orthogonal copies of $\mathfrak{sl}(2,\mathbb{R})$ by introducing the linear combinations defining a $\mathbb{Z}_2$-grading

$$\mathcal{L}_m = \frac{1}{2}\,(L_m + P_m)\,, \quad \bar{\mathcal{L}}_m = \frac{1}{2}\,(L_m - P_m)\,, \tag{2.2.6}$$

verifying

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}\,, \quad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0\,. \tag{2.2.7}$$

The quadratic Casimir operator of $\mathfrak{so}(2,2)$ introduced in eq. (2.2.11) is then expressed in terms of the Casimir operators of the two orthogonal $\mathfrak{sl}(2,\mathbb{R})$ algebras as

$$C_2 = L^2 + P^2 = 2\left(\mathcal{L}^2 + \bar{\mathcal{L}}^2\right), \tag{2.2.8}$$

where we used the components of the inverse of the $\mathfrak{sl}(2,\mathbb{R})$ Killing metric γ_{mn} to contract indices, e.g., $L^2 = \gamma^{mn}L_mL_n$ and $\mathcal{L}^2 = \gamma^{mn}\mathcal{L}_m\mathcal{L}_n$, with the convention

$$\gamma_{mn} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}. \tag{2.2.9}$$

In three space-time dimensions, the higher-spin algebra takes a much simpler form, since the generators $\mathbb{Y}_3(s-1,t)$ with $s>2$ and $t>1$ do not exist due to Young symmetry. This means that for every spin $s\geq 2$, there are only two generators, with symmetries $\mathbb{Y}_3(s-1)$ and $\mathbb{Y}_3(s-1,1)$, which will be interpreted as higher-spin generators extending translations and Lorentz transformations respectively. Moreover, the second one can be dualised by means of the three-dimensional Levi-Civita tensor ε_{abc} to give again a generator with complete symmetry in its indices.

Carrying on with the construction of the previous section, it can be proven that

$$\mathcal{I}_{(AB)}\sim 0 \quad\Longleftrightarrow\quad \mathcal{L}_m\bar{\mathcal{L}}_n\sim 0\,, \tag{2.2.10}$$

where $\sim$ is the equivalence relation defined by the factoring out of the ideal and where we omitted the anti-commutator since the two copies of $\mathfrak{sl}(2,\mathbb{R})$ naturally commute, see [107]. This explains why, in the construction of higher-spin algebras in three dimensions, it is enough to consider the direct sum of two copies of the UEA of $\mathfrak{sl}(2,\mathbb{R})$, since all mixed products are factored out, see, e.g., [134] and the review [150].[7] In terms of L_m and P_m generators, the relations (2.2.10) read, see [107]

$$L_m\,P_n-P_m\,L_n\sim 0\,,\quad P_m\,P_n-L_m\,L_n\sim 0\,,\quad L^2+P^2-\frac{\lambda^2-1}{2}\,id\sim 0\,. \tag{2.2.11}$$

On the other hand, contrary to the generic case, in three dimensions one does not need to factor out the element $\mathcal{I}_{[ABCD]}$. One can dualise it instead into a singlet, using the ambient space Levi-Civita symbol ε^{ABCD}. Although a singlet fits in the vector space of global symmetries of a massless spin-one field as the $U(1)$ phase transformation, the UEA construction already provides us with such a singlet: the identity. The two, however, do not match, since

$$W:=\frac{1}{16}\,\varepsilon^{ABCD}\mathcal{I}_{ABCD}=j^a p_a=\gamma^{mn}L_mP_n=\mathcal{L}^2-\bar{\mathcal{L}}^2\,, \tag{2.2.12}$$

while $C_2=2\left(\mathcal{L}^2+\bar{\mathcal{L}}^2\right)\propto\text{id}$. Moreover, factoring out $\mathcal{I}_{AB}$ implies a relation between W and C_2

$$W^2\sim\tfrac{1}{4}\,(C_2)^2\,. \tag{2.2.13}$$

Imposing $\mathcal{I}_{[ABCD]}=\frac{2}{3}\,\varepsilon_{ABCD}W\sim 0$ as in section 2.2.1 would imply $C_2\sim 0$, consistently with the D-dimensional result (2.2.14). However, we will choose to work with the weaker condition (2.2.13) that leaves the choice of the eigenvalue of the quadratic Casimir C_2 free. For the latter, we will use the convenient[8] parameterisation

$$C_2\sim\frac{\lambda^2-1}{2}\,\text{id}\,. \tag{2.2.14}$$

[7] An alternative construction, in which mixed products of the $\mathcal{L}$ and $\bar{\mathcal{L}}$ generators are allowed, leads to an extended algebra dubbed ‘large AdS higher-spin algebra’ in [186]. Similar extended algebras also appear in the description of partially-massless fields in three dimensions [187].

[8] The lower bound on the value of C_2 is related to unitarity constraints [136], and the case $\lambda\in\mathbb{N}$ reproduces some finite-dimensional truncations.

The condition $\mathcal{I}_{(AB)} \sim 0$ also guarantees that products of W with other elements in the UEA do not introduce new generators since the relations (2.2.10) imply

$$W\mathcal{L}_m \sim \tfrac{1}{2}\,C_2\,\mathcal{L}_m\,, \quad W\bar{\mathcal{L}}_m \sim -\tfrac{1}{2}\,C_2\,\bar{\mathcal{L}}_m\,. \tag{2.2.15}$$

The eq. (2.2.14) leads to the one-parameter family of higher-spin algebras that has been considered in the literature on massless fields in three dimensions [135, 188, 189, 150]. One can write

$$W \sim \frac{\lambda^2-1}{4}\,\eta\,, \tag{2.2.16}$$

where we introduced the twist operator η flipping the sign of one copy of $\mathfrak{sl}(2,\mathbb{R})$ while leaving the other untouched. Of course, $\eta^2 = \mathrm{id}$ which respects (2.2.13). This leads to the presentation of the one-parameter family of higher-spin algebras:

$$\mathrm{id}\oplus\eta\oplus\mathfrak{hs}_3[\lambda] := \frac{\mathcal{U}(\mathfrak{so}(2,2))}{\big\langle \mathcal{I}_{AB}\oplus\big(C_2-\frac{\lambda^2-1}{2}\,\mathrm{id}\big)\big\rangle}\,, \tag{2.2.17}$$

which is equivalently written as

$$\mathfrak{hs}_3[\lambda] = \mathfrak{hs}[\lambda]\ltimes\mathfrak{hs}[\lambda]\,, \tag{2.2.18}$$

where the algebra $\mathfrak{hs}[\lambda]$ is defined by

$$\mathbb{1}\oplus\mathfrak{hs}[\lambda] = \frac{\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))}{\big\langle \mathcal{C}_2-\frac{\lambda^2-1}{4}\,\mathbb{1}\big\rangle}\,, \tag{2.2.19}$$

where $\mathcal{C}_2$ denotes the $\mathfrak{sl}(2,\mathbb{R})$ Casimir operator (say $\mathcal{L}^2$ or $\bar{\mathcal{L}}^2$). When $\lambda = N \in \mathbb{N}$, its eigenvalue corresponds to that of a finite-dimensional irreducible representation and a further infinite-dimensional ideal appears. Factoring it out leads to the $\mathfrak{sl}(N,\mathbb{R})$ algebra, that can be interpreted as a higher-spin algebra involving a finite number of fields with spin $2 \le s \le N$.

It is worth revisiting the previous construction in the finite-dimensional case. The absence of mixed products of $\mathcal{L}$ and $\bar{\mathcal{L}}$ means that one can consider $\mathfrak{so}(2,2)$ representations of the form

$$\mathcal{L}_m = \begin{pmatrix} l_m & 0\\ 0 & 0\end{pmatrix}, \quad \bar{\mathcal{L}}_m = \begin{pmatrix} 0 & 0\\ 0 & l_m\end{pmatrix}, \tag{2.2.20}$$

where the finite-dimensional representations l_m of the $\mathfrak{sl}(2,\mathbb{R})$ algebra are the same in the two blocks. This is because the eigenvalue of l^2 in the upper and lower blocks are forced to assume the same eigenvalue if we want C_2 to be a multiple of the identity by virtue of eq. (2.2.14). Choosing that of dimension N, the Casimir operators of $\mathfrak{so}(2,2)$ take the form

$$C_2 = \frac{N^2-1}{2}\begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}, \quad W = \frac{N^2-1}{4}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}, \tag{2.2.21}$$

where W manifestly satisfies the relation (2.2.13) imposed by $\mathcal{I}_{AB} \sim 0$.

Historically, this class of higher-spin algebras was arrived at by other considerations. Indeed, by remembering that AdS_3 gravity can be formulated as a Chern-Simons theory based on the gauge group $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, with connections

$$A_m = \omega_m + R^{-1}\, e_m\,, \quad \bar{A}_m = \omega_m - R^{-1}\, e_m\,, \tag{2.2.22}$$

where e_m plays the role of the vielbein and ω_m of the spin connection. A natural extension of $\mathfrak{sl}(2,\mathbb{R})$ is $\mathfrak{sl}(N,\mathbb{R})$. Decomposing $\mathfrak{sl}(N,\mathbb{R})$ into irreducible $\mathfrak{sl}(2,\mathbb{R})$ components we get

$$\mathfrak{sl}(N,\mathbb{R}) = \underline{2N-1} \oplus \underline{2N-3} \oplus \cdots \oplus \underline{3}\,, \tag{2.2.23}$$

where the final $\underline{3}$ represents the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$ and $(2\,s-1)$ is the dimension of the spin-$(s-1)$ representation of $\mathfrak{sl}(2,\mathbb{R})$, i.e. the number of components of a fully symmetric tensor with $(s-1)$ indices under the three-dimensional Lorentz algebra which is identified with the components of a spin-s frame-like field (2.1.1). The family of algebras $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ is therefore appropriate to describe a higher-spin theory with fields of spin 2 to N included. The non-linear theory gauging this algebra was described in [137, 138], see also [190] and references therein.

2.3 Higher-spin algebras for massless theories in flat space

As mentioned in the introduction, the existence of flat space higher-spin theories is restricted by a number of no-go results, some of them constraining the low-energy behaviour of the interacting theory, others regarding the consistency of interactions, others lastly concerning the (in)existence of Lie algebras underlying the symmetry. For instance, as advertised in the introductory remarks of this chapter, the higher-spin algebra $\mathfrak{hs}_4$ built in [60] doesn't admit any flat-space contraction whose linearised curvatures reproduce the limit $R \to \infty$ of the initial conditions displayed in (2.1.28). Although this should resound as a no-go theorem for the possibility to build an interacting theory of higher-spin gravity in flat space along the same lines as [54, 59, 61, 67], there are a number of observations that suggest to reconsider this problem.

Firstly, this problem is evaded in $D = 3$. Although theories of massless higher-spin fields are degenerate in three space-time dimensions in the sense that there are no dynamical degrees of freedom, the existence of a flat contraction of the higher-spin algebras should be signalled as a yes-go. Moreover, the fact that higher-spin theories are constrained by low-energy no-go theorems in dimensions greater than three does not have to be related to the existence or not of a gauge algebra in dimensions greater than three, because the first of these statements is dynamical while the other is kinematical. At best, the existence of an algebra in $D = 3$ should be considered as coincident to the no-go theorems not being applicable.

Secondly, non-Abelian higher-spin algebras extending the Poincaré algebra actually exist in dimensions strictly greater than three. In the original article [60] displaying the construction of the higher-spin algebra $\mathfrak{hs}_4$, the authors discussed possible contractions of this algebra giving rise to a higher-spin algebra extending $\mathfrak{iso}(1,3)$. Although they found several, none of them was able to reproduce the limit $R \to \infty$ of the eqs. (2.1.28). One of these algebras was also reproduced more recently in [191], where it was remarked that using the spinorial representation of $SL(2,\mathbb{C})$, this flat-space algebra contained zero-norm state whose factoring out inevitably led to the trivialisation of the action of translations. In the following, we will identify one of the possible contractions of [60] as a candidate flat-space higher-spin algebra, explaining why the arguments mentioned above do not constitute in themselves a no-go to the existence of an interacting theory gauging it.

2.3.1 Three dimensions

We can perform an İnönü-Wigner contraction that takes one from the AdS higher-spin algebra $\mathfrak{hs}_3[\lambda]$ to the flat space higher-spin algebra $\mathfrak{ihs}_3[\lambda]$, as was already noticed for the specific value $\lambda = 3$, where the infinite-dimensional higher-spin algebra can be truncated to a finite-dimensional algebra isomorphic to two copies of $\mathfrak{sl}(3,\mathbb{R})$. In [78, 79], this İnönü-Wigner contraction was defined and shown to reproduce a theory of three-dimensional spin-three gravity formulated on flat spacetime. Later, the case where $\lambda \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ were considered in [192, 80].

The $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ algebra is spanned by generators L_m, P_m with $|m| \leq 1$ and U_n, V_n with $|n| \leq 2$ and its Lie brackets are given by

$$[L_m, U_n] = (2m-n)\,U_{m+n}\,, \quad [L_m, V_n] = (2m-n)\,V_{m+n}\,, \tag{2.3.1a}$$

$$[P_m, U_n] = (2m-n)\,V_{m+n}\,, \quad [P_m, V_n] = (2m-n)\,U_{m+n}\,, \tag{2.3.1b}$$

$$[U_m, U_n] = (m-n)\left(2m^2+2n^2-mn-8\right)L_{m+n}\,, \tag{2.3.1c}$$

$$[U_m, V_n] = (m-n)\left(2m^2+2n^2-mn-8\right)P_{m+n}\,, \tag{2.3.1d}$$

$$[V_m, V_n] = (m-n)\left(2m^2+2n^2-mn-8\right)L_{m+n}\,, \tag{2.3.1e}$$

in addition to eq. (2.2.5). By introducing a parameter $\epsilon \in \mathbb{R}$, replacing the generators P_m and V_m by $\epsilon^{-1} P_m$ and $\epsilon^{-1} V_m$ respectively, the limit $\epsilon \to 0$ reproduces the İnönü-Wigner contraction to the $\mathfrak{sl}(3,\mathbb{R}) \ltimes \mathfrak{sl}(3,\mathbb{R})$ algebra. The resulting algebra contains a three-dimensional Poincaré sub-algebra.

The procedure to define the İnönü-Wigner contraction $\mathfrak{ihs}_3[\lambda]$ from $\mathfrak{hs}_3[\lambda]$ follows from the observation that the structure of (2.3.1) is reproduced for any spin: the generators of $\mathfrak{hs}_3[\lambda]$ always split into two categories, one of them forming a sub-algebra. Therefore, one can perform an İnönü-Wigner contraction by redefining the generators in the complement of this sub-algebra with an inverse power of ϵ, as explained in [80], and sending ϵ to zero.

As explained in section 2.2.2, the algebra $\mathfrak{hs}_3[\lambda]$, with generators that we shall denote as $L_m^{(s)}$ and $P_m^{(s)}$ (of which L_m, P_m on the one hand and U_m, V_m on the other hand are the $s=2$ and $s=3$ instances respectively), possesses a $\mathbb{Z}_2$-grading and its generators can be built as the sum and differences of generators of two orthogonal copies of $\mathfrak{hs}[\lambda]$, that we shall denote as $\mathcal{L}_m^{(s)}$ and $\bar{\mathcal{L}}_m^{(s)}$. As such, there is a natural prescription for the İnönü-Wigner contraction

$$L_m^{(s)} = \mathcal{L}_m^{(s)} + \bar{\mathcal{L}}_m^{(s)}\,, \quad P_m^{(s)} = \epsilon\left(\mathcal{L}_m^{(s)} - \bar{\mathcal{L}}_m^{(s)}\right), \tag{2.3.2}$$

with commutators

$$\left[L_m^{(s)}, L_n^{(t)}\right] = \sum_{\substack{u=|s-t|+2\\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)}\,, \tag{2.3.3a}$$

$$\left[L_m^{(s)}, P_n^{(t)}\right] = \sum_{\substack{u=|s-t|+2\\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) P_{m+n}^{(u)}\,, \tag{2.3.3b}$$

$$\left[P_m^{(s)}, P_n^{(t)}\right] = \epsilon^2 \sum_{\substack{u=|s-t|+2\\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)}\,, \tag{2.3.3c}$$

where $g_{s+t-u}^{st}(m,n;\lambda)$ are some structure constants, displayed for instance in [135, 193]. The limit $\epsilon \to 0$ reproduces the algebra $\mathfrak{ihs}_3[\lambda] \simeq \mathfrak{hs}[\lambda] \ltimes \mathfrak{hs}[\lambda]$.

A natural question is if one can build the algebra $\mathfrak{ihs}_3[\lambda]$ from a UEA construction. If the answer turns out to be positive, this would mean that the AdS$_3$ higher-spin algebra and its flat-space counterpart should be thought of on the same footing, a statement that we can hope to generalise to higher dimensions.

To this end, we will start from the observation that the generators $L_m^{(s)}$ form a sub-algebra extending the Lorentz sub-algebra of Poincaré, and which is isomorphic to $\mathfrak{hs}[\lambda]$. This leads to the natural assumption that the generators $L_m^{(s)}$ are constructed as products of Lorentz generators only. From there, defining the other generators $P_m^{(s)}$ through the adjoint action of translations and checking that the commutators indeed reproduce $\mathfrak{ihs}_3[\lambda]$, we will encounter consistency conditions that will constitute the ideal one has to factor out in order to build $\mathfrak{ihs}_3[\lambda]$ as a coset.

Higher-Lorentz sector

We are working in the UEA of the Poincaré algebra $\mathfrak{iso}(1,2)$, with associative product $\star$ that we will omit in the following. Assume the following form for the generators $L_m^{(s)}$, reflecting the observation that they form a sub-algebra isomorphic to $\mathfrak{hs}[\lambda]$

$$L^{(s)}_{\pm(s-1)} := (L_{\pm1})^{s-1}\ , \quad L^{(s)}_{m\mp1} := \frac{\mp1}{s\pm m-1}\left[L_{\mp1}\,,\,L^{(s)}_m\right]. \tag{2.3.4}$$

For instance, for $s=3$ we find

$$L^{(3)}_{\pm2} = L_{\pm1}L_{\pm1}\,, \qquad L^{(3)}_{\pm1} = L_0L_{\pm1} \pm \frac{1}{2}\,L_{\pm1}\,, \qquad L^{(3)}_0 = L_0L_0 - \frac{1}{3}\,L^2\,. \tag{2.3.5}$$

To recover the $\mathfrak{hs}[\lambda]$ algebra we have to impose that the quadratic quantity L^2 of the Lorentz sub-algebra is proportional to the identity with the following λ dependence:

$$L^2 := \gamma^{mn}L_mL_n = L_0L_0 - \frac{1}{2}(L_1L_{-1}+L_{-1}L_1) \sim \frac{\lambda^2-1}{4}\,id\,, \tag{2.3.6}$$

where γ^{mn} is the inverse Killing metric of $\mathfrak{so}(2,1)\simeq\mathfrak{sl}(2,\mathbb{R})$ with the conventions of eq. (2.2.9). Notice that this is a rather strong constraint since L^2 does not commute with translations: we shall check its consistency later.

Higher-translation sector

The other class of generators, denoted by $P_m^{(s)}$, can be recovered from the $L_m^{(s)}$ via the adjoint action of the Poincaré sub-algebra. Indeed, from (2.3.3), we have that

$$\left[P_m\,,\,L^{(s)}_n\right] = \big((s-1)\,m-n\big)P^{(s)}_{m+n}\,, \tag{2.3.7}$$

and we can use this relation to define all $P_m^{(s)}$, that will thus be linear in P_m in their UEA realisation. For instance, for $s=3$ we have

$$P^{(3)}_{\pm2} = L_{\pm1}P_{\pm1}\,, \qquad P^{(3)}_{\pm1} = L_0P_{\pm1} \pm \tfrac{1}{2}\,P_{\pm1}\,, \qquad P^{(3)}_0 = L_0P_0 - \tfrac{1}{3}\,W\,, \tag{2.3.8}$$

where

$$W := \gamma^{mn}L_mP_n = L_0P_0 - \tfrac{1}{2}\,L_1P_{-1} - \tfrac{1}{2}\,L_{-1}P_1\,, \tag{2.3.9}$$

is the three-dimensional analogue of the Pauli-Lubanski vector. However, in this case, it is already a central element because it is scalar.

Eqs. (2.3.8) can be readily generalised to arbitrary values of the spin by noticing that the adjoint action of P_m is consistent with the definitions

$$P^{(s)}_{\pm(s-1)} := (L_{\pm1})^{s-2}\,P_{\pm1}\,, \quad P^{(s)}_{m\mp1} := \frac{\mp1}{s\pm m-1}\left[L_{\mp1}\,,\,P^{(s)}_m\right]. \tag{2.3.10}$$

The expression for $P^{(s)}_{\pm(s-1)}$ follows from the adjoint action of P_0 on $L^{(s)}_{\pm(s-1)}$ and the position of the operator $P_{\pm1}$ is irrelevant since $[L_{\pm1},P_{\pm1}]=0$. The other components are fixed by the known action of Lorentz transformations on $P_m^{(s)}$. Still, the consistency of the whole set of relations (2.3.4), (2.3.6) and (2.3.10) requires some additional constraints which will specify the ideal that has to be factored out from the UEA of $\mathfrak{iso}(1,2)$.

Universal enveloping construction of $\mathfrak{ihs}_3[\lambda]$

Let us observe from (2.3.3) that

$$\big[P_m\,,\,L_n^{(s)}\big] = \big((s-1)\,m-n\big)P_{m+n}^{(s)} = \big[L_m\,,\,P_n^{(s)}\big]\,. \tag{2.3.11}$$

These relations give rise to a first set of consistency conditions, since the two commutators must agree. We can obtain them from the analysis of the case $s=3$:

$$\left[P_\mp, L_{\pm 2}^{(3)}\right] \stackrel{!}{=} \left[L_\mp, P_{\pm 2}^{(3)}\right] \quad \Longrightarrow \quad L_\pm P_0 \sim P_\pm L_0\,, \tag{2.3.12a}$$

$$\left[P_\mp, L_{\pm 1}^{(3)}\right] \stackrel{!}{=} \left[L_\mp, P_{\pm 1}^{(3)}\right] \quad \Longrightarrow \quad L_\pm P_\mp \sim P_\pm L_\mp\,, \tag{2.3.12b}$$

which can be summed up in

$$L_m P_n \sim P_m L_n\,, \tag{2.3.13}$$

because $[L_m, P_m] = 0$. Here and in the following, we used the symbol $\sim$ instead of an equality to stress that the identity is valid only in a representation of the UEA of the Poincaré algebra. Equivalently, $\sim$ defines is an equivalence relation, where the left-hand side is equivalent to the right-hand side module terms in an ideal that we will now build.

We can verify that the remaining relations in (2.3.11) are identically satisfied: the cases with $s \geq 4$ give rise to the same conditions, multiplied on the left or the right by some elements of $\mathcal{U}(\mathfrak{iso}(1,2))$. Using eq. (2.3.13), we can check that

$$\left[L^2, P_m\right] \sim 0\,. \tag{2.3.14}$$

Therefore, in this setup L^2 commutes with all elements of the Poincaré algebra and this confirms the consistency of the relation (2.3.6), in which we imposed $L^2 \propto \mathrm{id}$.

We now have to check that the higher-translation generators previously defined form an Abelian factor and satisfy the limit $\epsilon \to 0$ of the commutation relations (2.3.3). Given the form of the $[L_m, P_m]$ commutator, it is clear that $[L_m^{(s)}, P_n^{(t)}]$ contains exactly one P generator, and so it belongs to the higher-translation sector. When developing $\left[P_m^{(s)}, P_n^{(t)}\right]$ in powers of L_m's and P_m's, any factors of P_m can be pushed to the right thanks to (2.3.13) and, for $s=2$ and $t=3$, one obtains the following set of consistency conditions:

$$\left[P_0\,,\,P^{(3)}_{\pm2}\right] \overset{!}{=} 0 \quad\Longrightarrow\quad P_{\pm1}P_{\pm1}\sim 0\,, \tag{2.3.15a}$$
$$\left[P_{\mp1}\,,\,P^{(3)}_{\pm2}\right] \overset{!}{=} 0 \quad\Longrightarrow\quad P_{\pm1}P_{0}\sim 0\,, \tag{2.3.15b}$$
$$\left[P_{\mp1}\,,\,P^{(3)}_{\pm1}\right] \overset{!}{=} 0 \quad\Longrightarrow\quad P_{\pm1}P_{\mp1}\sim 0\,. \tag{2.3.15c}$$

Taking advantage of the relation (2.3.13) to get $P^{(3)}_{\pm1}\sim P_0L_{\pm1}\pm\frac{1}{2}\,P_{\pm1}$ from (2.3.8), the last commutator also gives

$$\left[P_{\mp1}\,,\,P^{(3)}_{\pm1}\right] \overset{!}{=} 0 \quad\Longrightarrow\quad P_0P_0\sim 0\,. \tag{2.3.16}$$

In conclusion, the product of any two translation generators must vanish:

$$P_mP_n\sim 0\,. \tag{2.3.17}$$

In particular, this implies that the quadratic Casimir P^2 of the Poincaré algebra must vanish.

From the consistency conditions (2.3.13) and (2.3.17) one can also fix the action of the second quadratic Casimir W of the Poincaré algebra that we defined in eq. (2.3.9) on the generators of $\mathfrak{iso}(1,2)$:

$$WL_k=\gamma^{mn}L_mP_nL_k\sim\gamma^{mn}L_mL_nP_k=L^2P_k\sim\frac{\lambda^2-1}{4}\,P_k\,, \tag{2.3.18a}$$
$$WP_k=\gamma^{mn}L_mP_nP_k\sim 0\,. \tag{2.3.18b}$$

The whole set of consistency conditions,

$$\mathcal{P}_{mn}:=P_mP_n\sim 0\,, \tag{2.3.19a}$$
$$\mathcal{I}_{m,n}:=L_mP_n-P_mL_n\sim 0\,, \tag{2.3.19b}$$
$$L^2-\frac{\lambda^2-1}{4}\,\mathrm{id}\sim 0\,, \tag{2.3.19c}$$

defines an ideal because

$$[L_k,\mathcal{I}_{m,n}]=(k-m)\,\mathcal{I}_{m+k,n}+(k-n)\,\mathcal{I}_{m,n+k}\,, \tag{2.3.20a}$$
$$[L_k,\mathcal{P}_{mn}]=(k-m)\,\mathcal{P}_{(m+k)n}+(k-n)\,\mathcal{P}_{m(n+k)}\,, \tag{2.3.20b}$$

which show that $\mathcal{P}_{mn}$ and $\mathcal{I}_{m,n}$ transform as Lorentz tensors, and

$$[P_k,\mathcal{I}_{m,n}]=(k-m)\,\mathcal{P}_{(m+k)n}-(k-n)\,\mathcal{P}_{m(n+k)}\,, \tag{2.3.21a}$$
$$[P_k,\mathcal{P}_{mn}]=0\,. \tag{2.3.21b}$$

Note that $\mathcal{P}_{mn}$ is manifestly symmetric due to the fact that translations commute, and $\mathcal{I}_{m,n}$ is antisymmetric owing to the commutator of Lorentz transformations with translations. Furthermore, we already showed that L^2 is central in the quotient and in the following we shall see how one can recover the ideal (2.3.19) from the flat limit of the ideal introduced in the quotient construction of higher-spin algebra in AdS_3. In conclusion, the one-parameter family of flat-space higher-spin algebras can be obtained as

$$\mathrm{id} \oplus W \oplus \mathfrak{ihs}_3[\lambda] = \frac{\mathcal{U}(\mathfrak{iso}(1,2))}{\left\langle \mathcal{P}_{mn} \oplus \mathcal{I}_{m,n} \oplus \left(L^2 - \frac{\lambda^2-1}{4}\,\mathrm{id}\right)\right\rangle}\,, \tag{2.3.22}$$

where the relations defining the ideal are given in (2.3.19).

From a limit of the ideal

Let us work again in the UEA of $\mathfrak{so}(2,2)$ and introduce the dimensionless parameter ϵ, redefining the generator of translations $P_m \to \epsilon^{-1}\,P_m$ such that the limit $\epsilon \to 0$ reproduces the Poincaré algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)\,L_{m+n}\,,\\
[L_m, P_n] &= (m-n)\,P_{m+n}\,,\\
[P_m, P_n] &= \epsilon^2\,(m-n)\,L_{m+n}\,.
\end{aligned} \tag{2.3.23}$$

Under this rescaling of P_m, the quadratic relations in $\mathcal{U}(\mathfrak{so}(2,2))$ that define the ideal written in (2.2.11) become

$$L_m\,P_n - P_m\,L_n \sim 0\,, \quad P_m\,P_n - \epsilon^2\,L_m\,L_n \sim 0\,, \quad L^2 - \frac{\lambda^2-1}{4}\,\mathrm{id} \sim 0\,, \tag{2.3.24}$$

where we used the trace of the second relation to transform P^2 into $\epsilon^2\,L^2$ in the last one.

We may choose to represent higher-spin generators in the UEA of the AdS_3 algebra in different ways, as an example, using the relations in the ideal, the spin-three generator $L^{(3)}_{+2}$ can be represented either by its initial definition $L_{+1}\,L_{+1}$, or by $\epsilon^{-2}\,P_{+1}\,P_{+1}$. Clearly, one expression becomes singular when $\epsilon \to 0$ while the other stays regular. Notice also that the generators $L_{+1}\,L_{+1}$ and $P_{+1}\,P_{+1}$ have distinct commutators in the Poincaré algebra. Similarly, one can express the right-hand side of a commutator in $\mathfrak{hs}_3[\lambda]$ by different representatives in the equivalence class of generators, modulo relations in the ideal

$$\begin{aligned}
[P_{+1}, L_{+1}\,L_{+1}\,L_{+1}\,P_0] &= \epsilon^2\,L_{+1}\,L_{+1}\,L_{+1}\,L_{+1}\\
&\sim L_{+1}\,L_{+1}\,P_{+1}\,P_{+1}\\
&\sim \epsilon^{-2}\,P_{+1}\,P_{+1}\,P_{+1}\,P_{+1}\,,
\end{aligned} \tag{2.3.25}$$

where the right-hand side can become either zero, finite or divergent when $\epsilon \to 0$, depending on the expression. This means that the choice of a representative is essential when discussing the flat limit of the algebra at the level of the UEA construction.

Now, the direct limit $\epsilon \to 0$ in the left-hand side of the expressions in eq. (2.3.24) takes the form

$$L_m P_n - P_m L_n \,, \quad P_m P_n \,, \quad L^2 - \frac{\lambda^2 - 1}{4}\,\mathrm{id}\,, \tag{2.3.26}$$

and setting them to zero reproduces the consistency conditions first obtained in eqs. (2.3.19a) – (2.3.19c), and identified as the ideal one has to quotient out of the UEA of $\mathfrak{iso}(1,2)$ in order to reproduce $\mathfrak{ihs}_3[\lambda]$. This signals that in the flat case, the product of at least two translations should be identified to zero, while the product of two Lorentz transformations remains a free generator.

The upshot of the analysis of the previous sections is that taking the limit $\epsilon \to 0$ is a well-defined procedure both in the UEA definition and in the commutators, by choosing a representative for the higher-spin generators which are the products of Lorentz generators with at most one translation generator. On the other hand, expressions in the UEA written as the product of two or more translation generators will become factored out in the limit ϵ is sent to 0 .

Finite-dimensional matrix representation

One can treat the algebras $\mathfrak{hs}_3[\lambda]$ and $\mathfrak{ihs}_3[\lambda]$ in a similar way. As an example, for $\lambda = N \in \mathbb{N}$, both algebras admit finite-dimensional truncations of the form $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$. In the flat case, one can recover a matrix representation by evaluating the UEA of $\mathfrak{iso}(1,2)$ on the following finite-dimensional representation of the Poincaré algebra:

$$L_m = \begin{pmatrix} l_m & 0 \\ 0 & l_m \end{pmatrix} \,, \quad P_m = \begin{pmatrix} 0 & 0 \\ l_m & 0 \end{pmatrix} \,, \tag{2.3.27}$$

where the l_m are $N \times N$ matrices giving an irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$. Thanks to the lower-triangular form of the generators P_m , the conditions (2.3.19a) and (2.3.19b) in the definition of the ideal are clearly satisfied. Moreover,

$$L^2 = \begin{pmatrix} l^2 & 0 \\ 0 & l^2 \end{pmatrix} = \frac{N^2-1}{4} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \,, \quad W = \begin{pmatrix} 0 & 0 \\ l^2 & 0 \end{pmatrix} = \frac{N^2-1}{4} \begin{pmatrix} 0 & 0 \\ \mathbb{1} & 0 \end{pmatrix} \,, \tag{2.3.28}$$

so that the conditions (2.3.18) and (2.3.19c) are satisfied too. Notice that W is manifestly a central element, but it is not proportional to the identity either: this is consistent with the structure of the representation (2.3.27), which is not irreducible but indecomposable.

The semi-direct structure is realised by block 2×2 matrix multiplication, so that the whole set of generators forming the algebra known as $\mathfrak{isl}(2,\mathbb{R})$ takes the form

$$L_m^{(s)} = \begin{pmatrix} l_m^{(s)} & 0 \\ 0 & l_m^{(s)} \end{pmatrix} \,, \quad P_m^{(s)} = \begin{pmatrix} 0 & 0 \\ l_m^{(s)} & 0 \end{pmatrix} \,, \tag{2.3.29}$$

where the $l_m^{(s)}$ are the generators of $\mathfrak{sl}(N,\mathbb{R})$ in the defining representation, which can be built as products of l_m . Let us stress that the lower-triangular form of the generators with $\mathfrak{so}(1,2)$ representations on the diagonal blocks is in agreement with general results on the structure of finite-dimensional indecomposable representations of the Poincaré algebra [194]. We will also encounter a similar type of representations in the discussion on the possible holographic dual of this construction, presented in section 3.

Discussion

We found that there are several ways to write higher-spin generators in the UEA of the isometry algebra of AdS_3. For instance, classifying all possible spin-three generators and using relations in the ideal, one can represent the generator spin-three Lorentz transformations either by the product of two Lorentz generators L_m or two translations P_m . These two ways of representing the higher-spin generators are completely equivalent when the cosmological constant is non-zero, but lead to two distinct generators when it is sent to zero. The natural expectation would be that the spin-three translation generators are represented by the product of two P_m's, since they commute with translations. We explored this possibility in [107], and we showed that with this prescription, the spin-three Lorentz generator is given by the product of one L_m and one P_m. At this point, one has to face an inconsistency: we have exhausted the number of generators of the flat-space higher-spin algebra, but we still have the product of two L_m's at our disposal. The problem is that this combination cannot be part of the ideal, at the risk of getting rid of all higher-spin generators. Consider for instance the quadratic combinations written in eq. (2.3.5) and use the repeated adjoint action of the generators P_m : this gives all the combinations displayed in eq. (2.3.8), as well as the combinations $P_m\,P_n$; thus quotienting by the expressions in eq. (2.3.5) also quotients all quadratic combinations.

Therefore, one has to abandon the prescription of representing higher-translations by a product of translations only. While it seems counter-intuitive, focusing first on the higher-Lorentz transformations (that is products of only L_m) allows to resolve the previous tension. Representing the spin-three Lorentz transformations by the product of two L_m's, the spin-three translations must be given by the product of one L_m and one P_m. The product of two P_m's is an extra generator *which can be consistently factored out.*

We noticed that we can realise the full algebra $\mathfrak{ihs}_3[\lambda]$ as the quotient of the UEA of $\mathfrak{iso}(1,2)$ by a one-parameter ideal, and showed that the expressions entering this ideal could be recovered by taking the limit $\epsilon \to 0$ of the ideal that appears in AdS. Notice also that in AdS, one may represent higher-spin generators by the product of many L_m's and at most one P_m . In that case, the structure constants appearing on the right-hand side of the commutators take an expression such that the limit $\epsilon \to 0$ is well-behaved. We now want to extend what we learned to the higher-dimensional setting.

2.3.2 Higher dimensions

In the previous section, we learned how to construct a flat three-dimensional higher-spin algebra from an İnönü-Wigner contraction of one on AdS_3. We were also able to reproduce the contracted algebra from a Universal Enveloping Algebra construction. The contraction relies on the existence of a sub-algebra (called algebra of 'higher-Lorentz' transformations in the case of $D = 3$, here spanned by products of Lorentz generators) which allows to make a rescaling of the generators living in the complement. We learnt that, in order to repeat the contraction at the level of the Universal Enveloping Algebra requires to carefully pick a representative for the higher-spin generators containing no more than one translation generator. In turn, this implies that the ideal one has to quotient in the coset construction must contain all generators built as products containing at least two translations. This structure allows to precisely reproduce the flat-space higher-spin algebra and all its finite-dimensional truncations.

We will try to repeat this observation in $D \geq 4$: a candidate higher-spin algebra in Minkowski space should be built as an İnönü-Wigner contraction leaving a sub-algebra untouched, and as a quotient of the universal enveloping algebra of the Poincaré algebra.

As an İnönü-Wigner contraction

The most straightforward way of building a flat-space higher-spin algebra from the Eastwood-Vasiliev algebra is to proceed like in $D = 3$ by identifying an interesting İnönü-Wigner contraction. To this end, it is necessary and sufficient to find a sub-algebra of $\mathfrak{hs}_D$ containing the Lorentz algebra. Indeed, for any algebra $\mathfrak{g}$, if one can find a splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ such that $\mathfrak{h}$ is a sub-algebra of $\mathfrak{g}$ containing the Lorentz algebra, then one can then redefine elements of $\mathfrak{k}$ by multiplying them by a real dimensionless parameter $\tilde{\mathfrak{k}} := \epsilon\,\mathfrak{k}$, such that

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}\,, \quad [\mathfrak{h}, \tilde{\mathfrak{k}}] \subseteq \epsilon\,\mathfrak{h} + \tilde{\mathfrak{k}}\,, \quad [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}] \subseteq \epsilon^2\,\mathfrak{h} + \epsilon\,\tilde{\mathfrak{k}}\,, \tag{2.3.1}$$

and in the limit $\epsilon \to 0$, one finds that commutators involving only $\mathfrak{h}$ are untouched while $\tilde{\mathfrak{k}}$ becomes an Abelian ideal.

In the case of $\mathfrak{g} = \mathfrak{so}(2, D-1)$ and $\mathfrak{h} = \mathfrak{so}(1, D-1)$, the result of the İnönü-Wigner contraction is the Poincaré algebra. In the case of $\mathfrak{g} = \mathfrak{hs}_D$, it is not obvious what the sub-algebra $\mathfrak{h}$ should be, but it should be strictly bigger than the Lorentz algebra and therefore infinite-dimensional. Indeed, if $\mathfrak{h}$ is the Lorentz algebra then the İnönü-Wigner procedure results in an algebra where the only non-vanishing commutators are the ones involving only the generators of the Lorentz sub-algebra among themselves. In other words, the resulting theory will include higher-spin interactions that do not deform the gauge algebra, which is the opposite of what we set out to do. Therefore, the sub-algebra $\mathfrak{h}$ must also contain one higher-spin generator, and it follows from usual arguments that it must be infinite-dimensional.

Fortunately, there are a few observations that can help to find some interesting sub-algebras. Firstly, note that the only available tensor that can appear in the structure constants of the AdS higher-spin algebra is the flat metric η_{ab}. This means that the number of indices of the generators on the right-hand side of the Lie bracket is equal to the number of indices on the left-hand side of the Lie bracket, modulo a multiple of 2 . Secondly, the spin of the generators appearing on the right-hand side of the Lie bracket is equal to the sum of the spins of the generators on the left-hand side, modulo a multiple of 2 (this is a consequence of the Poincaré-Birkhoff-Witt theorem applied to the UEA construction of $\mathfrak{hs}_D$, and one even has the spin addition rules displayed in section 2.2).

These conditions can be schematically summarised as

$$\left[M_{a(s_1-1),b(s_1-t_1-1)}, M_{c(s_2-1),d(s_2-t_2-1)}\right] \propto \sum_{s_3=|s_1-s_2|+2}^{s_1+s_2-2} \sum_{t_3=0}^{s_3-1} M_{e(s_3-1),f(s_3-t_3-1)}\,, \quad (2.3.2)$$

where $s_1 + s_2 - s_3 \bmod 2 = 0$ and $t_1 + t_2 - t_3 \bmod 2 = 0$ and we omitted the precise form of the structure constants. Here and in the following, we changed notation with respect to section 2.1, so that t directly parameterises the depth.

These two basic observations already allow us to build a list of candidate sub-algebras. We will follow the terminology of sections 5 and 6 of [60] to classify infinite-dimensional sub-algebras of the AdS$_4$ higher-spin algebra (although there a specific realisation was assumed):

- $s \bmod 2 = 0$, which corresponds to the sub-algebra of even spin ;
- $s + t \bmod 2 = 0$, which corresponds to the generalisation to any dimensions of (the bosonic part of) the h_2 sub-algebra of [60] ;
- $s \bmod 2 = 0$ and $t \bmod 2 = 0$, which corresponds to the generalisation to any dimensions of (the bosonic part of) the f_{22} sub-algebra of [60] ;
- $t \bmod 2 = 0$, which corresponds to the generalisation to any dimensions of (the bosonic part of) the k sub-algebra of [60] .

Not all of them are of interest. For instance, the first item leads to the truncation to minimal higher-spin algebras, but does not lead to a higher-spin algebra extending the Poincaré one. The two most interesting items are the second and last. We will argue in the following that only the last one can be reproduced from a coset construction. We call the algebra resulting from an İnönü-Wigner contraction with respect to the even t sub-algebra $\mathfrak{ihs}_D$. It is clear that the structure constants of $\mathfrak{ihs}_D$ are exactly the same as those of $\mathfrak{hs}_D$, except for the commutators of two generators with t odd which vanish. Finally, the third item can be used to define a minimal truncation of $\mathfrak{ihs}_D$ to its even-spin sub-algebra.

In the following, we will show how to construct a flat-space higher-spin algebra in any dimensions, enlarging the Poincaré one with an infinite set of generators. We will construct it from a quotient of the UEA of the Poincaré algebra, in a similar way as we showed before for $\mathfrak{ihs}_3[\lambda]$. We will then argue that it reproduces $\mathfrak{ihs}_D$.

Annihilator of the scalar singleton

Reversing the logic we followed in section 2.3.1, we first identify this ideal by looking at how the limit of vanishing cosmological constant affects the ideal that one factors out in the AdS_D coset construction. We then check its consistency and track how the resulting algebras can also be recovered as İnönü-Wigner contractions of Eastwood-Vasiliev algebras. We also prove that, under reasonable assumptions, the ideal we obtain in the limit is the only one whose factoring out gives a coset algebra defined on the same vector space as the Eastwood-Vasiliev one. Let us recall once again that, in any $D \geq 4$, the contractions presented below can be interpreted either as flat limits of AdS_D higher-spin algebras or as ultra-relativistic, Carrollian limits of conformal higher-spin algebras in $D-1$ dimensions.

To study the flat-space limit of the AdS coset construction we first have to express the algebra $\mathfrak{so}(2, D-1)$ in a basis adapted to the limit. We shall later use the same basis to classify all cosets of the UEA of $\mathfrak{iso}(1, D-1)$ that give the same set of generators as in Eastwood-Vasiliev algebras.

Let us rewrite the AdS algebra $\mathfrak{so}(2, D-1)$, singling out the generator $P_a = \epsilon\, J_{a\,D}$

$$[J_{ab}\,,\ J_{cd}] = \eta_{bc}\, J_{ad} - \eta_{ac}\, J_{bd} - \eta_{bd}\, J_{ac} + \eta_{ad}\, J_{bc}\,, \tag{2.3.3a}$$

$$[J_{ab}\,,\ P_c] = \eta_{bc}\, P_a - \eta_{ac}\, P_b\,, \tag{2.3.3b}$$

$$[P_a\,,\ P_b] = \epsilon^2\, J_{ab}\,, \tag{2.3.3c}$$

where η_{ab} has signature $(-,+,\dots,+)$. We assign the same mass dimensions to the generators P_a and J_{ab} , so that ϵ is a dimensionless parameter and the Poincaré algebra $\mathfrak{iso}(1, D-1)$ is recovered by sending ϵ to 0 . It will also prove useful to have a contraction parameter ϵ distinct from the cosmological constant, since (as we will see in section 2.4) the limits $\epsilon \to 0$ and $R \to \infty$ will not coincide.

Quadratic combinations

In section 2.2.1 we factored out from the UEA of $\mathfrak{so}(2, D-1)$ an ideal generated by quadratic combinations of the J_{AB}, corresponding to the annihilator of the scalar singleton representation. In the basis (2.3.3), linearly-independent quadratic combinations of the generators can be conveniently classified according to their properties under permutations of their free indices. We get:

- Two independent scalars

$$P^2 := \frac{1}{2} P_a \odot P^a \,, \qquad J^2 := \frac{1}{4} J_{ab} \odot J^{ba} \,; \tag{2.3.4}$$

- One vector

$$\mathcal{I}_a := J_{ab} \odot P^b \,; \tag{2.3.5}$$

- Two traceless symmetric tensors of rank two

$$Q_{ab} := P_a \odot P_b - \tfrac{2}{D}\, \eta_{ab} P^2 \,, \qquad S_{ab} := J^c{}_a \odot J_{bc} - \tfrac{4}{D}\, \eta_{ab} J^2 \,; \tag{2.3.6}$$

- One irreducible and traceless tensor transforming as a hook Young diagram

$$M_{ab,c} := P_{(a} \odot J_{b)c} + \tfrac{1}{D-1} \left(\eta_{ab} \mathcal{I}_c - \eta_{c(a} \mathcal{I}_{b)} \right) ; \tag{2.3.7}$$

- One tensor transforming as a traceless two-row rectangular Young diagram

$$\begin{aligned} K_{ab,cd} := J_{c(a} \odot J_{b)d} &- \tfrac{1}{D-2} \left(\eta_{ab} S_{cd} + \eta_{cd} S_{ab} - 2\eta_{c(a} S_{b)d} \right) \\ &- \tfrac{4}{D(D-1)} \left(\eta_{ab} \eta_{cd} - \eta_{c(a} \eta_{b)d} \right) J^2 \,; \end{aligned} \tag{2.3.8}$$

- Two completely anti-symmetric tensors

$$\mathcal{I}_{[abc]} := J_{[ab} \odot P_{c]} \,, \qquad \mathcal{I}_{[abcd]} := J_{[ab} \odot J_{cd]} \,. \tag{2.3.9}$$

Note that, due to the commutation relations of the AdS isometry algebra, the combinations P^2, J^2, $\mathcal{I}_{[abc]}$ and $\mathcal{I}_{[abcd]}$ can actually be defined using the associative product rather than the symmetrised product, the latter differing by a factor of two from the former.

The tensors in eqs. (2.3.4) – (2.3.8) correspond to the branching in $\mathfrak{so}(1, D-1)$-irreducible components of the product $J_{C(A} \odot J_{B)D}$, while those in (2.3.9) correspond to the branching of $J_{[AB} \odot J_{CD]}$. Notice that it is not necessary to symmetrise explicitly the indices in $P_{(a} \odot P_{b)}$ and $J^c{}_{(a} \odot J_{b)c}$ because the symmetrised product automatically projects on the symmetric component.

The ideal (2.2.10) that we factored out in the AdS_D coset construction is generated by the following combinations, where we recall that the generator P_a has been rescaled by a factor of ϵ :

$$J^2 - \frac{D-1}{2}\,\epsilon^{-2}\,P^2 \sim 0\,, \tag{2.3.10a}$$

$$\epsilon^{-1}\,\mathcal{I}_a \sim 0\,, \tag{2.3.10b}$$

$$S_{ab} + \epsilon^{-2}\,Q_{ab} \sim 0\,, \tag{2.3.10c}$$

$$\epsilon^{-1}\,\mathcal{I}_{abc} \sim 0\,, \tag{2.3.10d}$$

$$\mathcal{I}_{abcd} \sim 0\,, \tag{2.3.10e}$$

$$C_2 := J^2 + \epsilon^{-2}\,P^2 \sim -\frac{(D+1)(D-3)}{4}\,\mathrm{id}\,. \tag{2.3.10f}$$

The first three expressions come from the branching of the symmetric traceless product $\mathcal{I}_{(AB)}$ defined in eq. (2.2.10) and the next two come from the branching of the completely anti-symmetric one, $\mathcal{I}_{[ABCD]}$. Finally, C_2 is the quadratic Casimir of $\mathfrak{so}(2,D-1)$, which is fixed by the factoring out of the previous expressions (see eq. (2.2.14)). Taking linear combinations of the first and the last equations we get

$$J^2 \sim \frac{D-1}{D+1}\,C_2 \sim -\frac{(D-1)(D-3)}{4}\,\mathrm{id}\,, \tag{2.3.11a}$$

$$\epsilon^{-2}\,P^2 \sim \frac{2}{D+1}\,C_2 \sim -\frac{D-3}{2}\,\mathrm{id}\,, \tag{2.3.11b}$$

which means that both P^2 and J^2 are central elements in the scalar singleton representation. This is the case because $[J^2, P_a] = \mathcal{I}_a \sim 0$.

Higher-spin generators

Among the quadratic combinations listed above, all are fixed or factored out except for Q_{ab} (or S_{ab}), $M_{ab,c}$ and $K_{ab,cd}$, which we identify as the spin-three generators of the Eastwood-Vasiliev algebra. All these tensors are fully traceless and irreducible, so that they transform as the following $\mathfrak{so}(1,D-1)$ irreducible representations:

$$Q_{ab}\,,\quad S_{ab} \;\simeq\; \mathbb{Y}_D(2)\,,\quad M_{ab,c} \;\simeq\; \mathbb{Y}_D(2,1)\,,\quad K_{ab,cd} \;\simeq\; \mathbb{Y}_D(2,2)\,. \tag{2.3.12}$$

In the rest of this section, we shall use either the name of the generators or their associated Young diagrams to denote them. Note that, at this stage, we may choose the spin-three fully-symmetric generator as either Q_{ab} or S_{ab} since the two expressions are identified by the relations that define the coset algebra $\mathfrak{hs}_D$. One has to be careful when working in the Poincaré UEA, since we shall see that the two expressions cannot be identified anymore.

The structure of the other generators of the algebra results from the branching rules of two-row Young diagrams of $\mathfrak{so}(2,D-1)$ into $\mathfrak{so}(1,D-1)$ diagrams: for

$s \geq 2$, spin-s generators are associated to all two-row diagrams of $\mathfrak{so}(1, D-1)$ with length $s-1$ and depth ranging from 0 to $s-1$

$$M_{a(s-1),b(s-t-1)} \quad \text{with } t \in \{0, \dots, s-1\}\,. \tag{2.3.13}$$

Once again, in analogy with section 2.3.1, there are multiple expressions for these generators as products of J's and P's which are equivalent in AdS (modulo relations of the ideal), but that will become inequivalent when ϵ is sent to 0.

We now first consider the limit $\epsilon \to 0$ of the ideal (2.3.10) and check that it defines an ideal in the Poincaré UEA. We then show that requiring the same set of generators as in (2.3.13) also identifies uniquely that ideal.

Flat ideal from the contraction limit

By multiplying each expression of the ideal (2.3.10) by the suitable power of ϵ so as to keep only the leading part, one can take a limit $\epsilon \to 0$ and get

$$J_{ab} \odot P^b \sim 0\,, \tag{2.3.14a}$$
$$P_a \odot P_b - \tfrac{2}{D}\,\eta_{ab} P^2 \sim 0\,, \tag{2.3.14b}$$
$$J_{[ab} \odot P_{c]} \sim 0\,, \tag{2.3.14c}$$
$$J_{[ab} \odot J_{cd]} \sim 0\,, \tag{2.3.14d}$$

together with

$$P^2 \sim 0\,, \qquad J^2 \sim -\frac{(D-1)(D-3)}{4}\,\mathrm{id}\,. \tag{2.3.15}$$

Combining eqs. (2.3.14b) and (2.3.15) one can eventually recast these expressions as

$$P_a\, P_b \sim 0\,, \tag{2.3.16a}$$
$$\mathcal{I}_a := J_{ab} \odot P^b \sim 0\,, \tag{2.3.16b}$$
$$\mathcal{I}_{[abc]} := J_{[ab} \odot P_{c]} \sim 0\,, \tag{2.3.16c}$$
$$\mathcal{I}_{[abcd]} := J_{[ab} \odot J_{cd]} \sim 0\,, \tag{2.3.16d}$$
$$J^2 + \frac{(D-1)(D-3)}{4}\,\mathrm{id} \sim 0\,. \tag{2.3.16e}$$

We verify now that these relations span an ideal, which we will abbreviate in $\mathcal{I}^\flat$. All tensors entering (2.3.16) form irreducible representations of the Lorentz group under the adjoint action of Lorentz generators, as displayed e.g. in eq. (2.1.28a). Therefore, the only thing we need to check is the commutation relations with the generator of translations. Remark that

$$[J^2, P_a] = \mathcal{I}_a\,, \tag{2.3.17}$$

so that J^2 is a central element of the Poincaré algebra if and only if $\mathcal{I}_a$ is factored out. Next

$$[\mathcal{I}_a, P_b] = 2\, P_a\, P_b - 2\, \eta_{ab}\, P^2\,. \tag{2.3.18}$$

On the other hand

$$[\mathcal{I}_{[abcd]}, P_e] = 4\, \eta_{e[a}\, \mathcal{I}_{bcd]}\,, \quad [\mathcal{I}_{[abc]}, P_d] = 0\,. \tag{2.3.19}$$

Notice that we recovered the condition $P_a\, P_b \sim 0$ that was already manifest in $D=3$, but we do not impose the stronger constraint $P_a \sim 0$ that characterises the flat limit of the scalar singleton proposed in [71]. Compared to the three-dimensional case, the eigenvalue of J^2 is instead fixed. Moreover, both the quadratic Casimir of Poincaré P^2 and the Pauli-Lubanski tensor [195]

$$\mathbb{W}_{[a_1\,\cdots\,a_{D-3}]} := \tfrac{1}{2}\, \varepsilon_{a_1\,\cdots\,a_{D-3}bcd}\, J^{bc}\, P^d\,, \tag{2.3.20}$$

vanish on account of the relations (2.3.16). This implies that all Casimir operators of the Poincaré algebra are set to zero in any representation satisfying eqs. (2.3.16a) and (2.3.16c), as one can appreciate by looking at their explicit expressions reported, e.g., in [196]. Therefore, we are looking at a massless, scalar representation.[9] However, eq. (2.3.16a) tells us that we are not dealing with an irreducible representation of the Poincaré group obtained in Wigner's classification (cf. [26, 27]), because if that were the case, there exists a state $|p\rangle$ characterised by its momentum p_a such that

$$P_a\, P_b\, |p\rangle = p_a\, p_b\, |p\rangle\,, \tag{2.3.21}$$

but the factoring out of the left-hand side can be obtained if and only if p_a is zero, meaning that we are in the zero-momentum representation. However, in this case $P_a\, |p\rangle = p_a |p\rangle = 0$ so that we are identifying the generator P_a to zero in the UEA of Poincaré. In part 3, we will propose a reducible but indecomposable representation of the Poincaré algebra satisfying eqs. (2.3.16).

We can now take the quotient of the Poincaré UEA by the two-sided ideal $\langle \mathcal{I}^\flat \rangle$ whose quadratic expressions are given in eqs. (2.3.16) and consider the resulting coset algebra as a flat-space higher-spin algebra in any dimensions D or as a Carrollian conformal higher-spin algebra in $(D-1)$ dimensions (see section 3.3):

$$\mathfrak{ihs}_D := \frac{\mathcal{U}(\mathfrak{iso}(1, D-1))}{\langle \mathcal{I}^\flat \rangle}\,. \tag{2.3.22}$$

The generators of this algebra are labelled by $M_{a(s-1),b(s-t-1)}$ like in $\mathfrak{hs}_D$, and spin-s generators are realised as products of $s-1$ generators of the Poincaré algebra.

[9] In the case of massless Poincaré representations, the Pauli-Lubanski pseudo-vector or its higher-dimensional generalisation is proportional to the generator of translations, where the proportionality constant encodes the helicity. In our case, $\mathbb{W}_{a_1\,\cdots\,a_{D-3}} \sim 0$, meaning that we are looking at a massless scalar.

As it is manifest from eq. (2.3.16a), these products contain at most one translation generator and, more precisely, none if t is even and one if t is odd. Since the t-even sub-algebra only contains products of J's, it can be viewed as a coset of the UEA of the Lorentz sub-algebra. Moreover, the completely anti-symmetric projection $\mathcal{I}_{[abcd]}$ is factored out, so that this sub-algebra is isomorphic to one of the higher-spin algebras for partially-massless fields (in $D-1$ dimensions and with de Sitter signature). The t-odd part can then be recovered by the adjoint action of P_a on the allowed products of J's, with the prescription that $\mathcal{I}_{[abc]} \sim 0$ and $\mathcal{I}_a \sim 0$. Computing an additional commutator with P_a does not produce new generators nor extra consistency conditions because the combination $P_a\, P_b$ is factored out.

Some commutators

Since the generators of $\mathfrak{ihs}_D$ are realised as products of Poincaré generators, they all transforms as Lorentz tensors. For instance, for $s=3$ one has

$$[J_{ab}, S_{cd}] = \eta_{bc}S_{ad} + \eta_{bd}S_{ac} - \eta_{ac}S_{bd} - \eta_{ad}S_{bc}\,, \tag{2.3.23a}$$

$$[J_{ab}, M_{cd,e}] = 2\,\eta_{b(c}M_{d)a,e} + \eta_{be}M_{cd,a} - 2\,\eta_{a(c}M_{d)b,e} - \eta_{ae}M_{cd,b}\,, \tag{2.3.23b}$$

$$[J_{ab}, K_{cd,ef}] = 2\left(\eta_{b(c}\,K_{d)a,ef} + \eta_{b(e}\,K_{f)a,cd} - \eta_{a(c}\,K_{d)b,ef} - \eta_{a(e}\,K_{f)b,cd}\right)\,, \tag{2.3.23c}$$

where we used the fact that $K_{ab,cd} = K_{cd,ab}$ to write the commutators in a compact form. On the other hand, their commutators with translations take a more 'exotic' form:

$$[P_a, S_{bc}] = 2\,M_{bc,a}\,, \tag{2.3.24a}$$

$$[P_a, M_{bc,d}] = 0\,, \tag{2.3.24b}$$

$$\begin{aligned}[P_a, K_{bc,de}] = {} & \eta_{ab}\,M_{de,c} + \eta_{ac}\,M_{de,b} + \eta_{ad}\,M_{bc,e} + \eta_{ae}\,M_{bc,d} \\ & + \tfrac{2}{D-2}\left(\eta_{d(b}M_{c)e,a} + \eta_{e(b}M_{c)d,a} - \eta_{bc}M_{de,a} - \eta_{de}M_{bc,a}\right).\end{aligned} \tag{2.3.24c}$$

Due to our choice of definition for the generators P_a and $M_{ab,c}$, the AdS radius does not appear explicitly in the commutators (2.3.23) and (2.3.24).

This structure generalises to any value of s, and we can always redefine the generators so that

$$\begin{aligned}[J_{cd}, M_{a(s-1),b(s-t-1)}] = {} & (s-1)\left(\eta_{da}\,M_{ca(s-2),b(s-t-1)} - \eta_{ca}\,M_{da(s-2),b(s-t-1)}\right) \\ & + (s-t-1)\left(\eta_{db}\,M_{a(s-1),cb(s-t-2)} - \eta_{cb}\,M_{a(s-1),db(s-t-2)}\right),\end{aligned} \tag{2.3.25}$$

and

$$\begin{aligned}[P_c, M_{a(s-1),b(s-2n-1)}] = {} & \eta_{c\{b}\,M_{a(s-1),b(s-2n-2)\}} \\ & + \beta_{s,s-2n}\,M_{a(s-1),b(s-2n-1)c}\,,\end{aligned} \tag{2.3.26a}$$

$$\left[P_c, M_{a(s-1),b(s-2n)}\right] = 0\,. \tag{2.3.26b}$$

The vanishing commutators in (2.3.26b) are clearly an exotic feature and do not respect the initial conditions for a flat-space higher-spin algebra displayed in (2.1.28). This implies that the curvatures of the algebra (2.3.22) do not reproduce upon linearisation the linear curvatures introduced in [58] to describe the free dynamics of higher-spin particles.

Nevertheless, in section 2.4, we will show that they allow to describe the propagation of free fields of arbitrary integer spin. These algebras are the first step of a possible new paradigm for the formulation of the free dynamics in order to lead to an interacting higher-spin gauge theory in Minkowski space via their gauging.

Classification of the possible ideals

In the previous pages we identified an ideal that allows us to obtain a higher-spin extension of the Poincaré algebra with the same spectrum as the Eastwood-Vasiliev algebra. We now prove that, under certain assumptions, this algebra is the only coset of the Poincaré UEA with the desired set of generators. In particular, we work under the hypothesis that all spin-s generators are built out of products of $(s-1)$ generators of $\mathfrak{iso}(1,D-1)$. This implies, by consistency, that all elements of the ideal to be factored out are homogeneous in the number of generators they contain. We shall thus ignore here the option to add dimensional-dependent terms as, e.g., those entering the ideal in $D=5$ whose peculiarities are discussed in [107].

As a first step, we have to identify an ideal such that, after its factoring out, only the generators transforming as the $\mathfrak{so}(1,D-1)$ Young diagrams $\mathbb{Y}_D(2)$, $\mathbb{Y}_D(2,1)$ and $\mathbb{Y}_D(2,2)$ in eq. (2.3.12) are left from the quadratic combinations of Poincaré generators listed in eqs. (2.3.4) – (2.3.9). Achieving this goal requires to factor out the fully anti-symmetric combinations $\mathcal{I}_{abc}$ and $\mathcal{I}_{abcd}$. From the commutation relations presented in eq. (2.3.19) it can be seen they form in themselves an ideal, consistently with their interpretation as the components of $\mathcal{I}_{ABCD}$. Therefore, we can consistently factor out these two combinations.

Among the remaining quadratic combinations, only $K_{ab,cd}$ transforms as a $\mathbb{Y}_D(2,2)$ Young diagram, so that we have to keep it. Similarly, only $M_{ab,c}$ fits the role of the $\mathbb{Y}_D(2,1)$ generator. The delicate point is that both Q_{ab} and S_{ab} display the correct Lorentz transformations to fill the role of the remaining spin-three generator. However, we still have to handle the vector $\mathcal{I}_a$, that cannot belong to the set of generators of the higher-spin algebra since the vector P_a already plays this role. Keeping $\mathcal{I}_a$ would thus both introduce an unwanted multiplicity and violate our hypothesis on the structure of the generators. Requiring $\mathcal{I}_a \sim 0$ then implies that both Q_{ab} and P^2 have to vanish as well when quotienting the ideal because of eq. (2.3.18).

Summarising, factoring out $\mathcal{I}_a$, $\mathcal{I}_{abc}$ and $\mathcal{I}_{abcd}$ from the UEA of $\mathfrak{iso}(1,D-1)$, as required to match the Eastwood-Vasiliev spectrum, implies as well the condition $P_a\, P_b \sim 0$.

What remains to be determined is the fate of J^2 . As it is manifest in (2.3.17), it becomes a central element thanks to the previous conditions. It is thus natural to set it proportional to the identity so as to avoid multiplicities in the spectrum. Its eigenvalue is then fixed by

$$\mathcal{I}_{abc}\, J^{bc} - \frac{2}{3}\, J_{ab}\, \mathcal{I}^{b} + \frac{D-3}{3}\, \mathcal{I}_{a} = -\frac{4}{3}\left(J^2 + \frac{(D-1)(D-3)}{4}\,\mathrm{id}\right) P_a\,, \qquad (2.3.27)$$

and the requirement that the left-hand side be factored out.

In conclusion, if one wants to build a higher-spin extension of the Poincaré algebra with the Eastwood-Vasiliev spectrum (2.1.10) as a quotient of its UEA by a two-sided ideal, one can only obtain the coset algebra (2.3.22).

Notice that one can proceed along the same lines to recover the Eastwood-Vasiliev algebra as a coset of the UEA of $\mathfrak{so}(2, D-1)$, but eq. (2.3.18) is substituted by

$$\epsilon^{-2}\,[P_a\,,\,\mathcal{I}_b] = -\left(S_{ab} + \epsilon^{-2} Q_{ab}\right) - \frac{4}{D}\,\eta_{ab}\left(J^2 - \frac{D-1}{2}\,\epsilon^{-2} P^2\right)\,, \qquad (2.3.28)$$

and thus implies (2.3.10a) and (2.3.10c).

Finally, let us mention that the splitting of the AdS$_D$ ideal into Lorentz-irreducible parts (2.3.10) was already displayed in [70], and that its flat limit in $D = 4$ was already taken in [191], but was discarded as a candidate algebra for higher-spin symmetry in flat space, since the nilpotent character of translations was deemed too problematic.

2.4 Free dynamics from the higher-spin curvatures

Defining the algebra $\mathfrak{ihs}_D$, we noticed that its structure constants do not reproduce the $R \to \infty$ limit of those advertised in (2.1.28b). In turn, this means that the linearised curvatures of this algebra are not the $R \to \infty$ limit of those identified in (2.1.26).

The burning question is then what system can these linearised curvatures describe. In order to do this, we will begin by analysing the form of the linearised curvature $\bar{F}^{a(s-1),b(t)}$ for the gauge potentials $\omega^{a(s-1),b(t)}$ and show that, by imposing the same equations of motion as in the usual case (2.1.19a) and (2.1.19b), our set of linearised curvatures are still equivalent to the Lopatin-Vasiliev set of equations [64] describing the propagation of a spin-s particle on Minkowski background.

The reason for this rather surprising result is actually quite simple: one does not need to impose the full set of equations (2.1.19a) and (2.1.19b) in order to describe the Fronsdal equation whose components are carried by the frame-like $e^{a(s-1)}$. Instead, one may choose the tower of equations at any point we like and reconstruct the missing equations by the Poincaré lemma. This fact was already noticed in more abstract terms in [165, 166, 167], and we provide here a step-by-step explanation of its inner working.

A crucial element in this discussion is that the gauge field $\omega^{a(s-1)}$, which has the symmetries to be identified with the frame-like field (2.1.1), turns out to be pure gauge as soon as $s > 2$, which is also true of most of the other gauge potentials. Instead, the first non-pure-gauge connection of the tower is $\omega^{a(s-1),b(s-2)}$, which for $s = 2$ coincides with the vielbein.

2.4.1 New higher-spin curvatures in flat space

We recall that the commutators $[M_{a(s_1-1),b(s_1-t_1-1)}, M_{c(s_2-1),d(s_2-t_2-1)}]$ of $\mathfrak{ihs}_D$ are the same as the corresponding commutators in $\mathfrak{hs}_D$ when t_1 or t_2 are even, but vanish when t_1 and t_2 are odd. In particular when $s_1 = 2$ and $t_1 = 1$, the commutators are given by eq. (2.1.28b) when t_2 is even (where the generators have been redefined so that the AdS radius does not appear) and vanish when t_2 is odd.

Let us consider a potential one-form taking values in the Lie algebra $\mathfrak{ihs}_D$,

$$A = \sum_{s=1}^{\infty} \sum_{t=0}^{s-1} \omega^{a(s-1),b(s-t-1)} M_{a(s-1),b(s-t-1)} \,, \tag{2.4.1}$$

and its Yang-Mills curvature,

$$\mathrm{d}A + A \wedge A = \sum_{s=1}^{\infty} \sum_{t=0}^{s-1} F^{a(s-1),b(s-t-1)} M_{a(s-1),b(s-t-1)} \,. \tag{2.4.2}$$

The one-forms ω^a and $\omega^{a,b}$ correspond to the space-time vielbein and spin connection respectively. Consider the perturbative expansion $\omega^a = h^a + e^a$ and, for simplicity, let us choose Cartesian coordinates so that the background vielbein is $h_\mu{}^a = \delta_\mu{}^a$ and the background spin connection vanishes.

The linearisation (denoted with a bar) of the curvatures $F^{a(s-1),b(s-t-1)}$ around the Minkowski background only depend on the commutators between the higher-spin generators $M_{a(s-1),b(s-t-1)}$ and those of the Poincaré sub-algebra

$$\bar{F}^{a(s-1),b(s-2n-1)} := \mathrm{d}\omega^{a(s-1),b(s-2n-1)} \,, \tag{2.4.3a}$$

$$\begin{aligned} \bar{F}^{a(s-1),b(s-2n)} := {} & \mathrm{d}\omega^{a(s-1),b(s-2n)} + h_c \wedge \omega^{a(s-1),b(s-2n)c} \\ & + \beta_{s,s-2n}\, h^{\{b} \wedge \omega^{a(s-1),b(s-2n-1)\}} \,, \end{aligned} \tag{2.4.3b}$$

where $0 \leq n \leq \left\lfloor \frac{s-1}{2} \right\rfloor$ in the first equation and $1 \leq n \leq \left\lfloor \frac{s}{2} \right\rfloor$ in the second one. Note that, although they involve the same type of terms as the linearised Lopatin-Vasiliev equations, they are in fact quite different. Indeed, they contain both σ_+ and σ_- terms when t is odd, and none when t is even.

These linearised curvatures can be obtained as a limit $R \to \infty$ of the linearised curvatures of Lopatin and Vasiliev, provided one rescales the fields $\omega^{a(s-1),b(s-t-1)}$ by R^t if t is even, and R^{t-1} if t is odd (equivalently, one can keep the cosmological constant fixed to a finite value and rescale the fields with t odd, including the background vielbein h^a, by a factor ϵ then sent to 0).

The linearised curvatures (2.4.3) are invariant under the gauge transformations

$$\delta\omega^{a(s-1),b(s-2n-1)} = \mathrm{d}\lambda^{a(s-1),b(s-2n-1)}\,, \tag{2.4.4a}$$

$$\begin{aligned}\delta\omega^{a(s-1),b(s-2n)} &= \mathrm{d}\lambda^{a(s-1),b(s-2n)} + h_c\,\lambda^{a(s-1),b(s-2n)c} \\ &\quad + \beta_{s,s-2n}\,h^{\{b}\,\lambda^{a(s-1),b(s-2n-1)\}}\,. \end{aligned}\tag{2.4.4b}$$

In order to proceed, we need to impose equations of the type of the ones usually implemented in [64] and recalled in section 2.1.2 that were shown to describe the propagation of a spin-s particle. However, it is not at all obvious which one we should impose at this stage. The spin-three case is instructive to start with.

2.4.2 Equations on the curvatures

Spin-three case

For a spin-three particle eqs. (2.4.3) read

$$\bar{F}^{ab} := \mathrm{d}e^{ab}\,, \tag{2.4.1a}$$

$$\bar{F}^{ab,c} := \mathrm{d}\omega^{ab,c} + h_d \wedge X^{ab,cd} + \beta_{3,1}\,h^{\{c} \wedge e^{ab\}}\,, \tag{2.4.1b}$$

$$\bar{F}^{ab,cd} := \mathrm{d}X^{ab,cd}\,, \tag{2.4.1c}$$

where, for clarity, we renamed the fields $\omega^{ab} \to e^{ab}$ and $\omega^{ab,cd} \to X^{ab,cd}$. These equations are invariant under

$$\delta e^{ab} = \mathrm{d}\xi^{ab}\,, \tag{2.4.2a}$$

$$\delta\omega^{ab,c} = \mathrm{d}\lambda^{ab,c} + h_d\,\rho^{ab,cd} + \beta_{3,1}\,h^{\{c}\,\xi^{ab\}}\,, \tag{2.4.2b}$$

$$\delta X^{ab,cd} = \mathrm{d}\rho^{ab,cd}\,. \tag{2.4.2c}$$

The first curvature $\bar{F}^{ab}$ only involves the exterior derivative of the frame-like field e^{ab}, therefore setting it to zero would mean that the latter becomes pure-gauge thanks to the Poincaré lemma

$$\bar{F}^{ab} \stackrel{!}{=} 0 \quad \Longrightarrow \quad e^{ab} = \mathrm{d}S^{ab}\,, \tag{2.4.3}$$

where S^{ab} is a zero-form field with the same symmetry in its fibre indices, and whose gauge variation is given by

$$\delta S^{ab} = \xi^{ab}\,. \tag{2.4.4}$$

Let us pick a gauge where $S^{ab} = 0$, which fixes the gauge parameter ξ^{ab}. In this gauge, the remaining equations read

$$\bar{F}^{ab,c} := \mathrm{d}\omega^{ab,c} + h_d \wedge X^{ab,cd}\,, \tag{2.4.5a}$$
$$\bar{F}^{ab,cd} := \mathrm{d}X^{ab,cd}\,, \tag{2.4.5b}$$

and the corresponding gauge variations

$$\delta\omega^{ab,c} = \mathrm{d}\lambda^{ab,c} + h_d\,\rho^{ab,cd}\,, \tag{2.4.6a}$$
$$\delta X^{ab,cd} = \mathrm{d}\rho^{ab,cd}\,. \tag{2.4.6b}$$

Notice that the last two curvatures take exactly the same form as the last two curvatures of the Lopatin-Vasiliev tower of unfolded equations. Driven by this analogy, we impose

$$\bar{F}^{ab,c} \stackrel{!}{=} 0\,, \quad \bar{F}^{ab,cd} \stackrel{!}{=} h_e \wedge h_f\, C^{abe,cdf}\,, \tag{2.4.7}$$

where $C^{abe,cdf}$ is a Lorentz-irreducible zero-form.

In the usual case, the auxiliary fields $\omega^{ab,c}$ and $X^{ab,cd}$ are expressed in terms of the first and second derivative of a frame-like field e^{ab}. Here, this is seemingly not the case since the latter is pure gauge. Nevertheless, we can reconstruct such a field by noticing that

$$h_c \wedge \bar{F}^{ab,c} = h_c \wedge \mathrm{d}\omega^{ab,c} + h_c \wedge h_d \wedge X^{ab,cd} = h_c \wedge \mathrm{d}\omega^{ab,c} = -\mathrm{d}\left(h_c \wedge \omega^{ab,c}\right)\,, \tag{2.4.8}$$

where we used that $h_c \wedge h_d$ is anti-symmetric in c and d, and that $\mathrm{d}h_c = 0$. The Poincaré lemma then allows one to introduce the one-form $\tilde{e}^{ab}$ such that

$$-h_c \wedge \omega^{ab,c} = \mathrm{d}\tilde{e}^{ab}\,, \tag{2.4.9}$$

which has the gauge variation

$$\delta\tilde{e}^{ab} = \mathrm{d}\tilde{\xi}^{ab} + h_c \wedge \omega^{ab,c}\,. \tag{2.4.10}$$

The existence of the field $\tilde{e}^{ab}$ and associated parameter $\tilde{\xi}^{ab}$ is guaranteed by the Poincaré lemma, but has nothing to do with the gauge field e^{ab} and parameter ξ^{ab}, which are both zero in our gauge. Therefore, we were able to reconstruct the full tower of Lopatin-Vasiliev equations of motion in the limit $R \to \infty$, presented in eqs. (2.1.19a) and (2.1.19b), for the fields $\tilde{e}^{ab}$, $\omega^{ab,c}$ and $X^{ab,cd}$

$$\mathrm{d}\tilde{e}^{ab} + h_c \wedge \omega^{ab,c} = 0\,, \tag{2.4.11a}$$
$$\mathrm{d}\omega^{ab,c} + h_d \wedge X^{ab,cd} = 0\,, \tag{2.4.11b}$$
$$\mathrm{d}X^{ab,cd} = h_e \wedge h_f\, C^{abe,cdf}\,, \tag{2.4.11c}$$

with the gauge symmetries

$$\delta\tilde{e}^{ab} = \mathrm{d}\tilde{\xi}^{ab} + h_c\,\lambda^{ab,c}\,, \tag{2.4.12a}$$
$$\delta\omega^{ab,c} = \mathrm{d}\lambda^{ab,c} + h_d\,\rho^{ab,cd}\,, \tag{2.4.12b}$$
$$\delta X^{ab,cd} = \mathrm{d}\rho^{ab,cd}\,. \tag{2.4.12c}$$

One can then extract a Fronsdal field $\varphi_{\mu\nu\rho} = h_{(\mu}{}^a\, h_\nu{}^b\, \tilde{e}_{\rho)ab}$ as usual and observe that the first two equations impose the Fronsdal equation, as discussed in section 2.1.2.

Note, however, that the field $\tilde{e}^{ab}$ is not part of the original gauge description of the free theory, and is rather a by-product of the consistency of the linearised equations.

Arbitrary spin

When the spin $s \geq 4$ is arbitrary, one can apply a similar mechanism. First, note that we can get rid of the fields $\omega^{a(s-1),b(s-2n-1)}$ with $1 \leq n \leq \left\lfloor\frac{s-1}{2}\right\rfloor$ by imposing the equations

$$\bar{F}^{a(s-1),b(s-2n-1)} \stackrel{!}{=} 0 \quad\Longrightarrow\quad \omega^{a(s-1),b(s-2n-1)} = \mathrm{d}S^{a(s-1),b(s-2n-1)}\,, \tag{2.4.13}$$

for some zero-forms $S^{a(s-1),b(s-2n-1)}$ thanks to the Poincaré lemma. The latter are shifted by the gauge parameters

$$\delta S^{a(s-1),b(s-2n-1)} = \lambda^{a(s-1),b(s-2n-1)}\,, \tag{2.4.14}$$

so we can pick a gauge where they are both zero. In this gauge, which is the generalisation of $S^{ab} = 0$ to arbitrary spin, all other curvatures but the last two become

$$\bar{F}^{a(s-1),b(s-2n)} = \mathrm{d}\omega^{a(s-1),b(s-2n)} \qquad (\text{gauge-fixed})\,, \tag{2.4.15}$$

for $2 \leq n \leq \left\lfloor\frac{s}{2}\right\rfloor$, and setting these curvatures to zero allows one to gauge away the corresponding connections

$$\bar{F}^{a(s-1),b(s-2n)} \stackrel{!}{=} 0 \quad\Longrightarrow\quad \omega^{a(s-1),b(s-2n)} = \mathrm{d}S^{a(s-1),b(s-2n)}\,, \tag{2.4.16}$$

with

$$\delta S^{a(s-1),b(s-2n)} = \lambda^{a(s-1),b(s-2n)}\,, \tag{2.4.17}$$

Therefore, the only non-zero curvatures in this gauge are

$$\bar{F}^{a(s-1),b(s-2)} = \mathrm{d}\omega^{a(s-1),b(s-2)} + h_c \wedge \omega^{a(s-1),b(s-2)c} \quad (\text{gauge-fixed})\,, \tag{2.4.18a}$$
$$\bar{F}^{a(s-1),b(s-1)} = \mathrm{d}\omega^{a(s-1),b(s-1)}\,, \tag{2.4.18b}$$

which, again, correspond with those emerging from the limit $R \to \infty$ of Lopatin-Vasiliev equations of motion. From there, we impose

$$\bar{F}^{a(s-1),b(s-2)} \stackrel{!}{=} 0\,, \tag{2.4.19a}$$

$$\bar{F}^{a(s-1),b(s-1)} \stackrel{!}{=} h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d} \tag{2.4.19b}$$

like in the usual case. The Bianchi identities in this case read

$$\mathrm{d}\bar{F}^{a(s-1),b(s-2n-1)} = 0\,, \tag{2.4.20}$$

for $1 \leq n \leq \left\lfloor \frac{s-1}{2} \right\rfloor$ and

$$\mathrm{d}\bar{F}^{a(s-1),b(s-2n)} = -h_c \wedge \bar{F}^{a(s-1),b(s-2n)c} - \beta_{s,s-2n}\, h^{\{b} \wedge \bar{F}^{a(s-1),b(s-2n-1)\}}\,, \tag{2.4.21}$$

for $1 \leq n \leq \left\lfloor \frac{s}{2} \right\rfloor$. They are consistent with eqs. (2.4.19), and the consistency of $\bar{F}^{a(s-1),b(s-1)} = h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}$ in the zero-form sector are the same as the ones in the usual case, see eq. (2.1.22).

We can verify that we can reconstruct the whole hierarchy of one-forms corresponding to the limit $R \to \infty$ of Lopatin and Vasiliev equations of motion

$$0 = h_c \wedge \bar{F}^{a(s-1),b(s-3)c} = -\mathrm{d}\left(h_c \wedge \omega^{a(s-1),b(s-3)c}\right)\,, \tag{2.4.22}$$

which leads, thanks to the Poincaré lemma, to

$$-h_c \wedge \omega^{a(s-1),b(s-3)c} = \mathrm{d}\tilde{\omega}^{a(s-1),b(s-3)}\,, \tag{2.4.23}$$

and so on. The final set of equations reproduces the limit $R \to \infty$ of Lopatin and Vasiliev

$$\mathrm{d}\tilde{\omega}^{a(s-1),b(t)} + h_c \wedge \tilde{\omega}^{a(s-1),b(t)c} = 0\,, \qquad 0 \leqslant t \leqslant s-2\,, \tag{2.4.24a}$$

$$\mathrm{d}\tilde{\omega}^{a(s-1),b(s-1)} = h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}\,, \tag{2.4.24b}$$

where we also renamed for simplicity $\omega^{a(s-1),b(s-1)} \to \tilde{\omega}^{a(s-1),b(s-1)}$ and $\omega^{a(s-1),b(s-2)} \to \tilde{\omega}^{a(s-1),b(s-2)}$. Of the family of gauge potentials $\omega^{a(s-1),b(s-t-1)}$, only the $t = 0$ and $t = 1$ members are not pure-gauge in the final description of the unfolded dynamics.

Since the full theory is $\mathfrak{so}(1, D-1)$-invariant, we can generalise the previous equations to an arbitrary system of coordinates by replacing $h_\mu{}^a = \delta_\mu{}^a$ with any background vielbein and the exterior derivative d into the Lorentz-covariant derivative ∇ of Minkowski, verifying $\nabla h^a = 0$ and $\nabla^2 = 0$.

2.5 Discussion

In this chapter, we discussed the construction of an algebra for higher-spin symmetry in Minkowski space-time, and showed that its gauging reproduces, at the linear level, the Lopatin-Vasiliev equations of motion. There are still a number of open questions with this new formalism, related to the fact that we proceeded counter flow (define the algebra first then the equations of motion) compared to the usual case described in section 2.1 (find an algebra for the symmetries of the equations of motion). Let us stress, however, that our algebra and the corresponding curvatures are nothing more than the limit $\epsilon \to 0$ of the curvatures of Lopatin and Vasiliev [64] where we rescaled the fields with t odd by a factor of ϵ^{-1}, which is distinct from the limit $R \to \infty$ of eq. (2.1.26).

One still open question is if there exist an alternative description of the free theory, either in the metric-like or in the frame-like formulation, that possesses the same spectrum of reducibility parameters as the Fronsdal case, but where the Lie derivative along a constant vector reproduces the bracket of eq. (2.3.26)?

The reducibility parameters of the system described in (2.4.3) are given by the rigid symmetries of the system, i.e. the symmetries of the vacuum solution $\omega^{a(s-1),b(s-t-1)} = 0$ in eq. (2.4.4) for all $t \in \{0, \ldots, s-1\}$. We find

$$\lambda^{a(s-1),b(s-2n-1)} = \Lambda^{a(s-1),b(s-2n-1)}\,, \tag{2.5.1a}$$

$$\begin{aligned}\lambda^{a(s-1),b(s-2n)} &= \Lambda^{a(s-1),b(s-2n)} - x_c\,\Lambda^{a(s-1),b(s-2n)c} \\ &\quad - \beta_{s,s-2n}\,x^{\{b}\,\Lambda^{a(s-1),b(s-2n-1)\}}\,,\end{aligned} \tag{2.5.1b}$$

for some constant tensors $\Lambda^{a(s-1),b(s-t-1)}$. In this form, it is clear that the Lie bracket of spin-two isometries with the parameters of global isometries $\lambda^{a(s-1),b(s-t-1)}$ reproduces the Lie brackets (2.3.25) and (2.3.26).

However, it is less clear at the moment how to find a second-order field equation possessing the $\lambda^{a(s-1),b(s-t-1)}$ of eq. (2.5.1) as reducibility parameters. Clearly, one has to start with a new description of the dynamics making use of another field than the Fronsdal. One candidate to replace the Fronsdal field is the symmetric component of the connection ${\omega_\mu}^{a(s-1),b(s-2)}$

$${h_\mu}^a \cdots {h_\mu}^a\,{h_\nu}^b \cdots {h_\nu}^b\,\omega_{\rho\,a(s-1),b(s-2)}\,. \tag{2.5.2}$$

which has mixed symmetry, and one can draw inspiration from the case of Labastida fields [39] in order to build an equation of motion.

We also investigated an action principle resembling the one using the first two fields, aiming at reproducing directly the equations of motion (2.4.19)

$$\begin{aligned}\int \mathrm{d}^D x\,\left(\mathrm{d}\omega^{a_1 a(s-2),b(s-2)} + \tfrac{1}{2}h_c \wedge \omega^{a_1 a(s-2),b(s-2)c}\right)& \\ \wedge\,{{\omega^{a_2}}_{a(s-2),b(s-2)}}^{a_3} \wedge K_{a_1a_2a_3}\,,&\end{aligned} \tag{2.5.3}$$

where $K_{a_1a_2a_3}$ is defined near eq. (2.1.13), which is a natural generalisation of the action presented in [57]. However, variation with respect to $\omega^{a(s-1),b(s-1)}$ yields only a projection of the torsion constraint

$$T^{a(s-1),b(s-2)}\big|_{\mathbb{Y}_D(s,s-1)\,\oplus\,\mathbb{Y}_D(s-1,s-2)\,\oplus\,\mathbb{Y}_D(s-1,s-1,1)} = 0\,, \tag{2.5.4}$$

because $\omega^{a(s-1),b(s-1)}$ contains less Lorentz-irreducible components than $T^{a(s-1),b(s-2)}$, while this is not an issue for the action in eq. (2.1.13) because $\omega^{a(s-1),b}$ contains more Lorentz-irreducible components than $T^{a(s-1)}$. It might be worth trying to reproduce the non-gauge-fixed version of these equations of motion instead, i.e. including contributions from the other fields with $t \geq 2$.

The fact that the 'last' connections within the frame-like approach $\omega^{a(s-1),b(s-2)}$ and $\omega^{a(s-1),b(s-1)}$ play a more fundamental role than the 'first' ones $\omega^{a(s-1)}$ and $\omega^{a(s-1),b}$ may seem counter-intuitive, but it mirrors independent observations in chiral higher-spin gravity [197], that proposes an alternative way to build fully interacting higher-spin theories on self-dual four-dimensional manifolds.[10]

If all the connections $\omega^{a(s-1),b(s-t-1)}$ with $t \geq 2$ are actually pure gauge in our system, one might wonder if it is really necessary to introduce them in the first place. However, if one remains within the UEA approach, one is forced to do so. Indeed, from the classification of section 2.3.2 at the quadratic level, one can deduce that a higher-spin algebra containing the generator $M_{a(2),b(2)}$ and $M_{a(2),b}$ must also contain the generator $M_{a(2)}$, and therefore reproduce the algebra $\mathfrak{ihs}_D$, which is an additional argument in favour of its role as a higher-spin algebra in flat space.

Let us mention that process of reconstruction a Lopatin-Vasiliev branch out of a single torsion constraint is in fact quite general and not restricted to our algebra, since it relies on the application of identifying pure-gauge fields and applying the Poincaré lemma. Any algebra satisfying the Jacobi identity and which allows for this process to happen may be qualified as a potentially interesting algebra for flat space higher-spin symmetry. For instance, taking the 'exotic' contraction based on the sub-algebra of even $s+t$ (which corresponds to the h_2 sub-algebra of [60]), the curvatures are the same as (2.4.3a) and (2.4.3b) for even spin, but one obtains for odd spin

$$\begin{aligned}\bar{F}^{a(s-1),b(s-2n-1)}_{\text{exotic}} &:= \mathrm{d}\omega^{a(s-1),b(s-2n-1)} + h_c \wedge \omega^{a(s-1),b(s-2n-1)c} \\ &\quad + \beta_{s,s-2n-1}\, h^{\{b} \wedge \omega^{a(s-1),b(s-2n-2)\}}\,, \end{aligned}\tag{2.5.5a}$$

$$\bar{F}^{a(s-1),b(s-2n)}_{\text{exotic}} := \mathrm{d}\omega^{a(s-1),b(s-2n)}\,. \tag{2.5.5b}$$

By imposing still the same equations as before

$$\bar{F}^{a(s-1),b(s-t-1)}_{\text{exotic}} \stackrel{!}{=} 0\,, \qquad \bar{F}^{a(s-1),b(s-1)}_{\text{exotic}} \stackrel{!}{=} h_c \wedge h_d\, C^{a(s-1)c,b(s-1)d}\,, \tag{2.5.6}$$

[10] In this approach, the full power of spinor notation is leveraged, so all the various spin-connections have chiral and anti-chiral components with a total of $(2s-2)$ frame indices, making it clear that they play an equivalent role.

for $1 \leq t \leq s-1$, one can play a similar game as before: the fields $\omega^{a(s-1),b(s-2n)}$ are pure gauge for $1 \leq n \leq \frac{s-1}{2}$, so we can choose a gauge where they vanish. In this gauge, we can also gauge way $\omega^{a(s-1),b(s-2n-1)}$ for $1 \leq n \leq \frac{s-1}{2}$ so that the only remaining equation is

$$\bar{F}^{a(s-1),b(s-1)}_{\text{exotic}} \stackrel{!}{=} h_c \wedge h_d \, C^{a(s-1)c,b(s-1)d} \,, \tag{2.5.7}$$

which is enough to reconstruct a new tower of connections, ending with a frame-like field from the identity

$$0 = h_c \wedge \bar{F}^{a(s-1),b(s-2)c}_{\text{exotic}} = -\mathrm{d}\left(h_c \wedge \omega^{a(s-1),b(s-2)c}\right) , \tag{2.5.8}$$

and the Poincaré lemma, recursively. Therefore this exotic unfolded set of equations propagates again the correct dynamics at the linearised level, even though the underlying algebra cannot be built upon a quotient of the UEA of the Poincaré algebra.

Once the free theory is identified, the next logical step is to introduce interactions. In [130], it was noticed that a consistent non-Abelian coupling of a spin-three field with a spin-two field (the $3-3-2$ vertex) that deforms the algebra of gauge symmetries exists, at the price of introducing higher-derivative terms. In [55], the analysis was pushed to the coupling of a spin-s field with a spin-2 field and the same pattern was verified. This coupling introduces $(2s-2)$ derivatives and can be viewed as the non-uniform limit $R \to \infty$ of the Fradkin-Vasiliev top vertex, also containing $(2s-2)$ covariant derivatives but completed with a decreasing number of derivatives, due to non-zero commutation relations.

Concerning the cubic interactions of our system, we have already access to a class of non-Abelian ones from the structure constants of the algebra $\mathfrak{ihs}_D$. In particular, a class of gravitational couplings of the form $s-s-2$ is given by the brackets $[M^{a(s-1),b(s-1)}, M^{c(s-1),d(s-2)}] \propto P^e + \dots$ which are present both in the AdS and flat-space algebras. As explained in [108], they encode the non-minimal gravitational coupling found in [130, 55] by BV-BRST arguments. This is so because the $(2s-2)$-derivative vertex is the one that possesses the highest number of derivatives among those that constitute the Fradkin-Vasiliev gravitational coupling in AdS, that has later been reproduced within the unfolded formulation. Only this top vertex survives the flat limit that coincides with the high-energy limit of the Fradkin-Vasiliev action. The advantage of using the penultimate spin connection as a fundamental field is that, on-shell, it encodes not a Fronsdal field but $(s-2)$ derivatives thereof. Therefore, a two-derivative cubic $2-s-s$ coupling using this connection is in fact a $(2s-2)$-derivative coupling in terms of a Fronsdal field, in agreement with the existing classification.

Another approach to quartic and higher-order interactions is to use the formal construction of [198, 199]. The possible interactions are encoded in the Hochschild cohomology class of the higher-spin algebra, which measures its potential to be deformed. Our algebra retains many features of the AdS higher-spin algebra, which are known to be deformed. In addition, it contains an Abelian ideal, which usually signals more possibilities.

Chapter 3

Higher-spin symmetry of Carrollian Conformal Field Theories

In this part, we will be concerned with the appearance of higher-spin symmetry at the asymptotic boundary of space-time, which is a first step towards a *holographic* realisation of the putative theory in flat space described in the previous part. The AdS/CFT correspondence – which is a particular realisation of the idea of holography – was first derived in the context of string theory [73], but its influence has permeated through to many other aspects of physics, and in particular to gravitational physics in a broad sense. In its original form, it establishes a duality between type IIB string theory formulated in the *bulk* (i.e. the interior) of AdS_5 space-time times an internal manifold S^5, with a 'dual' theory formulated on its *boundary* $\partial\mathrm{AdS}_5 = \mathbb{R}^{1,3}$, which is a certain field theory enjoying conformal invariance, namely $\mathcal{N} = 4$ super-symmetric Yang-Mills. This equivalence was proven in the limit where the string coupling constant in bulk is small, corresponding to the limit where the number of colors in the Yang-Mills theory (the size of its matrix group) is large. The conjecture that this correspondence holds outside of this regime of parameters is the object of intense research, since it could provide a description of the quantum regime of gravity. The holographic correspondence has also been extended to other gravitational theories aside from type IIB string theory, including (but not limited to) three-dimensional gravity with a negative cosmological constant and the description of black-hole horizons.

As far as higher-spin holography is concerned, a lot of progress has been made in understanding what the dual theory to AdS higher-spin gravity is. In its simplest form, the holographic dual of AdS higher-spin gravity is given by a single free scalar field living on its conformal boundary, while more refined models involve the large N limit of a collection of N scalar fields interacting via a quartic coupling term [74, 75]. One of the basic reasons why holography works can be found in symmetry: the algebra of rigid isometries of AdS_D and the algebra of conformal transformations of $\mathbb{R}^{1,D-2}$ are both isomorphic to $\mathfrak{so}(2, D-1)$. In the bulk, gravity *gauges* space-time isometries, while the fact that the boundary theory preserves conformal isometries guarantees that its stress-energy tensor is conserved, symmetric and traceless. This statement extends to higher-spin holography, where the gauged higher-spin symmetry in the bulk giving rise to AdS higher-spin gravity corresponds to an extension of the rigid conformal symmetries of the boundary theory to include conserved higher-spin currents. In the AdS/CFT dictionary, bulk higher-spin gauge fields then couple to higher conserved currents (for a review, see e.g., [200, 62, 201]). The requirement that the boundary field theory has exact higher-spin symmetry is quite constraining, while theories with slightly-broken (i.e. exact up to order $1/N$ in a large-N expansion) higher-spin symmetry gives more flexibility and would correspond to a situation where higher-spin symmetry in the bulk is broken by quantum effects. In the free theory with N scalars, the value of N and of the dimension of the boundary play little role. However, when the boundary theory becomes interacting thanks to the addition of a quartic coupling, it flows to an interacting fixed point in the IR which is stable only when the boundary space-time dimension is three, see e.g., [202].

Evidence for the holographic character of higher-spin gravity can also be gathered from other viewpoints. As an example, the spectrum of higher-spin gravity in AdS described in the previous part of this thesis, that is composed of bulk massless fields of all spins with multiplicity one, can be recovered from group-theoretical arguments, by considering the decomposition in irreducible representations of the tensor product of two scalar representations of a certain type, called the *singleton* and reviewed in section 3.1. This is known as the Flato-Fronsdal theorem [71], and it states that the tensor product of two singleton modules can be decomposed into the direct sum of massless higher-spin irreducible unitary representations of AdS_D with multiplicity one. The singleton will play a fundamental role in the following, since it is the free boundary scalar field upon which higher-spin symmetry is realised as higher-differential operators, following an argument by Eastwood [65]. The singleton, which we already encountered in section 2.2 near eq. (2.2.14) is a short irreducible representation of the $\mathfrak{so}(2, D-1)$ algebra. It also admits a realisation as an AdS scalar field, and its ambient space reformulation allows to prove easily that its algebra of higher-differential symmetries is $\mathfrak{hs}_D$.

Much less is known about holography in asymptotically flat space-time. Early on, it was realised that the asymptotic symmetries of gravity are not the expected group of Poincaré symmetries of the vacuum Minkowski space, but an infinite-dimensional enhancement called BMS, following the work of Bondi, van der Burg, Metzner and Sachs [97, 98]. The BMS group is the semi-direct product of usual Lorentz transformations and an infinite-dimensional Abelian factor called *super-translations*, enhancing the usual space-time translations. By relaxing the conditions on the form of the boundary metric, it is possible to enhance the BMS group further to include local conformal transformations of the two-sphere S^2, obtaining the generalised BMS group $\mathrm{Diff}(S^2) \ltimes \mathcal{C}^\infty(S^2, \mathbb{R})$, where the generators of $\mathrm{Diff}(S^2)$ are called *super-rotations* [99].[1]

In arbitrary dimension, the co-dimension one conformal boundary of asymptotically Minkowski$_D$ space-time is non-Lorentzian and described by a pair of null manifolds, called past (resp. future) null infinity $\mathscr{I}^-$ (resp. $\mathscr{I}^+$, abbreviated to $\mathscr{I}$ when talking about either one of them) whose geometry is locally diffeomorphic to the product of the real line $\mathbb{R}$ and the sphere S^d, where $d = D-2$, and whose coordinates are called the advanced time u (resp. retarded time v) and d angular coordinates $\mathbf{x}^i$. The metric is degenerate in the direction of u, e.g., $\mathrm{d}s^2 = 0 \times \mathrm{d}u^2 + \gamma_{ij}(\mathbf{x})\, \mathrm{d}\mathbf{x}^i \mathrm{d}\mathbf{x}^j$ (see e.g. appendix A).

Such null manifolds are often called Carrollian, since their symmetries are given by the eponymous group following the work of Lévy-Leblond [203] on the limit of vanishing speed of light of the Poincaré group. Interestingly, the isometries of the Carrollian conformal manifold $\mathbb{R} \times S^d$ (see appendix A) were shown to reproduce the restriction of BMS symmetries to $\mathscr{I}$ [87, 104, 204, 205, 206], where super-translations are angle-dependant shifts in the null time, thus confirming that the study of Carrollian conformal field theories constitute a promising route to flat-space holography. There is also a stream of independent evidence pointing to Carrollian theories for the description of gravity in asymptotically Minkowski space [94, 95]. For instance in the fluid-gravity correspondence, the AdS bulk theory is dual to a boundary theory which is ruled by relativistic fluid equations and the limit $R \to \infty$ in the bulk corresponds to the limit $k \to 0$ on the boundary, where the speed of light k is directly proportional to $\frac{1}{R^2}$ [92, 207].

In the following, we will argue that the correct framework to describe a holographic dual of the flat-space higher-spin gravity theory presented in part 2, if it exists, indeed fits within the context of Carrollian holography. To this end, we will show that the algebra of higher-spin symmetries $\mathfrak{ihs}_D$ can be realised as (a sub-algebra of) the algebra of higher-differential operators preserving the action of a *Carrollian* scalar field on $\mathscr{I}$ [114], as presented in section 3.2. The full algebra of symmetry is actually much bigger. Work on the asymptotic symmetries of Fronsdal fields propagating on Minkowski background [101, 102, 103] allowed to identify the analogue of super-translations and super-rotations for any spin, and we show that our algebra also contains a sub-algebra displaying the same spectrum of asymptotic symmetry generators, albeit with a different expression, while providing a concrete algebraic realisation.

[1] Another possibility is given by the extended BMS algebra of Barnich and Troessaert [89], which enhances the group of conformal transformations of the two-sphere to transformations of the Riemann sphere by arbitrary holomorphic and anti-holomorphic functions.

Carrollian field theories are usually defined through the limit $c \to 0$ of relativistic field theories [208, 110, 116, 111]. In [110], it was noticed that Carrollian field theories usually come in (at least) two kinds, dubbed *electric* (or time-like) and *magnetic* (or space-like), named by analogy with the 'electric-like' and 'magnetic-like' limits of Maxwell theory. The electric theory is usually obtained by taking directly the limit $c \to 0$ in a Lagrangian formulation, while the magnetic theory is obtained after a Legendre transformation and redefining the fields and conjugate momentum in the Hamiltonian formulation.

The field playing the role of a flat-space analogue of the singleton module in CFT, defined in section 3.2, will be tentatively called the *simpleton*. It is characterised by the equation of motion ${\partial_u}^2 \phi = 0$, making it one of the simplest possible field theories on the boundary side. We will then show in section 3.3 that the other realisation of the simpleton, characterised by the equations of motion $\partial_u \phi = 0$ and $\partial_u \pi = \hat{\nabla}^2 \phi$, also admits a realisation as a bulk field.

Throughout this part, we will denote the dimension of the celestial sphere by d, related to the dimension of the bulk by $D = d + 2$, so that the dimension of the boundary manifold is $d + 1$. We will make ample use of ambient space techniques, and in order to make it easier to distinguish between objects and the space they live in, we will denote:

- ambient space coordinates by X^A where $A \in \{0, \dots, d+2\}$ and fields by $\Phi(X)$, $\Psi(X)$, ... ;
- bulk space-time coordinates by x^a where $a \in \{0, \dots, d+1\}$ and fields by $\varphi(x)$;
- boundary coordinates by x^μ where $\mu \in \{0, \dots, d\}$ and fields by $\phi(x)$.

In addition, when the boundary is the Carrollian manifold $\mathbb{R} \times S^d$, the coordinates x^μ will be split into u for the null time and $\mathbf{x}^i$ with $i \in \{1, \dots, d\}$ for the angles on the celestial sphere S^d. Thus, the Bondi coordinates for Minkowski space-time will be denoted by $(r, u, \mathbf{x})$ and its metric in Bondi gauge is

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2\,\gamma_{ij}(\mathbf{x})\,\mathrm{d}\mathbf{x}^i \mathrm{d}\mathbf{x}^j\,. \tag{3.0.1}$$

3.1 The singleton module

The description of the singleton in four-dimensional (A)dS space-time dates back to Dirac [112]. It can be described in any dimensions by means of representation theory of the conformal algebra $\mathfrak{so}(2, d+1)$ defined in eq. (3.2.1) [113, 209]. By looking at highest-weight representations satisfying the unitarity bound, one finds that a scalar field $\phi(x)$ on the boundary defining a state $|\phi\rangle$ is unitary if its scaling dimension, that is the eigenvalue of the operator of dilations, is greater than the bound $\frac{d-1}{2}$. In the language of representations of the conformal algebra $\mathfrak{so}(2, d+1)$, a (quasi-)primary state is a highest-weight representation, meaning that it is annihilated by the action of weight-lowering generators (i.e. the special conformal transformations K_μ), and the action of the level-zero generators of the conformal algebra, i.e. Lorentz transformations $J_{\mu\nu}$ and dilations D, act diagonally

$$K_\mu|\phi\rangle = 0\,, \quad J_{\mu\nu}|\phi\rangle = 0\,, \quad D|\phi\rangle = \Delta|\phi\rangle\,. \tag{3.1.1}$$

The generators K_μ, $J_{\mu\nu}$ and D form a parabolic subalgebra of the conformal algebra which have non-positive level by convention.

From there, one can build a generalised Verma module $\mathcal{V}(\Delta, 0)$, where the first entry in $\mathcal{V}$ represents the conformal weight and the second the irreducible representation of $\mathfrak{so}(d+1)$. It is given by the successive action of weight-raising operators, the translation generators P_μ

$$\mathcal{V}(\Delta, 0) = \operatorname{span}\{P_{\mu_1} \cdots P_{\mu_s}|\phi\rangle\}_{s\geq 0}\,. \tag{3.1.2}$$

As explained, e.g., in [209], this Verma module is not irreducible for $\Delta = \frac{d-1}{2}$, as it admits the sub-module $\mathcal{V}(\Delta+2, 0)$

$$\mathcal{V}(\Delta+2, 0) = \operatorname{span}\left\{P_{\mu_1} \cdots P_{\mu_s} P^2|\phi\rangle\right\}_{s\geq 0}\,. \tag{3.1.3}$$

Taking a quotienting of the module $\mathcal{V}(\Delta, 0)$ by 'on-shell-trivial states' contained in $\mathcal{V}(\Delta+2, 0)$, meaning that we get rid of the ideal $\mathcal{V}(\Delta+2, 0)$ which is trivial when the wave equation $\partial^2\phi = 0$ is satisfied, defines the singleton module

$$\mathcal{D}(\Delta, 0) := \mathcal{V}(\Delta, 0) \,/\, \mathcal{V}(\Delta+2, 0) \simeq \operatorname{span}\left\{P_{\mu_1} \cdots P_{\mu_s}|\phi\rangle \;\middle|\; P^2|\phi\rangle \sim 0\right\}_{s\geq 0}\,. \tag{3.1.4}$$

In what follows, we will work directly with the fields using [210, 211] to provide a summary of the various definitions of the singleton field. Starting with the definition of a singleton, i.e. a free conformal scalar field living at the boundary of AdS$_{d+2}$ in section 3.1.1, we will review Eastwood's argument to construct its higher-symmetries in section 3.1.2. Subsequently, we will define the singleton in the bulk of AdS$_{d+2}$ space-time in 3.1.3 as a shortened scalar field, and propose a third definition 3.1.4 which links the previous two by means of ambient space geometry. Finally, we will then check that the higher-spin algebra $\mathfrak{hs}_{d+2}$ is realised on the singleton in section 3.1.5.

3.1.1 Boundary definition

Consider a free complex scalar field $\phi(x)$ with scaling dimension Δ

$$S = \frac{1}{2}\int \mathrm{d}^{d+1}x\; \bar{\phi}(x)\,\partial^2\phi(x)\,. \tag{3.1.1}$$

Recall that a field $\phi(x)$ has scaling dimension Δ if it satisfies $\phi(\lambda\, x) = \lambda^{-\Delta}\, \phi(x)$ for $x^\mu \in \mathbb{R}^{1,d}$ and $\lambda > 0$. By performing a rigid rescaling of space-time $x^\mu \to \lambda\, x^\mu$ in the action (3.1.1), we observe that the action is multiplied by an overall factor $\lambda^{d-1-2\,\Delta}$. To ensure that this action is invariant under rigid scale transformations, we will therefore tune $\Delta = \frac{d-1}{2}$. It is well-known that this action is also invariant under the action of rigid conformal transformations

$$\delta\phi = i\left[a^\mu\partial_\mu + \omega^{\mu\nu}x_\nu\partial_\mu + c\,(x^\mu\partial_\mu + \Delta) + b^\mu\left(2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta - x^2\partial_\mu\right)\right]\phi\,, \quad (3.1.2)$$

where a^μ, b^μ, c and $\omega^{[\mu\nu]}$ are constants, representing the action of translations, special conformal transformations, dilations and Lorentz transformations respectively.

3.1.2 Higher symmetries of the singleton

Let us here reproduce the argument of Eastwood [65]. A *higher-differential symmetry* of a differential operator A is a differential operator $\hat{\mathcal{D}}$ such that there exists another differential operator $\hat{\delta}$ verifying

$$A \circ \hat{\mathcal{D}} = \hat{\delta} \circ A\,, \quad (3.1.1)$$

where $\circ$ denotes the composition of differential operators. For instance, when A is the Laplacian or the d'Alembertian ∂^2, symmetries of A are differential transformations $\delta\phi = i\,\hat{\mathcal{D}}\,\phi$ mapping solutions of the equation of motion $\partial^2\,\phi = 0$ to themselves.

We will call a *non-trivial*[2] higher-differential symmetry of A a higher-differential symmetry $\hat{\mathcal{D}}$ of A such that in addition, there does not exist an $\hat{\mathcal{D}}'$ such that $\hat{\mathcal{D}} = \hat{\mathcal{D}}' \circ A$. Indeed, in this case, $\hat{\delta} = A \circ \hat{\mathcal{D}}'$ suffices to satisfy eq. (3.1.1).

Finally, an *on-shell* (non-trivial) symmetry of A is a (non-trivial) higher-differential symmetry of A that preserves the action $\frac{1}{2}\,\langle\phi|A\,\phi\rangle$, where the inner product $\langle -|-\rangle$ is defined for compactly supported functions by

$$\langle\phi|\psi\rangle = \int \mathrm{d}^{d+1}x\ \bar{\phi}(x)\psi(x)\,. \quad (3.1.2)$$

It is easy to check that an on-shell higher-symmetry $\hat{\mathcal{D}}$ of A is a higher-symmetry $\hat{\mathcal{D}}$ of A such that $\hat{\delta} = \hat{\mathcal{D}}^\dagger$, where $\dagger$ denotes Hermitian conjugation with respect to the inner product $\langle -|-\rangle$.

One can apply the previous argument to the case of the singleton, by looking at the higher symmetries of the action (3.1.1), which are differential operators $\hat{\mathcal{D}}$

[2] Alternatively, one may consider higher-differential symmetries, up to the quotient by the equivalence relation $\sim$ defined by $\hat{\mathcal{D}}_1 \sim \hat{\mathcal{D}}_2$ if and only if there exists $\hat{\mathcal{P}}$ such that $\hat{\mathcal{D}}_1 - \hat{\mathcal{D}}_2 = \mathcal{P} \circ A$.

such that the transformation $\delta\phi = i\,\hat{\mathcal{D}}\,\phi$ leaves the action invariant. One obtains the condition

$$\partial^2 \circ \hat{\mathcal{D}} = \hat{\mathcal{D}}^\dagger \circ \partial^2\,, \tag{3.1.3}$$

where it is sometimes said that such an operator $\hat{\mathcal{D}}$ commutes *weakly* with the d'Alembertian.

It is clear that these symmetries form an associative algebra, from the observation that if $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ are isometries, then $i\,[\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2] = i\,\Big(\hat{\mathcal{D}}_1 \circ \hat{\mathcal{D}}_2 - \hat{\mathcal{D}}_2 \circ \hat{\mathcal{D}}_1\Big)$ is an isometry as well. Indeed,

$$\begin{aligned}\partial^2 \circ \Big(i\,[\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2]\Big) &= i\,\partial^2 \circ \hat{\mathcal{D}}_1 \circ \hat{\mathcal{D}}_2 - i\,\partial^2 \circ \hat{\mathcal{D}}_2 \circ \hat{\mathcal{D}}_1 \\ &= i\,\hat{\mathcal{D}}_1^\dagger \circ \hat{\mathcal{D}}_2^\dagger \circ \partial^2 - i\,\hat{\mathcal{D}}_2^\dagger \circ \hat{\mathcal{D}}_1^\dagger \circ \partial^2 = \Big(i\,[\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2]\Big)^\dagger \circ \partial^2\,.\end{aligned} \tag{3.1.4}$$

Moreover, $\Big\{\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2\Big\} = \hat{\mathcal{D}}_1 \circ \hat{\mathcal{D}}_2 + \hat{\mathcal{D}}_2 \circ \hat{\mathcal{D}}_1$ as well

$$\partial^2 \circ \Big\{\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2\Big\} = \Big\{\hat{\mathcal{D}}_1^\dagger, \hat{\mathcal{D}}_2^\dagger\Big\} \circ \partial^2 = \Big\{\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2\Big\}^\dagger \circ \partial^2\,, \tag{3.1.5}$$

so that we can construct higher-order isometries from products of lower-order ones. This is in fact equivalent to the observation that Killing tensors are products of Killing vectors, and that higher-spin algebras can be built as universal enveloping algebras.

By considering the class of differential operators of the form

$$\hat{\mathcal{D}} = V^{\mu_1 \cdots \mu_{s-1}}(x)\,\partial_{\mu_1}\,\cdots\,\partial_{\mu_{s-1}} + \text{ lower-order terms}\,, \tag{3.1.6}$$

where $V^{\mu_1 \cdots \mu_{s-1}}$ is a symmetric and traceless[3] tensor and lower-order differential terms are expressed in terms of $V^{\mu_1 \cdots \mu_{s-1}}$ only, Eastwood [65] classified all such symmetries, and showed that the spectrum of non-trivial higher-symmetries is in one-to-one correspondence with the traceless conformal Killing tensors of $(d+1)$-dimensional Minkowski space $\xi_{\mu_1 \cdots \mu_{s-1}}$ such that

$$\partial_{(\mu_1}\xi_{\mu_2 \cdots \mu_s)} - \frac{s-1}{d+2s-3}\,\eta_{(\mu_1\mu_2}\partial\cdot\xi_{\mu_3 \cdots \mu_s)} = 0\,, \quad \xi'{}_{\mu_3 \cdots \mu_{s-1}} = 0\,, \tag{3.1.7}$$

itself in bijection with the reducibility parameters of a collection of free massless higher-spin fields in AdS_{d+2}.

Therefore, non-trivial higher symmetries of the d'Alembertian [132, 133, 65] give rise to an algebra which is a candidate algebra for the role of higher-spin symmetry in AdS_{d+2}, or equivalently, of conformal higher-spin symmetry [212] in $\mathbb{R}^{1,d}$. In fact we will see in section 3.1.5 that it is the same algebra $\mathfrak{hs}_{d+2}$.

[3] The requirement that $\hat{\mathcal{D}}$ is non-trivial means that we can always look for $V^{\mu_1 \cdots \mu_{s-1}}$. Indeed, if $V^{\mu_1 \cdots \mu_{s-1}} = \eta^{(\mu_1\mu_2}W^{a_3 \cdots \mu_{s-1})}$, then the highest-order of $\hat{\mathcal{D}}$ is trivial. This means that $\hat{\mathcal{D}}$ is *equivalent* to another differential operator of strictly lower degree, in the definition of footnote 2.

3.1.3 Bulk definition

Let us consider a free scalar field $\varphi(x^\mu)$ in AdS$_{d+2}$ space-time of radius R, with metric g_{ab}

$$\left(\nabla^2 - m^2\right)\varphi = 0\,, \tag{3.1.1}$$

where $\nabla^2\varphi = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,g^{ab}\,\partial_b\,\varphi\right)$ is the Laplace-Beltrami operator in AdS$_{d+2}$, and the mass parameter m^2 satisfies the mass-shell condition

$$m^2 = \frac{\Delta(\Delta-d-1)}{R^2} \quad \Leftrightarrow \quad \Delta = \Delta_\pm = \tfrac{d+1}{2} \pm \sqrt{\left(\tfrac{d+1}{2}\right)^2 + (mR)^2}\,, \tag{3.1.2}$$

where the two branches with $\Delta_\pm$ correspond to two different scaling dimensions of the field $\varphi(x)$ near the boundary of AdS$_{d+2}$.

We can perform an expansion of the field $\varphi(x)$ near the boundary using coordinates $x^a = (\rho, x^\mu)$, with $\rho \in [0, +\infty[$, and the metric

$$\mathrm{d}s^2 = \frac{R^2}{4\rho^2}\mathrm{d}\rho^2 + \frac{1}{\rho}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{3.1.3}$$

The conformal boundary of AdS$_{d+2}$ is located at the limit $\rho \to 0$. By writing the Ansatz

$$\varphi(\rho, x^\mu) = \rho^{\Delta_-/2}\varphi_-(\rho, x^\mu) + \rho^{\Delta_+/2}\varphi_+(\rho, x^\mu)\,, \tag{3.1.4}$$

we obtain

$$\left(\nabla^2 - m^2\right)\varphi = 0 \quad \Leftrightarrow \quad \partial_\mu\partial^\mu\varphi_\pm + 2\left(d+3-2\Delta_\mp + 2\rho\partial_\rho\right)\partial_\rho\varphi_\pm = 0\,. \tag{3.1.5}$$

Assuming the two fields $\varphi_\pm(\rho, x^\mu)$ are analytic near $\rho = 0$ and performing a Taylor expansion

$$\varphi_\pm(\rho, x^\mu) = \sum_{n\geq 0}\rho^n\,\phi_\pm^{(n)}(x^\mu)\,, \tag{3.1.6}$$

and plugging inside of (3.1.5) we obtain the set of equations, order by order in ρ

$$2n\left(2\Delta_\mp - d - 1 - 2n\right)\phi_\pm^{(n)} = \partial^2\phi_\pm^{(n-1)}\,. \tag{3.1.7}$$

If there exists an $\ell \geq 1$ such that $2\Delta_+ - d - 1 = 2\ell$, then the pre-factor in the left hand side of eq. (3.1.7) for the φ_- branch becomes zero when $n = \ell$. This signals that the function $\phi_-^{(\ell-1)}$ verifies a second-order differential equation and the set of functions $\phi_-^{(n)}$ for $n \geq \ell$ is independent from the value of $\phi_-^{(\ell-1)}$. Therefore, all these functions can be consistently set to zero.[4] As an example, for $\ell = 1$, $\Delta_+ = \frac{d+3}{2}$ and $\Delta_- = \frac{d-1}{2}$, the $n = 1$ instance of eq. (3.1.7) imposes $\partial_\mu\partial^\mu\phi_-^{(0)} = 0$. One can set to zero all the functions $\phi_-^{(n\geq 1)}$, which also coincides with the φ_+ branch, and notice that the field $\phi_-^{(0)}$ satisfies the same equation of motion as the field ϕ in section 3.1.1.

[4]In this case, it is customary to add a logarithmic branch [213] so that the holographic reconstruction procedure can be performed without obstruction, but in the case of the singleton we precisely want to keep things that way so that a truncation of the spectrum appears, corresponding to defining an 'ultra-short' representation of the conformal group.

3.1.4 Ambient space definition

We can incorporate both of the previous definitions inside of a third one, using ambient space geometry. We can describe the singleton by a set of three equations acting on a field $\Phi(X)$ depending on $X^A \in \mathbb{R}^{2,d+1}$, with $A \in \{0, \dots, d+2\}$

$$\eta^{AB}\,\partial_A\,\partial_B\,\Phi = 0\,, \quad \big(X^A\,\partial_A + \Delta\big)\,\Phi = 0\,, \quad \Phi \simeq \Phi + X^2\,\Psi\,. \tag{3.1.1}$$

The first equation is simply the fact that Φ verifies a wave equation in ambient space, while the second fixes its homogeneity degree to be the real number Δ. Finally, the last one signals that we are quotienting $\Phi(X)$ by terms that vanish when $X^2 = 0$, represented by the relation $\simeq$.

This system of equations is self-consistent only when $\Delta = \frac{d-1}{2}$. Indeed, this can be seen by considering the bracket of the first and the third conditions acting on the quotiented contribution Ψ with scaling dimension $\Delta + 2$

$$\left[\partial^2, X^2\right]\Psi = 2\left(X^A\partial_A + \frac{d+3}{2}\right)\Psi\,, \tag{3.1.2}$$

which is factored out only if the scaling dimension of Ψ is $\Delta + 2 = \frac{d+3}{2}$. In this case, the definition of eq. (3.1.1) can be seen as a set of constraints generating an algebra isomorphic to $\mathfrak{sp}(2)$ which commutes with the generators of ambient space isometries J_{AB}, forming what is called as a Howe dual pair [214, 202].

The advantage of working in ambient space is that one can have access both to the bulk of AdS and to its conformal boundary. On the one hand, as explained in section 2.2.1, one can recover the bulk of AdS of radius R by selecting the manifold

$$\mathrm{AdS}_{d+2} := \left\{X^2 = -R^2 \,\middle|\, X \in \mathbb{R}^{2,d+1}\right\}. \tag{3.1.3}$$

On the other hand, to recover the conformal boundary of AdS, on has to define the projective light-cone $X^2 = 0$, where points along the (positive) light-rays are identified

$$\partial\mathrm{AdS}_{d+2} := \left\{X^2 = 0 \,\middle|\, X \in \mathbb{R}^{2,d+1}\,,\ X \sim \lambda\,X \text{ for all } \lambda > 0\right\}, \tag{3.1.4}$$

which is a $(d+1)$-dimensional manifold, identified with the boundary of AdS.

To recover the two previously introduced representations of the singleton, the one in AdS_{d+2} and the one on $\mathbb{R}^{1,d-1}$, one has to look at the pullback of the equations of motion (3.1.1) on the two different manifolds defined in eqs. (3.1.3) and (3.1.4).

On the one hand, $\Phi(X)$ is the unique extension of $\varphi(x)$ to the region $X^2 < 0$ of ambient space. AdS space-time can be embedded in ambient space by going to light-cone coordinates

$$X^2 = \eta_{AB}X^AX^B = 2\,X^+X^- + \eta_{\mu\nu}\,X^\mu X^\nu\,, \tag{3.1.5}$$

and using Poincaré coordinates parameterising $X^2 = -R^2$

$$X^+ = \frac{1}{\sqrt{\rho}}\,, \quad X^\mu = \frac{1}{\sqrt{\rho}}x^\mu\,, \quad X^- = -\frac{1}{2}\left(R^2\sqrt{\rho} + \frac{1}{\sqrt{\rho}}x^2\right). \tag{3.1.6}$$

The pullback of the flat metric $\mathrm{d}s^2 = \eta_{AB}\,\mathrm{d}X^A\mathrm{d}X^B$ then takes the form of eq. (3.1.3). Using homogeneity of the field Φ, we can write

$$\Phi(X) = \left(-\frac{X^2}{R^2}\right)^{-\Delta/2}\bar{\Phi}\left(\frac{R}{\sqrt{-X^2}}X\right)\,, \tag{3.1.7}$$

where $\bar{\Phi}$ is defined on $X^2 = -R^2$ and $\Delta = \frac{d-1}{2}$. With the parameterisation of eq. (3.1.6), we can define a bulk field

$$\varphi(\rho, x^\mu) = \bar{\Phi}(X)\big|_{X^2=-R^2}\,. \tag{3.1.8}$$

One can then prove that the ambient d'Alembertian equation in (3.1.1) then pulls back to the wave equation shown in (3.1.1), with the correct value of the mass term $R^2\,m^2 = -\frac{(d+1)(d+3)}{4}$, while the quotient condition in eq. (3.1.1) tells us that the Δ_+ branch has to be set to zero, see, e.g., [210, 211].

On the other hand, the field $\Phi(X)$ is the unique extension of $\phi(x)$ outside the light-cone $X^2 = 0$, which is the conformal boundary of AdS_{d+2}. We will parameterise it using the same light-cone metric, but this time with

$$X^+ = \frac{1}{\sqrt{\rho}}\,, \quad X^\mu = \frac{1}{\sqrt{\rho}}x^\mu\,, \quad X^- = -\frac{1}{2}\frac{1}{\sqrt{\rho}}x^2\,. \tag{3.1.9}$$

Using these coordinates and homogeneity of Φ, we have that

$$\Phi(X)|_{X^2=0} = \Phi\left(X^+, X^+x^\mu, -\tfrac{1}{2}X^+x^2\right) := (X^+)^{-\Delta}\phi(x^\mu) \tag{3.1.10}$$

where the field ϕ lives purely on the boundary. It can be shown that it satisfies the d'Alembertian equation $\partial_\mu\partial^\mu\phi = 0$, and therefore it can be identified with the one used in eq. (3.1.1).

Due to the simple and unifying nature of the ambient space description, it will be the preferred language to look for a flat counterpart of the singleton.

3.1.5 Higher-spin algebra of ambient isometries

The ideal defined near eq. (2.2.8) that is quotiented in the definition of the higher-spin algebra $\mathfrak{hs}_{d+2}$ corresponds to the annihilator (i.e. the trivial symmetries) of the singleton module [70]. As a result, Eastwood's algebra of higher symmetries provides an explicit representation of $\mathfrak{hs}_{d+2}$ as an algebra of differential operators. Conversely, the algebra of higher symmetries of a free scalar field in $(d+1)$-dimensional Minkowski space-time is isomorphic to the higher-spin algebra in $(d+2)$-dimensional

AdS space-time, a statement that can also be extended to partially-massless theories by replacing the wave equation $\partial^2\phi = 0$ by a higher-order wave equation (also called polywave) $\partial^{2\ell}\phi = 0$ [215, 216, 217, 211, 218, 181].

The ambient space description of the singleton can serve as a basis to prove the isomorphism between Eastwood's construction of the higher-spin algebra and $\mathfrak{hs}_{d+2}$. Indeed, in ambient space, AdS isometries are realised as simple differential operators (where we replaced infinitesimal canonical generators of Lorentz transformations by i times themselves)

$$J_{AB} = X_A\,\partial_B - X_B\,\partial_A\,, \tag{3.1.1}$$

therefore the completely anti-symmetric projection automatically vanishes

$$J_{[AB} \circ J_{CD]}\,\Phi = 4\,X_{[A}\,\eta_{BC}\,\partial_{D]}\Phi + 4\,X_{[A}\,X_B\,\partial_C\,\partial_{D]}\Phi = 0\,. \tag{3.1.2}$$

Moreover, using eq. (3.1.1)

$$J^C{}_{(A} \circ J_{B)C}\,\Phi = \big(X^C\partial_{(A} - X_{(A}\partial^C\big) \circ \big(X_{B)}\partial_C - X_C\partial_{B)}\big)\,\Phi \simeq -\Delta\,\eta_{AB}\,\Phi\,, \tag{3.1.3}$$

where we recall that $\simeq$ meant 'equal up to terms proportional to X^2', and

$$\tfrac{1}{2}J^{AB} \circ J_{BA}\,\Phi = \Delta\,(\Delta - 1 - d)\,\Phi\,, \tag{3.1.4}$$

so that the whole ideal is indeed factored out. Thus, using ambient space geometry, it is extremely simple to check that the higher-spin algebra is realised as the algebra of higher-symmetries of the ambient space field Φ verifying eqs. (3.1.1).

3.2 The simpleton module

After this review of the AdS (or, equivalently, conformal) case, we now switch to the Minkowski (Carrollian conformal) case, loosely following [114]. As a preliminary remark, let us recall the isomorphism between the Poincaré algebra and the Carrollian contraction of the conformal algebra.

The algebra of Carrollian conformal transformations emerges as the $c \to 0$ limit of conformal transformations [203, 219, 205].[5]

The conformal algebra is generated by Lorentz transformations $J_{\mu\nu}$, translations P_μ, special conformal transformations K_μ and dilations D and satisfy the Lie brackets

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho}\,J_{\mu\sigma} - \eta_{\mu\rho}\,J_{\nu\sigma} - \eta_{\nu\sigma}\,J_{\mu\rho} + \eta_{\mu\sigma}\,J_{\nu\rho}\,,\\
[J_{\mu\nu}, P_\rho] &= \eta_{\nu\rho}\,P_\mu - \eta_{\mu\rho}\,P_\nu\,, \quad [D, P_\mu] = P_\mu\,,\\
[J_{\mu\nu}, K_\rho] &= \eta_{\nu\rho}\,K_\mu - \eta_{\mu\rho}\,K_\nu\,, \quad [D, K_\mu] = -K_\mu\,,\\
[K_\mu, P_\nu] &= 2\,\eta_{\mu\nu}\,D - 2\,J_{\mu\nu}\,.
\end{aligned} \tag{3.2.1}$$

[5] In its original form, the Carroll group of coordinate transformations is the $c \to 0$ contraction of the Poincaré group, and plays a dual role to the Galilei group obtained as the $c \to \infty$ contraction.

with $\mu, \nu, \rho, \sigma \in \{0, \ldots, d\}$ and where the metric $\eta_{\mu\nu}$ has signature $(-, +, \ldots, +)$.

Splitting the generators along the time direction $\mu = 0$ and rescaling the generators as $B_i = c\, J_{i0}$, $H = c\, P_0$ and $K = c\, K_0$, the Carrollian conformal algebra arises as the $c \to 0$ limit of the previous brackets, whose generators will be denoted by J_{ij}, B_i, P_i, H, K_i, K and D (see [205]). The non-zero Lie brackets of the Carrollian conformal algebra are

$$
\begin{aligned}
&[J_{ij}, J_{kl}] = \delta_{jk}\, J_{il} - \delta_{ik}\, J_{jl} - \delta_{jl}\, J_{ik} + \delta_{il}\, J_{jk}\,, \quad [J_{ij}, B_k] = \delta_{jk}\, B_i - \delta_{ik}\, B_j\,,\\
&[J_{ij}, P_k] = \delta_{jk}\, P_i - \delta_{ik}\, P_j\,, \quad [B_i, P_j] = -\delta_{ij} H\,,\\
&[J_{ij}, K_k] = \delta_{jk}\, K_i - \delta_{ik}\, K_j\,, \quad [B_i, K_j] = -\delta_{ij} K\,,\\
&[D, H] = H\,, \quad [D, P_i] = P_i\,, \quad [D, K] = -K\,, \quad [D, K_i] = -K_i\,,\\
&[K_i, H] = -2\, B_i\,, \quad [K, P_i] = 2\, B_i\,, \quad [K_i, P_j] = 2\, \delta_{ij}\, D - 2\, J_{ij}\,,
\end{aligned}
\tag{3.2.2}
$$

where $i, j, k, l \in \{1, \ldots, d\}$.

This algebra is of prime importance in the approach to holography that we are advocating in this thesis (more can be found in [94, 95]), as it is isomorphic to the Poincaré algebra in one dimension more.[6] Let us define the generators

$$
\mathbb{P}_0 = \tfrac{1}{2}(H - K)\,, \quad \mathbb{P}_i = -B_i\,, \quad \mathbb{P}_{d+1} = \tfrac{1}{2}(H + K)\,. \tag{3.2.3}
$$

and

$$
\mathbb{J}_{i0} = \tfrac{1}{2}(P_i - K_i)\,, \quad \mathbb{J}_{ij} = J_{ij}\,, \quad \mathbb{J}_{i\,d+1} = \tfrac{1}{2}(P_i + K_i)\,, \quad \mathbb{J}_{0\,d+1} = D\,. \tag{3.2.4}
$$

We can see that they verify the Lie brackets of the Poincaré algebra in $d+2$ dimensions (see e.g. [107, 95])

$$
\begin{aligned}
&[\mathbb{J}_{ab}, \mathbb{J}_{cd}] = \eta_{bc}\, \mathbb{J}_{ad} - \eta_{ac}\, \mathbb{J}_{bd} - \eta_{bd}\, \mathbb{J}_{ac} + \eta_{ad}\, \mathbb{J}_{bc}\,,\\
&[\mathbb{J}_{ab}, \mathbb{P}_c] = \eta_{bc}\, \mathbb{P}_a - \eta_{ac}\, \mathbb{P}_b\,,\\
&[\mathbb{P}_a, \mathbb{P}_b] = 0\,,
\end{aligned}
\tag{3.2.5}
$$

where $a, b, c, d \in \{0, \ldots, d+1\}$ and the metric η_{ab} has signature $(-, +, \ldots, +)$.

From this correspondence, in addition to the unitary irreducible representations of the Poincaré group classified by Wigner, one can also define some highest-weight (possibly non-unitary, see [140]) representations of the Poincaré algebra, understood as modules of the Carrollian conformal algebra presented above. These modules can be defined starting from modules of the conformal algebra and taking the limit $c \to 0$, as explained in [205]. Representations that are highest-weight and scalar remain highest-weight and scalar in the sense that

$$
K_i|\phi\rangle = 0\,, \quad K_0|\phi\rangle = 0\,, \quad J_{ij}|\phi\rangle = 0\,, \quad B_i|\phi\rangle = 0\,, \quad D|\phi\rangle = \Delta|\phi\rangle\,. \tag{3.2.6}
$$

[6] Let us also mention that the algebra of isometries of any null projective hypersurface (for instance null infinity $\mathscr{I}$) is the BMS algebra [104, 220], which is a straightforward extension of the conformal Carroll algebra.

where B_i are Carrollian boosts. For the singleton, the value $\Delta = \frac{d-1}{2}$ stays the same in the limit, and we can define the quotient of ultra-relativistic Verma modules in the same way as before

$$\text{span}\left\{P_{i_1}\,\cdots\,P_{i_s}\,|\phi\rangle\right\}_{s\geq 0}\;\oplus\;\text{span}\left\{P_{i_1}\,\cdots\,P_{i_s}\,P_0\,|\phi\rangle\right\}_{s\geq 0}\,, \tag{3.2.7}$$

where we discarded the product of more than one P_0 on $|\phi\rangle$ in the UEA of the Carrollian conformal algebra by account of the quotient by $P_{i_1}\,\cdots\,P_{i_s}\,P_0\,P_0\,|\phi\rangle$ for any $s\geq 0$. We can readily see the emergence of a semi-direct structure in the algebra, related to the presence of an Abelian factor generated by the action of the UEA of the Carrollian conformal algebra of eq. (3.2.2) on the second part of eq. (3.2.7).

3.2.1 Boundary definition

Let us start from the action for a massless *relativistic* complex scalar field ϕ living on $\mathbb{R}\times S^d$ with coordinates $(u,\mathbf{x})$

$$S_{\rm r} = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\,\bar{\phi}\left(\partial_u{}^2 - c^2\,\hat{\nabla}^2\right)\phi\,, \tag{3.2.1}$$

where $\gamma_{ij}(\mathbf{x})$ is the round metric on the d-dimensional round sphere S^d. In addition, the differential operator

$$\hat{\nabla}^2 := \nabla^2 - \frac{d-1}{4\,d}\mathcal{R}[\gamma] = \nabla^2 - \left(\frac{d-1}{2}\right)^2\,, \tag{3.2.2}$$

is the conformal completion of the Laplacian on $\mathbb{R}\times S^d$, where $\mathcal{R}[\gamma]$ represents the scalar curvature of the metric given by the line element $\mathrm{d}s^2 = -c^2\mathrm{d}u^2 + \gamma_{ij}\mathrm{d}\mathbf{x}^i\mathrm{d}\mathbf{x}^j$, and which is distinct from the Laplace-Yamabe operator $\nabla^2_{\rm LY}$ on the sphere S^d with metric γ defined in eq. (B.1.6) (more details in appendices B.1 and B.2). The action has also been rescaled by an overall factor of $-c^2$ so that the limit $c\to 0$ is well-defined.

Consider the action resulting from the direct limit $c\to 0$ of the relativistic scalar in $(d+1)$ dimensions, in the Lagrangian formulation

$$S = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\,\bar{\phi}\,\partial_u{}^2\phi\,, \tag{3.2.3}$$

which is now formulated on the Carrollian manifold $\mathbb{R}\times S^d$, which has the topology of $\mathscr{I}$. This is the electric Carrollian scalar field of [205, 221, 110, 109, 222, 111], and we propose this action for the flat limit of the singleton, which we will call in the following the simpleton. The equations of motion are simply

$$\partial_u{}^2\,\phi = 0\,, \tag{3.2.4}$$

so that on-shell

$$\phi(u, \mathbf{x}) \approx \phi_0(\mathbf{x}) + u\, \phi_1(\mathbf{x})\,. \tag{3.2.5}$$

One can already understand why the simpleton is a good candidate to factor out the ideal of $\mathfrak{ihs}_{d+2}$, in particular the condition $P_a\, P_b \sim 0$, uncovered in eq. (2.3.16). Indeed, bulk translations and Lorentz transformations are realised on the boundary as the subset of isometries of the Carrollian conformal manifold (see, e.g. [104] or appendix A) acting on the scalar field ϕ of scaling dimension Δ

$$P_a\, \phi = f_a(\mathbf{x}) \partial_u\, \phi\,, \tag{3.2.6a}$$
$$J_{ab}\, \phi = \left[\xi^i_{ab}(\mathbf{x})\partial_i + \tfrac{1}{d}\, \nabla \cdot \xi_{ab}(\mathbf{x})\, (u\partial_u + \Delta)\right] \phi\,, \tag{3.2.6b}$$

where f_a and ξ^i_{ab} verify

$$\left(\nabla_{(i}\nabla_{j)} - \tfrac{1}{d}\gamma_{ij}\nabla^2\right) f_a = 0\,, \quad \nabla_{(i}\xi_{j)}{}^{ab} - \tfrac{1}{d}\gamma_{ij}\nabla \cdot \xi^{ab} = 0\,. \tag{3.2.7}$$

It is easy to see that the part of the ideal corresponding to the product of two translations is automatically factored out on-shell $P_a\, P_b\, \phi = f_a\, f_b\, {\partial_u}^2\, \phi = 0$.

We will prove that the whole ideal of eq. (2.3.16) is factored out using ambient space techniques in section 3.2.2. It is important to note that the operators J_{ab} correspond to the standard realisation of the generators of the conformal algebra $\mathfrak{so}(1, d{+}1)$ acting on a primary scalar field on the sphere S^d with dimension Δ, up to the replacement of the number Δ by the first-order operator $\Delta{+}u\, \partial_u$ (thereby taking into account the scaling property of u). Therefore, when acting on a 'Carrollian primary' scalar ϕ on $\mathscr{I}$ with scaling dimension Δ, we have

$$J^2\, \phi = (\Delta + u\partial_u)\, (\Delta + u\partial_u - d)\, \phi\,, \tag{3.2.8}$$

which originates from the usual result $J^2 = \Delta\, (\Delta - d)$ for a conformal primary scalar of dimension Δ, taking into account the shift $\Delta \to \Delta + u\, \partial_u$. Expanding in powers of u and ∂_u, we have

$$J^2\, \phi = \left[u^2 {\partial_u}^2 + (2\Delta - d + 1)\, u\, \partial_u + \Delta(\Delta - d)\right] \phi\,, \tag{3.2.9}$$

Assuming that the ideal (2.3.16) is factored out means that the expression on the right-hand side of (3.2.9) must be a multiple of ϕ with eigenvalue fixed in (2.3.16). Moreover, the adjoint action of $P_0 = \partial_u$ on eq. (3.2.9) yields

$$\left[2\, u{\partial_u}^2 + (2\Delta - d + 1)\, \partial_u\right] \phi = 0\,, \quad {\partial_u}^2\, \phi = 0\,. \tag{3.2.10}$$

One can recognise the last condition as the Carrollian equation of motion for ϕ, whereas the previous one fixes the scaling dimension to be the one of the simpleton $\Delta = \frac{d-1}{2}$. Finally, the first equation fixes the eigenvalue of J^2 to be precisely $-\frac{d^2-1}{4}$.

3.2.2 Ambient definition and factoring out of the ideal

Recall the ambient description of the singleton (3.1.1), where we split the ambient space coordinate X^A into u and y^a where $a \in \{1, \ldots, d+1\}$, and with powers of c explicitly written

$$\left(\partial_u{}^2 - c^2 \partial_a \partial^a\right) \Phi = 0\,, \; (u\partial_u + y^a \partial_a + \Delta)\, \Phi = 0\,, \; \Phi \simeq \Phi + \left(y^2 - c^2 u^2\right) \Psi\,, \quad (3.2.1)$$

where $\Delta = \frac{d-1}{2}$. The ambient description of the electric simpleton is simply the limit $c \to 0$ of the previous equations

$$\partial_u{}^2\, \Phi = 0\,, \quad (u\partial_u + y^a \partial_a + \Delta)\, \Phi = 0\,, \quad \Phi \simeq \Phi + y^2\, \Psi\,. \qquad (3.2.2)$$

The geometry of ambient space is Carrollian, since the (covariant) metric is now degenerate in the direction of the vector ∂_u. Contrary to the relativistic case, the value of Δ is not fixed anymore by the requirement that the three relations in eq. (3.2.2) form an algebra when evaluated on a quotiented contribution, since

$$\left[u\partial_u + y^a \partial_a + \Delta, \partial_u{}^2\right] \Psi = -2\, \partial_u{}^2\, \Psi\,, \qquad (3.2.3a)$$
$$\left[u\partial_u + y^a \partial_a + \Delta, y^2\right] \Psi = +2\, y^2\, \Psi\,, \qquad (3.2.3b)$$
$$\left[\partial_u{}^2, y^2\right] \Psi = 0\,, \qquad (3.2.3c)$$

which closes on $\mathfrak{iso}(1,1)$, which is a contraction of $\mathfrak{sp}(2) \simeq \mathfrak{so}(2,1)$, for any value of Δ. We will keep the same value of Δ for eq. (3.2.2) as the relativistic parent eq. (3.1.1), and we will also see that it is necessary to fix this value to factor out the ideal (2.3.16).

Remark that the $c \to 0$ limit of ambient space isometry generators gives,[7] upon a proper rescaling of the generator $P_a = J_{a0}$

$$J_{ab} = 2\, y_{[a} \partial_{b]}\,, \quad P_a = y_a \partial_u\,, \qquad (3.2.4)$$

which are indeed the isometries of the sub-manifold $y^2 =$ constant, that commute with the generators defined in eq. (3.2.2).

Let us verify that the ideal (2.3.16) is indeed factored out. The first two conditions

$$J_{[ab} \circ J_{cd]}\, \Phi = 0\,, \quad J_{[ab} \circ P_{c]}\, \Phi = 0\,, \qquad (3.2.5)$$

are obvious, since both the coordinates y^a and the partial derivatives ∂_a commute with themselves. The remainder can be proven by a direct computation

$$\frac{1}{2}\, J_{ab} \circ J^{ba}\, \Phi \simeq \Delta\, (\Delta - d)\, \Phi\,, \qquad (3.2.6)$$

[7] This definition is well-suited for the boundary description of the simpleton, where bulk rigid translations are represented on the boundary by the vector fields $f_a(\mathbf{x})\, \partial_u$, with $f_a(\mathbf{x})$ a function of the celestial sphere verifying the good-cut equation defined in (3.2.7).

which gives the correct eigenvalue only for $\Delta = \frac{d\pm1}{2}$, and

$$\left(J_{ab} \circ P^b + P^b \circ J_{ab}\right) \Phi \simeq (-2\Delta + d - 1)\, y_a \partial_u \Phi \,, \tag{3.2.7}$$

which vanishes only for $\Delta = \frac{d-1}{2}$. We used everywhere that terms proportional to y^2 can be quotiented, as well as the homogeneity condition. Finally, as advertised

$$P_a \circ P_b \, \Phi = y_a \, y_b \, \partial_u{}^2 \, \Phi = 0 \,. \tag{3.2.8}$$

One can then realise the boundary simpleton by taking the null projection of the ambient field along the light-cone $y^2 = 0$. Let us choose a light-cone parameterisation for the coordinates $y^a = (y^+, y^i, y^-)$, with metric

$$\eta_{ab} \, y^a y^b = 2y^+ y^- + \gamma_{ij} y^i y^j \,, \tag{3.2.9}$$

with γ_{ij} the metric on the d-dimensional sphere. The locus $y^2 = 0$ in the region $y^+ > 0$ can be parameterised by

$$y^i = y^+ \mathbf{x}^i \,, \quad y^- = -\frac{1}{2} y^+ \gamma_{ij} \mathbf{x}^i \mathbf{x}^j \,. \tag{3.2.10}$$

Using the homogeneity of $\Phi(u, y^a)$, we have

$$\Phi \left(y^+ u, y\right) = (y^+)^{-\Delta} \phi(u, \mathbf{x}) \,, \tag{3.2.11}$$

where $\phi(u, \mathbf{x})$ depends only on $(u, \mathbf{x}) \in \mathbb{R} \times S^d$, and satisfies $\partial_u{}^2 \, \phi(u, \mathbf{x}) = 0$ like its ambient space parent.

3.2.3 Higher-spin symmetries of the simpleton

Following the philosophy of [65] (see also [132, 133]), we now show that a real form of the higher-spin algebra $\mathfrak{ihs}_{d+2}$ is a sub-algebra of the higher symmetries of the conformal Carrollian scalar. It was already noticed that the higher symmetries of (3.2.3) contain all generators of the form $f(\mathbf{x})\, \partial_u$ without any constraints on the functions $f(\mathbf{x})$ [205] (see also [223, 224]), so that they include, at least, super-translations. Similarly, in classifying the higher symmetries of (3.2.3), we shall obtain infinite-dimensional extensions of the algebra $\mathfrak{ihs}_{d+2}$ incorporating (extended) BMS symmetries and BMS-like higher-spin symmetries similar to those in [101, 103, 225].

The higher symmetries of the electric action (3.2.3) are differential operators $\hat{D}$ that commute weakly with the kinetic operator

$$\partial_u{}^2 \circ \hat{D} = \hat{D}^\dagger \circ \partial_u{}^2 \,. \tag{3.2.1}$$

In the following, we will omit the hat notation and use the short-hands $\dot{D} = [\partial_u, D]$ and $\ddot{D} = \left[\partial_u, \dot{D}\right]$. Since $[\partial_u{}^2, D] = \ddot{D} + 2\,\dot{D} \circ \partial_u$, the condition (3.2.1) is equivalent to

$$\ddot{D} + 2\,\dot{D} \circ \partial_u = \left(D^\dagger - D\right) \circ \partial_u{}^2 \,. \tag{3.2.2}$$

By implementing the on-shell identification $D \sim D + B \circ \partial_u{}^2$ for any differential operator B, one can write down without loss of generality an Ansatz for D of the form

$$D = D_0 + D_1 \circ \partial_u \,, \tag{3.2.3}$$

where D_0 and D_1 are independent of ∂_u . Then, eq. (3.2.2) translates into

$$\begin{aligned} &2\left(\dot{D}_0 + \dot{D}_1 \circ \partial_u\right) \circ \partial_u + \left(\ddot{D}_0 + \ddot{D}_1 \circ \partial_u\right) \\ &\quad = \left[\left(D_0{}^\dagger - D_0\right) - \left(D_1{}^\dagger + D_1\right) \circ \partial_u - \dot{D}_1{}^\dagger\right] \circ \partial_u{}^2 \,, \end{aligned} \tag{3.2.4}$$

which decomposes into powers of ∂_u as

$$D_1{}^\dagger + D_1 = 0 \,, \qquad 2\,\dot{D}_1 = D_0{}^\dagger - D_0 - \dot{D}_1{}^\dagger \,, \tag{3.2.5a}$$

$$2\,\dot{D}_0 + \ddot{D}_1 = 0 \,, \qquad \ddot{D}_0 = 0 \,. \tag{3.2.5b}$$

The general solution is

$$D_0 = K_0 - iK_{+1}\,u \,, \quad D_1 = iK_{-1} + \left(K_0{}^\dagger - K_0\right)u + iK_{+1}\,u^2 \,, \tag{3.2.6}$$

where the K_m ($m = -1, 0, +1$) are independent of u and the $K_{\pm 1}$ are Hermitian. Decomposing K_0 into Hermitian and anti-Hermitian parts, $K_0 = L_{-1} - iL_{+1}$, we have

$$D = K_{-1} \circ H_{-1} + L_{-1} \circ \mathrm{id} + 2\,L_{+1} \circ H_0 + K_{+1} \circ H_{+1} \,, \tag{3.2.7}$$

where the identity generator id has been inserted to emphasize the role of $L_{\pm 1}$ and $K_{\pm 1}$ as coefficients multiplying the symmetry generators

$$H_{-1} = i\,\partial_u \,, \quad H_0 = i\left(u\,\partial_u - \tfrac{1}{2}\right) \,, \quad H_{+1} = i\,u\,(u\,\partial_u - 1) \,, \tag{3.2.8}$$

satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra (or $\mathfrak{gl}(2,\mathbb{R})$ if one includes the identity generator) and representing the conformal isometries of the real line.

The non-trivial higher symmetries of the action (3.2.3) thus span a real Lie algebra isomorphic to

$$\mathcal{H}\left(S^d\right) \otimes \mathfrak{gl}(2,\mathbb{R}) \,, \tag{3.2.9}$$

with $\mathcal{H}(S^d)$ the Lie algebra of Hermitian differential operators on the celestial sphere S^d, while the set $\{\mathrm{id}, H_{-1}, H_0, H_{+1}\}$ forms a basis of $\mathfrak{gl}(2,\mathbb{R})$. The commutator (times the imaginary unit i) of differential operators defines a Lie bracket on this vector space. As anticipated, this algebra is much bigger than $\mathfrak{ihs}_{d+2}$ and we now discuss some relevant sub-algebras.

Large $\mathfrak{u}(1)$ transformations

We begin by considering symmetries which are differential operators of order zero. They are real functions on the sphere S^d, $D = \alpha(\mathbf{x})$, corresponding to local phase transformations

$$\delta_\alpha \phi = i\,\alpha(\mathbf{x})\,\phi\,. \tag{3.2.10}$$

In a putative holographic correspondence, this symmetry should signal the presence of large $\mathfrak{u}(1)$ transformations of Maxwell theory as in [100] within the asymptotic symmetries of the bulk theory.

Generalised BMS symmetry

We now move to first-order symmetries. We write $L_{-1} = -i\,Y^i(\mathbf{x})\,\partial_i - \frac{i}{2}\,\nabla\cdot Y(\mathbf{x})$ and $K_{-1} = -T(\mathbf{x})$ with real functions Y^i and T, while choosing $L_{+1} = -\frac{1}{2d}\,\nabla\cdot Y(\mathbf{x})$. In this way, we obtain

$$i\,D_{\mathfrak{gbms}} = T\,\partial_u + Y^i\partial_i + \tfrac{1}{d}\,\nabla\cdot Y\,(\Delta + u\,\partial_u)\,, \tag{3.2.11}$$

which is the form of a differential operator of order one generating super-translations via T and super-rotations via Y^i when acting on a scalar density of scaling dimension $\Delta = \frac{d-1}{2}$. Note that we did not impose any constraint on the vectors Y^i: therefore the super-rotations in (3.2.11) generate the whole $\mathfrak{diff}(S^d)$ algebra as in the extended BMS algebra of [99].

Enhanced BMS symmetry algebra

Still looking at first-order differential operators, there are two more available functions generating (super-)dilations and (super-)conformal boosts in the u direction:

$$D_{\mathfrak{bms}^+} = D_{\mathfrak{gbms}} + W(\mathbf{x})\,H_0 + Z(\mathbf{x})\,H_{+1}\,, \tag{3.2.12}$$

with real W and Z. From the bulk viewpoint, in which one seeks to interpret these transformations as being associated to asymptotic symmetries, the action of $W(\mathbf{x})\,H_0$ should correspond to 'BMS-Weyl' transformations as those considered in [226], while the action of $Z(\mathbf{x})\,H_{+1}$ is isomorphic to that of the Newman-Unti[8] group at level 3 (which is the largest Newman-Unti group with finite level) [104, 220]. The latter has a natural interpretation in [89] as the group of generalised BMS transformations allowing for Weyl rescalings of the metric of the sphere that are affine in u. We denote the full first-order sub-algebra of (3.2.9) as $\mathfrak{bms}^+_{d+2}$ and we remark that it does not seem to be isomorphic to any of the proposed conformal

[8]The Newman-Unti group at level k is generated by vector fields X preserving the metric up to a conformal factor, i.e. $\mathcal{L}_X g = \lambda g$ and such that ${\mathcal{L}_\xi}^k X = 0$ where ξ is the fundamental Carrollian vector fields defined in appendix A.

extensions of the BMS algebra in three of higher dimensions [227, 228]. Instead, the extra generator H_{+1} can be interpreted as a super-translations at the infinity of $\mathscr{I}$ (remember that a special conformal transformation is the combination of an inversion, a translation and an inversion).

Algebra of higher-order operators

The symmetrised product of differential operators satisfying (3.2.1) is a differential operator satisfying the same condition (similarly to the commutator times the imaginary unit, that defines the Lie bracket on (3.2.9)). Higher-order symmetries can thus be realised as symmetrised products of first-order ones. This simple observation guarantees that $\mathfrak{ihs}_{d+2}$ is a Lie sub-algebra of (3.2.9). Indeed, as it is also manifest in (3.2.11), the generators (3.2.6) belong to the symmetries of the simpleton and their symmetrised products acting on the simpleton give, by construction, the algebra $\mathfrak{ihs}_{d+2}$.

As discussed in section 3.2.3, the constraints (3.2.7) that select the algebra $\mathfrak{ihs}_{d+2}$ are not necessary to identify a symmetry of the simpleton. Therefore, one can also consider products of the operators in (3.2.11) with unconstrained $T(x)$ and $Y^i(x)$, which form a sub-algebra. In this way one obtains an infinite-dimensional extension of the algebra $\mathfrak{ihs}_{d+2}$, that we dub $\mathfrak{hsbms}_{d+2}$. It corresponds to realising the $\mathfrak{gbms}_{d+2}$ UEA on the simpleton module, while the infinite-dimensional extension of [225] realises it on the unitary Sachs module defined for $\Delta = \frac{d}{2}$. As a result, in that case as well as in [101, 103] polynomials of any order in u appear in the differential operators, while here the u-dependence only comes from the operators (3.2.8), compatibly with the observation that $\mathfrak{ihs}_{d+2}$ is not a sub-algebra of the higher-spin algebras of [225].

Concretely, higher-order symmetries are obtained composing the operators (3.2.8) with higher-order differential operators on the sphere, see (3.2.9). The latter can then be expanded as

$$L_{-1} = \sum_{s\geq 3} i^{s-1}\, Y^{j_1\cdots j_{s-1}}(\mathbf{x})\, \nabla_{(j_1} \cdots \nabla_{j_{s-1})} + \text{lower derivative} \tag{3.2.13}$$

with real $Y^{j_1\cdots j_{s-1}}$, and where lower-derivative terms are required to obtain Hermitian operators. The operators K_{-1}, L_{+1} and K_{+1} in (3.2.7) admit similar expansions. We only need to consider symmetric tensors $Y^{j_1\cdots j_{s-1}}$ because any antisymmetrisation of the indices will translate into the commutator of covariant derivatives on the sphere, which gives contributions proportional to the Riemann tensor and takes away two derivatives. Since the sphere is a space of constant curvature, these terms can be reabsorbed into lower-order differential operators.

A higher-spin generalisation of (super-)translations, i.e. an Abelian ideal, is given by $K_{-1} \circ H_{-1}$, while, in analogy with (3.2.11), the remaining generators of $\mathfrak{hsbms}_{d+2}$

are given by linear combinations of $L_{-1} \circ \mathrm{id}$ and $L_{+1} \circ H_0$, each one depending only on $Y^{i_1 \cdots i_{s-1}}$. This symbol, in its turn, can be recovered as a symmetrised product of vectors Y^i.

In spite of the differences discussed above regarding the u-dependence of the gauge parameters, decomposing $Y^{i_1 \cdots i_{s-1}}$ and the corresponding tensors $T^{i_1 \cdots i_{s-2}}$ of K_{-1} into traceless parts, one recovers the same set of generators as in the asymptotic symmetries of Fronsdal fields for the weakest boundary conditions considered in [103]. We wish to emphasize that the difference in the realisation of the asymptotic symmetries here with respect to [103] are of the same nature as the differences in the realisation of rigid (bulk) isometries between eqs. (2.5.1) and (2.1.7).

We conclude by stressing that one can also consider products of the operators (3.2.12), so that the higher symmetries of the simpleton actually provide a higher-spin extension of the algebra $\mathfrak{bms}^+_{d+2}$ of section 3.2.3.

3.3 Magnetic simpleton theory

The discussion of the previous section allowed us to identify a boundary candidate for the flat analogue of the simpleton, which is for the moment only realised on $\mathscr{I}$ and in ambient space. Its naive bulk realisation, which will be discussed in section 3.4, seems more adapted to describe the Carrollian limit of the bulk singleton, living in AdS-Carroll$_{d+2}$ space-time (the $c \to 0$ contraction of AdS$_{d+2}$ space-time[9]), which can be explained by the dual role played by the Poincaré algebra, as both the isometry algebra of flat space-time and of AdS-Carroll space-time.

In the following, we will follow an alternative route which is more adapted to define the simpleton in Minkowski space, by starting again from a different definition of the boundary dynamics.

3.3.1 Boundary definition

In addition to the electric, or time-like, limit $c \to 0$ of the relativistic scalar, one can define another scalar field theory, which is the Carrollian limit of the relativistic scalar field theory in Hamiltonian form. Consider the action (3.2.1) in Hamiltonian form

$$S_{\mathrm{H}} = \frac{1}{2} \int \mathrm{d}u\, \mathrm{d}^d\mathbf{x}\, \sqrt{\gamma} \left(\bar{\pi} \partial_u \phi + \pi \partial_u \bar{\phi} - c^2\, \bar{\pi}\pi + \bar{\phi} \hat{\nabla}^2 \phi \right), \tag{3.3.1}$$

with the field π playing the role of conjugate momentum to $\bar{\phi}$ and vice-versa (under this form, the action is manifestly real), and where $\hat{\nabla}^2$ was defined in eq. (3.2.2).

[9] AdS-Carroll space-time can be defined as the homogeneous space associated to the Klein pair $ISO(1, d+1)/ISO(d+1)$ in the language of [229, 230], rather than Minkowski$_{d+2}$ whose Klein pair is given by $ISO(1, d+1)/SO(1, d+1)$.

Upon integration by parts, the relativistic theory (3.2.1) can be recast as

$$S_{\mathrm{H}} = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\,\boldsymbol{\phi}^\dagger\,\boldsymbol{K}_c\,\boldsymbol{\phi}\,, \tag{3.3.2}$$

with $\boldsymbol{\phi} = \begin{pmatrix}\phi\\ \pi\end{pmatrix}$ and the kinetic operator $\boldsymbol{K}_c = \begin{pmatrix}\hat{\nabla}^2 & -\partial_u\\ \partial_u & -c^2\end{pmatrix}$ which is manifestly self-adjoint under the product $\langle\boldsymbol{\phi}|\boldsymbol{\psi}\rangle = \int \mathrm{d}u\,\mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\,\boldsymbol{\phi}^\dagger\,\boldsymbol{\psi}\,.$

The kinetic operator $\boldsymbol{K}_c$ has a non-vanishing determinant for all values of c

$$\det\left(\boldsymbol{K}_c\right) = {\partial_u}^2 - c^2\,\hat{\nabla}^2\,, \tag{3.3.3}$$

and the limit $c \to 0$ converges to the magnetic, or space-like, Carrollian theory

$$S_{\mathrm{m}} = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\left(\bar{\pi}\partial_u\phi + \pi\partial_u\bar{\phi} + \bar{\phi}\hat{\nabla}^2\phi\right). \tag{3.3.4}$$

also described in [110, 109, 111]. In appendix B.1, we prove that this action, when coupled to an arbitrary Carrollian background, is Weyl-invariant for this choice of $\hat{\nabla}^2$, and in appendix B.2, we prove that this action is invariant under rigid Carrollian conformal transformations. The equations of motion of this theory are

$$\dot{\phi}(u,\mathbf{x}) = 0\,, \quad \dot{\pi}(u,\mathbf{x}) = \hat{\nabla}^2\phi(u,\mathbf{x})\,. \tag{3.3.5}$$

On-shell, the magnetic scalar is parameterised by two arbitrary functions of the celestial sphere, $\phi_0(\mathbf{x})$ and $\pi_0(\mathbf{x})$, such that

$$\phi(u,\mathbf{x}) \approx \phi_0(\mathbf{x})\,, \quad \pi(u,\mathbf{x}) \approx \pi_0(\mathbf{x}) + u\hat{\nabla}^2\phi_0(\mathbf{x})\,. \tag{3.3.6}$$

Although the action in eq. (3.3.4) is different from the one in eq. (3.2.3), we will prove in appendix B.5 that it also realises (an infinite-dimensional enhancement of) the higher-spin algebra $\mathfrak{ihs}_{d+2}$, isomorphic to $\mathcal{H}\left(S^d\right) \otimes \mathfrak{gl}(2,\mathbb{R})$ and is therefore also a candidate to be a flat-space analogue of the singleton. Moreover, the extra symmetries that were called $W(\mathbf{x})\,H_0$ and $Z(\mathbf{x})\,H_{+1}$ find an interpretation in the bulk of Minkowski space-time: the first one can be interpreted as the relic of a Weyl symmetry, while the latter encodes the possibility of gluing the past of future null infinity with the future of past null infinity [95].

The reason why both algebras are isomorphic, in essence, is due to the fact that we are classifying on-shell symmetries of an action and the magnetic theory has the same solution space as the electric theory, i.e. two arbitrary functions of the angles, modulo the conformal completion of the Laplacian $\hat{\nabla}^2$. The latter is almost always invertible, since the eigenvalue of $\hat{\nabla}^2$ evaluated on the subspace of d-dimensional spherical harmonics with principal quantum number ℓ is

$$-\ell(\ell+d-1) - \tfrac{(d-1)^2}{4} = -\left(\ell + \tfrac{d-1}{2}\right)^2\,, \tag{3.3.7}$$

which is zero only when $d = 1$ and $\ell = 0$.

3.3.2 Ambient definition and factoring out of the ideal

We presented the magnetic simpleton theory on $\mathscr{I}$ with the aim of finding a bulk description of the simpleton. To this end, we first propose an alternative ambient space realisation of the simpleton, which we show in section 3.3.3 and in appendix B to be equivalent to the theory of eq. (3.3.4).

We start from another ambient realisation of the simpleton, related to the former by a Fourier transform (equivalently, an exchange $y^a \leftrightarrow \partial_a$ and $u \leftrightarrow \partial_u$)

$$\partial_a \partial^a \Phi = 0\,, \quad (x^a \partial_a + \varsigma \partial_\varsigma + \Delta)\,\Phi = 0\,, \quad \Phi \simeq \Phi + \varsigma^2 \Psi\,, \tag{3.3.1}$$

where the generators defining the constraints (3.3.1) form again the algebra $\mathfrak{iso}(1,1)$ and commute with the isometries

$$J_{ab} = 2\,x_{[a}\partial_{b]}\,, \quad P_a = \varsigma\,\partial_a\,, \tag{3.3.2}$$

generating the Poincaré algebra. We will take again $\Delta = \frac{d-1}{2}$. The first generator is exactly the canonical realisation of Lorentz transformations, while the second looks like the canonical realisation of translations, up to the factor of ς. Contrary to section 3.2.2, the geometry of ambient space is *Galilean*[10] rather than Carrollian.

The ideal of the simpleton is still factored out on this module, due to the dual role of the coordinates and their momenta. Explicitly,

$$J_{[ab} \circ J_{cd]}\,\Phi = 0\,, \quad J_{[ab} \circ P_{c]}\,\Phi = 0\,, \tag{3.3.3}$$

is guaranteed by the differential realisation of isometries, and the rest follows from a direct computation

$$\tfrac{1}{2} J_{ab} \circ J^{ba}\,\Phi \simeq -\tfrac{d^2-1}{4}\Phi\,, \tag{3.3.4a}$$

$$\big(J_{ab} \circ P^b + P^b \circ J_{ab}\big)\,\Phi \simeq 0\,, \tag{3.3.4b}$$

$$P_a \circ P_b\,\Phi \simeq 0\,, \tag{3.3.4c}$$

where in the last three identities we used the quotient condition $\Phi \simeq \Phi + \varsigma^2\,\Psi$ (the proof is completely analogous to that of section 3.2.2).

3.3.3 Bulk definition

The bulk description of the realisation (3.3.1) is quite elegant to perform: we simply evaluate $\Phi(x,\varsigma)$ at any finite value of ς, for instance $\varsigma = 1$. Performing again a

[10] Galilean geometry is the dual of Carrollian geometry of appendix A, in the sense that it is characterised by an inverse metric, which is degenerate in the direction of a nowhere-vanishing one-form. In our case, the ambient inverse metric with signature $(\underbrace{-,+,\cdots,+}_{a},0)$ and the one-form $d\varsigma$ play this role.

Taylor expansion of $\Phi(x,\varsigma)$ and truncating to second order in ς

$$\Phi(x,\varsigma) = \varphi_-(x) + \varsigma\, \varphi_+(x)\,, \tag{3.3.1}$$

we find that the bulk value of the field Φ can be arranged in a doublet, where fields are listed according to their power of ς

$$\boldsymbol{\varphi}(x) = \begin{pmatrix} \varphi_-(x) \\ \varphi_+(x) \end{pmatrix}, \tag{3.3.2}$$

and each component satisfies

$$\partial_a \partial^a \varphi_\pm(x) = 0\,, \quad (x^a\, \partial_a + \Delta_\pm)\, \varphi_\pm = 0\,, \tag{3.3.3}$$

where we defined $\Delta_- = \frac{d-1}{2}$ and $\Delta_+ = \Delta_- + 1 = \frac{d+1}{2}$ (note that the value of $\Delta_\pm$ is unrelated to the one of section 3.1.3). In this form, it is obvious that the ς-dependence is completely factored out (it was an auxiliary direction anyway, used to embed Minkowski space in ambient space), and that the action of translations is nilpotent, since

$$J_{ab} = \begin{pmatrix} 2\, x_{[a}\partial_{b]} & 0 \\ 0 & 2\, x_{[a}\partial_{b]} \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & 0 \\ \partial_a & 0 \end{pmatrix}. \tag{3.3.4}$$

Compared to the singleton, which can be described by a single scalar field in AdS_{d+2}, we have here a realisation in terms of a pair of scalar fields, with a slightly more complicated structure. We nevertheless show in appendix B.3 that the same structure can be achieved for the bulk AdS singleton, in which case the flat limit is actually well-defined and corresponds to eq. (3.3.3). On the other hand, the structure in pair is adapted to describe (the limit $c \to 0$ of) the boundary magnetic theory displayed in eq. (3.3.1).

3.3.4 Asymptotics of the bulk field

A motivation

The intuition why we propose the definition (3.3.3) (equivalently the ambient space description (3.3.1)) to be the bulk dual of the magnetic theory is that the equations of motion of a free bulk scalar field in Bondi coordinates reproduce the equations of motion (3.3.5) at lowest order in a $1/r$-expansion of the d'Alembert equation.

Indeed, recall that in Bondi coordinates, a field $\psi(r,u,\mathbf{x})$ with the $1/r$ expansion

$$\psi(r,u,\mathbf{x}) = \frac{1}{r^{\Delta'}} \sum_{n\geq 0} \frac{\psi^{(n)}(u,\mathbf{x})}{r^n}\,, \tag{3.3.1}$$

where Δ' is some real number that is for the moment arbitrary. The wave equation $\partial_a\partial^a\psi = 0$ in Minkowski space gives, order by order in the $1/r$ expansion [225]

$$(d-2\,n-2\,\Delta')\,\dot{\psi}^{(n)} = \nabla^2\psi^{(n-1)} - (\Delta'+n)\,(d-1-\Delta'-n)\,\psi^{(n-1)}\,, \tag{3.3.2}$$

where a dot stands for a derivative with respect to u. These equations can be solved order by order using u integrals, bringing along some 'integration constants', that are arbitrary functions of the angles.

There are several interesting values one can take for Δ'. If $\Delta' = \frac{d}{2} - n$ for some integer $n \geq 0$, then the coefficient in the left-hand side of eq. (3.3.2) vanishes, meaning that the corresponding field $\psi^{(n)}$ can have an arbitrary u-dependence. In particular, when $n = 0$ the leading order is an arbitrary field. This solution is called *radiative*, since it corresponds to the case where a function of arbitrary u-dependence reaches null infinity.

The simpleton (neither the electric nor the magnetic one) does not verify this property since its scaling dimension is $\frac{d-1}{2}$. However, for this value of Δ' and this value only, the two instances $n = 0$ and $n = 1$ of eq. (3.3.2) are precisely the equations of motion of the magnetic theory, as can be seen from eq. (3.3.5) with the identification $\psi^{(0)}(u,\mathbf{x}) := \phi(u,\mathbf{x})$ and $\psi^{(1)}(u,\mathbf{x}) := -\pi(u,\mathbf{x})$. This means that, even if the equations of motion $n = 0$ and $n = 1$ in eq. (3.3.2) describe a Poincaré scalar for any value of Δ', only for the value $\frac{d-1}{2}$ can these equations of motion be derived from a conformal Carroll-invariant action defined on $\mathscr{I}$.

In order to identify the field $\psi(r,u,x)$ as the bulk equivalent of the simpleton, we need to match the solution spaces of the two theories. This means that we need implement the condition that there are no free data (no 'integration constants') hiding in the orders $\psi^{(k)}$ for $k \geq 2$. This condition corresponds to a shortening of the degrees of freedom of the free scalar in Minkowski space, akin to the shortening condition in AdS_{d+2} allowed by the fine-tuning of the scaling dimension [211]. This can be implemented through a homogeneity condition and the doubling of the fields, in the way presented in eq. (3.3.3), which we describe in the rest of this section.

Boundary value of the bulk simpleton

Performing a $1/r$ expansion of both fields $\varphi_\pm(r,u,\mathbf{x})$ of the bulk magnetic theory in Bondi coordinates, we find that

$$\varphi_\pm(r,u,\mathbf{x}) = \frac{1}{r^{\Delta_\pm}}\sum_{n=0}^{\infty}\frac{\phi_\pm^{(n)}(u,\mathbf{x})}{r^n}\,, \tag{3.3.3}$$

where, solving the d'Alembert equation, we can express the part of $\phi_\pm^{(n)}$ homogeneous to u^n in terms of higher differential operators on the celestial sphere acting on $\phi_\pm := \phi_\pm^{(0)}$, and we can get rid of the 'integration constants' by homogeneity.

$$\phi_-^{(n)}(u,\mathbf{x}) = \frac{(2u)^n}{(2n)!}\prod_{k=0}^{n-1}(\Delta_-^2 - \nabla^2 - k^2)\phi_-(\mathbf{x})\,, \tag{3.3.4a}$$

$$\phi_+^{(n)}(u,\mathbf{x}) = \frac{(2u)^n}{(2n+1)!}\prod_{k=1}^{n}(\Delta_-^2 - \nabla^2 - k^2)\phi_+(\mathbf{x})\,. \tag{3.3.4b}$$

Thus, the magnetic simpleton is characterised at infinity by two functions of the celestial sphere $\phi_\pm(\mathbf{x})$, and one can reconstruct its full bulk value in a $1/r$ expansion without encountering any more arbitrary data. In other words, given the boundary value $\lim_{r\to\infty}\boldsymbol{\varphi} = \boldsymbol{\phi}$ (keeping u fixed) of the doublet of fields, it is enough to reconstruct the doublet of fields $\boldsymbol{\varphi}$ in the whole Minkowski space-time.

Remark that

$$\begin{aligned}
\partial_u\,\varphi_-(r,u,\mathbf{x}) &= \partial_u\left(\frac{1}{r^{\Delta_-}}\sum_{n=0}^{\infty}\frac{\phi_-^{(n)}(u,\mathbf{x})}{r^n}\right)\,,\\
&= \frac{1}{r^{\Delta_-}}\partial_u\sum_{n=0}^{\infty}\frac{1}{r^n}\frac{(2u)^n}{(2n)!}\prod_{k=0}^{n-1}(\Delta_-^2 - \nabla^2 - k^2)\phi_-(\mathbf{x})\,,\\
&= \frac{-1}{r^{\Delta_+}}\sum_{n=0}^{\infty}\frac{1}{r^n}\frac{(2u)^n}{(2n+1)!}\left[\prod_{k=1}^{n}(\Delta_-^2 - \nabla^2 - k^2)\right]\hat{\nabla}^2\phi_-(\mathbf{x})\,,
\end{aligned}\tag{3.3.5}$$

which corresponds to the Bondi expansion of a field of scaling dimension Δ_+ and boundary value $\left(\Delta_-^2 - \nabla^2\right)\phi_-(\mathbf{x}) = -\hat{\nabla}^2\,\phi_-(\mathbf{x})$, consistently with the identification $\pi = -\phi_+$ and the equation of motion for a magnetic Carrollian scalar $\dot{\pi} = \hat{\nabla}^2\phi$.

Note that, in order to define the boundary value of the magnetic simpleton, we could also have considered the pullback of the ambient definition of eq. (3.3.1) along the projective hyperplane $\varsigma = 0$, $(x^a,\varsigma) \sim \lambda\,(x^a,\varsigma)$, as one usually does in order to define a boundary value from an ambient description (see section 3.1.4). This is done in appendix B.4, where we also find that the boundary value is characterised by two arbitrary functions of the angles, in agreement with eq. (3.3.6).

3.4 Discussion

In this part, we have presented a construction allowing to realise the algebra $\mathfrak{ihs}_{d+2}$ as the algebra of symmetries of a Carrollian conformal scalar field theory. It relies on an ultra-relativistic cousin of the free scalar field, named *simpleton*, and can be described equally as a Carrollian field theory formulated on $(d+1)$-dimensional null infinity $\mathscr{I}$, as a scalar field theory in ambient space $\mathbb{R}^{2,d+1}$, or as a 'shortened' pair of free scalar fields in Minkowski$_{d+2}$ space. Its most naive description eq. (3.2.3) realises an extension of the generalised BMS$_{d+2}$ algebra that also incorporates generators that can be interpreted as higher-spin asymptotic symmetries, while the two-field description in eq. (3.3.4) is better suited for a realisation in the bulk of Minkowski space, see eq. (3.3.3). Since the higher-spin symmetries are computed on-shell, and the two theories have the same solution space, it should come as no surprise that their symmetry algebras are the same (see appendix B.5), albeit realised differently.

Although the electric and magnetic theories have the same symmetries, their bulk realisation are however quite different. Following the logic of the relativistic case, we can define the bulk value of the ambient space field Φ defined in eq. (3.2.2) by pulling back on the hypersurface $y^2 = -R^2$, defining the bulk of space-time. This hypersurface has the topology of $\mathbb{R} \times \mathbb{H}_{d+1}$, the product of null time with a two-sheeted hyperboloid, which has a natural interpretation as AdS-Carroll$_{d+2}$ [231, 232, 233, 230, 229].

One can split the non-Carrollian coordinates y^a in light-cone coordinates $y^a = (y^+, y^i, y^-)$ such that the metric reads $\eta_{ab}y^a y^b = 2\, y^+\, y^- + \gamma_{ij}\, y^i\, y^j$ and parameterise $y^2 = -R^2$ by

$$y^+ = \frac{1}{\sqrt{\rho}}\,, \quad y^i = \frac{1}{\sqrt{\rho}}\mathbf{x}^i\,, \quad y^- = -\frac{1}{2}\left(\sqrt{\rho}\,R^2 + \frac{1}{\sqrt{\rho}}\mathbf{x}^2\right)\,, \tag{3.4.1}$$

so that, using homogeneity

$$\Phi(u, y^a) = \left(-\frac{y^2}{R^2}\right)^{-\Delta/2} \bar{\Phi}\left(\frac{R}{\sqrt{-y^2}}u, \frac{R}{\sqrt{-y^2}}y^a\right)\,, \tag{3.4.2}$$

and $\bar{\Phi}$ verifies the property of being defined only on $y^2 = -R^2$. Then, one can define the field $\varphi(u, \rho, \mathbf{x})$ which is the pull-back of $\bar{\Phi}$

$$\varphi(u, \rho, \mathbf{x}) = \bar{\Phi}(u, y^a)\big|_{y^2=-R^2}\,, \tag{3.4.3}$$

verifying the second-order equation

$$\partial_u{}^2\, \varphi = 0\,. \tag{3.4.4}$$

Therefore, it is natural to interpret φ as the Carrollian limit of the singleton in AdS$_{d+2}$. The fact that we can define an ultra-relativistic contraction of $\mathfrak{hs}_{d+2}$ and that it is isomorphic to $\mathfrak{ihs}_{d+2}$ should not come as a surprise, since the algebra of isometries of Minkowski$_{d+2}$ and the Carroll contraction of the algebra of isometries of AdS$_{d+2}$ are isomorphic (see [234]), and one can prove that the limit $c \to 0$ of the ideal gives rise to the same expressions given in eq. (2.3.16) (consider that the discussion under eq. (2.2.6) is unchanged if instead of picking $\frac{\partial}{\partial X^D}$ as a stabiliser, we pick $\frac{1}{c}\frac{\partial}{\partial X^0}$ and send c to 0).

Although the algebras are the same, the homogeneous spaces Minkowski$_{d+2}$ and AdS-Carroll$_{d+2}$ defined as cosets of their respective kinematical Lie groups are different [234]. Surprisingly, the 'blow-up' of time-like infinity of Minkowski$_{d+2}$ defined in

[235] has the structure of an AdS-Carroll$_{d+2}$ space-time [233], which indicates that the AdS-Carroll field defined in eq. (3.4.4) may also play a role in flat higher-spin holography, albeit in a different region of space than the field defined in (3.3.3). The fact that Carroll field theories admit in general two distinct limits, and that one of them seems relevant for the description of the asymptotics of AdS-Carroll, while the other seems relevant for the description of the asymptotics of Minkowski space-time is rather curious and it would be necessary to perform more investigations in order to see if it constitutes a general statement.

In the discussion of Carroll-conformal-invariant field theories, let us remark that the option to define a BMS-invariant theory at null infinity was considered in [236], where a flat contraction of Liouville theory was shown to realise the BMS charge algebra (with non-zero central charge) in $d = 1$. The theory in question is a non-linear deformation of the free magnetic scalar by a term e^{ϕ}, although this theory was not checked to be explicitly conformal Carroll invariant. Realising the BMS algebra on free scalar fields was also investigated in [223, 237].[11]

In higher-spin holography, a key role is played by the boundary higher-spin currents $J_{\mu(s)}(x)$, built from the free scalar field theory of section 3.1 [72]. These higher-spin currents are conserved and traceless and are built from bilinears of $\phi(x)$ with s derivatives. We suspect that the Carrollian analogue of the currents $J_{\mu(s)}$ can be built by extending the setup of [92, 109, 238, 222, 111] to higher-spins, where the $s = 2$ instance corresponds to the stress-energy tensor $T^{\mu}{}_{\nu}$, which is displayed for both the electric and magnetic theories in [111]. The expected conservation laws are inherited from the relativistic ones, with the proviso that the currents have the structure of a 1-contravariant, $(s-1)$-covariant tensor $J^{\mu}{}_{\nu(s-1)}$ verifying $\nabla_{\mu} J^{\mu}{}_{\nu(s-1)} = 0$, $J^{\alpha}{}_{\alpha\nu(s-2)} = 0$ and $g_{\mu\rho} v^{\sigma} J^{\rho}{}_{\sigma\nu(s-2)} = 0$, where $g_{\mu\nu}$ represents the Carrollian metric, which is degenerate along v^{μ} (see appendix A and B.1).

However, the construction of these currents are still a loose end of our construction, in the sense that the Carrollian conformal scalar field (either electric or magnetic) has an affine u-dependence on-shell, meaning that the bilinear currents one can build out of them will clearly be at most quadratic in u. This signals that the usual AdS/CFT dictionary stating that the boundary value of AdS higher-spin fields couple to higher-spin currents through the minimal term $\int \mathrm{d}^{d+1}x\ \varphi^{\mu(s)} J_{\mu(s)}$ cannot, in this state, account for the gravitational and higher-spin radiation reaching null-infinity, encoded in the gravitational shear and its higher-spin analogue (the magnetic Carroll theory of section 3.3 is often called non-radiative for this reason). A way out could be to consider an *improved* stress-energy tensor, including arbitrary pieces allowed by the conservation laws. A situation of this sort has been

[11] In [225], the realisation of a higher-spin extension of the BMS$_{d+2}$ algebra on the asymptotic data of a free Minkowski scalar field was investigated, and it was proposed that the algebra realised for the value of the scaling dimension $\Delta = \frac{d}{2} - 1$ gave rise to a candidate asymptotic symmetry algebra for unconstrained higher-spin theories in Minkowski space-time.

encountered in the setup of the fluid-gravity correspondence applied to Carrollian hydrodynamics [92], where part of the heat current is not related to the geometry and can *a priori* assume any value. This falls under the general prescription of [95], where it was remarked that one needs to include source terms to account for the radiation reaching null infinity. A Carroll-invariant field theory accounting for the boundary degrees of freedom of gravity is still an outstanding question in Carrollian holography, and we hope that the elements laid in this part constitute a promising basis upon which to proceed.

Chapter 4

Conclusions

In this thesis we presented a new, Lorentz-covariant description of the dynamics for free massless higher-spin fields in Minkowski space-time of any dimensions, which is based on the gauging of a non-Abelian higher-spin algebra dubbed $\mathfrak{ihs}_D$.

The Lagrangian formulation of the free dynamics by Fronsdal makes use of a completely symmetric, doubly-traceless field $\varphi_{\mu(s)}$, and the starting point of its reformulation in terms of curvatures is the 'frame-like' field $e_\mu{}^{a(s-1)}\,\mathrm{d}x^\mu$, which is a space-time one-form with $(s-1)$ completely symmetrised indices, varying under the gauge transformations

$$\delta e^{a(s-1)} = \mathrm{d}\xi^{a(s-1)} + h_b\,\lambda^{a(s-1),b}\,. \tag{4.0.1}$$

Instead, our formulation uses a mixed-symmetry irreducible one-form with $(2s-3)$ indices $\omega_\mu{}^{a(s-1),b(s-2)}\,\mathrm{d}x^\mu$, with gauge variation

$$\delta\omega^{a(s-1),b(s-2)} = \mathrm{d}\lambda^{a(s-1),b(s-2)} + h_c\,\lambda^{a(s-1),b(s-2)c} + h^{\{b}\,\lambda^{a(s-1),b(s-3)\}}\,,$$

where braces denote a traceless and Young projection discussed in eq. (2.1.27).

This field is actually already part of the original frame-like description of the dynamics [58]. Where our formulation differs is that, while it used to play a purely auxiliary role in the original case, here it becomes fundamental.

Upon imposing unfolded equations of motion, this field can be identified with $(s-2)$ derivatives of a frame-like field, generalising the vielbein with additional indices so that its symmetric projection is a Fronsdal field. This is made possible by the observation that the equation

$$h_c \wedge \mathrm{d}\omega^{a(s-1),b(s-3)c} = 0\,,$$

which follows from the equations discussed in section 2.4 and which, for $s \geq 3$, automatically lead to the existence of a frame-like field $\tilde{e}^{a(s-1)}$ which obeys the equations of motion spelled out in [64]. In turn, the field $\tilde{e}^{a(s-1)}$ can be viewed as a Fronsdal field, although it is not necessary in our construction.

This novel formulation of the dynamics was made possible by the identification of a non-Abelian higher-spin algebra in flat-space and in any dimensions, $\mathfrak{ihs}_D$, which can be constructed as an İnönü-Wigner contraction of the non-Abelian algebra of rigid symmetries for higher-spin fields in AdS $\mathfrak{hs}_D$. Alternatively, this algebra can also be constructed by an appropriate quotient of the universal enveloping algebra of space-time isometries, similarly to the construction applying when the cosmological constant does not vanish, and is unique under certain assumptions. The structure constants of this algebra exhibit unusual features and do not satisfy the set of 'initial conditions' historically imposed in the quest for a non-Abelian higher-spin extension of the Poincaré algebra and that were dictated by the known formulation of the frame-like dynamics. Thus, its existence does not contradict the corresponding no-gos, but forced us to look for another formulation of the dynamics.

The equations that we impose on the linearisation of the curvatures taking values in this algebra have the same form as the ones proposed by Lopatin and Vasiliev in 1987 [64] when expressed in terms of higher-spin curvatures, but this time they force a plethora of fields to be pure-gauge, including the completely symmetric gauge field usually identified as the frame-like equivalent of the Fronsdal field. Crucially, we realised that it is possible to reconstruct a Fronsdal field from the remaining connections, and imposing the same equations of motion brings us back to the usual case. Although the new linearised curvatures can be obtained from a simple contraction of the ones in AdS, the mechanism used to eliminate the extra fields in the flat case is now radically different.

As anticipated, the new formulation reserves for the gauge field $\omega^{a(s-1),b(s-2)}$ a central role. The latter identifies with the vielbein for $s = 2$, and is one of the only two fields which are not pure-gauge, alongside $\omega^{a(s-1),b(s-1)}$. On-shell, they are identified with $(s-2)$ and $(s-1)$ derivatives of a Fronsdal field respectively, which is not part of the original set of connections gauging the algebra. This means that in our approach, the Fronsdal field is not directly accessible to build interactions, rather only higher-derivative combinations thereof. In turn, this provides an explanation as to why the only (cubic) interactions of higher-spin fields with gravity that were found to deform the gauge transformations are higher-derivative.

Since, this new formulation is based on the linearised curvatures gauging the higher-spin algebra $\mathfrak{ihs}_D$, the full curvatures may serve as the starting point for a consistent theory of higher-spin fields interacting with themselves and gravity in flat space-time, e.g., by following the algebraic approach of [199]. Although our construction starts with the Eastwood-Vasiliev algebra, and the equations gauging this algebra are known to be singular in the limit $R \to \infty$, we do not need to rely on a limiting procedure to construct the curvatures of eq. (2.4.3) and we can describe everything in flat space from the get-go. We hope that this approach will allow to eliminate most of the obstructions and subtleties associated with higher-spin interactions in flat-space.

In a second part, driven by the correspondence between the algebra of rigid symmetries of higher-spin fields propagating in AdS space-time and the algebra of higher differential symmetries of a free scalar field on Minkowski space, in the context of higher-spin holography, we put forward an analogue correspondence between our higher-spin algebra and the higher differential symmetries of a Carrollian scalar field theory that we tentatively call the *simpleton*, living on a null manifold with topology $\mathbb{R} \times S^d$. In its simplest realisation, it involves a single scalar 'electric' scalar field $\phi(u, \mathbf{x})$, which satisfies the equation of motion ${\partial_u}^2 \phi = 0$, while another, more exotic realisation involves a pair of fields $\phi(u, \mathbf{x})$, $\pi(u, \mathbf{x})$ which satisfy the equations of motion of the 'magnetic' Carrollian scalar field, $\partial_u \phi = 0$, $\partial_u \pi = \left(\nabla^2 - \frac{(d-1)^2}{4}\right) \phi$, with no propagating degrees of freedom. This pair of fields was then realised to be in one-to-one correspondence with the asymptotic data of a pair of massless fields in Minkowski space-time, which is the direct flat limit of a singleton.

In both cases, the algebra of higher differential operators contains $\mathfrak{ihs}_{d+2}$ as a sub-algebra. It also features an infinite-dimensional enhancement for generators of every spin, that are the higher-spin analogue of the generators of the generalised BMS algebra of [99]. The full algebra actually contains even more generators, which close on a higher-spin extension of the level-three Newman-Unti algebra [220]. What is still lacking at the moment is a realisation of the infinite-dimensional symmetries of the simpleton as the asymptotic symmetries of a higher-spin field theory in the bulk of Minkowski space-time. A natural candidate are the equations of motion obtained by gauging the flat-space higher-spin algebra discussed in this thesis, and we hope to come back to this issue soon.

Let us conclude with some research directions. A first direct extension of our results would be to include the treatment of fermionic higher-spin fields in the bulk. We expect the starting point to be the same: the identification of the rigid symmetries of the Fang-Fronsdal field [179, 180], given by Killing spinor-tensors. The next step would be to construct a super-symmetric extension of $\mathfrak{ihs}_4$, as the İnönü-Wigner contraction of the full super-higher-spin algebra $\mathfrak{shs}_4$ in AdS$_4$ space-time [60], rather than its bosonic commutator sub-algebra. Spin-$(s + \frac{1}{2})$ representations in the bulk of AdS$_4$ space-time then arise from the application of the Flato-Fronsdal theorem to the tensor product of a scalar and a Dirac singleton, therefore the Carrollian limit of the free Dirac field [239] is expected to play a role in the holographic description of this system. A second extension is related to the deformation of the boundary free theory to include quartic interactions ϕ^4 [240]. Finally, one can wonder about the applicability of our procedure to other types of representations of the Poincaré group than the completely symmetric ones [39, 31, 166, 241]. A non-Abelian higher-spin algebra for mixed-symmetry fields in AdS [119, 120, 242] was identified in [182], and it would be interesting to see if a flat-space contraction exists. This might have a more direct application to the program of describing the tensionless limit of string field theory, since mixed-symmetry fields are ubiquitous as soon as the dimension of space-time exceeds six, and constitute the main portion of the string spectra.

In view of the historical difficulties encountered in building an interacting theory of higher-spin gravity in Minkowski space starting from the Fronsdal formulation, and the alternative approach suggested in this thesis, one can question the physical relevance of the Fronsdal field in Minkowski space. Indeed, in our formulation of the dynamics, it is only a by-product, while the gauge connection that gives rise to it should be considered as the more fundamental field. However, in AdS, the Fronsdal field plays a preponderant role in higher-spin holography. Indeed, in the usual AdS/CFT dictionary, the latter couples to conserved higher-spin currents which are completely symmetric and traceless. If higher-spin holography in flat space-time were to work in the same way as it does in AdS space-time, one would need to perform $(s-2)$ integrations to have access to a Fronsdal field. This tension could be solved, for instance, by a careful treatment of the higher-spin currents in the dual field theory, or by the emergence of a new dictionary in flat-space higher-spin holography, see, e.g., [92, 243, 95].

A more general question, which is perhaps philosophical in nature, is if one should expect higher-spin holography to work in the same way both in AdS and flat space-time. Even though our approach features a *bona fide* extension of the Poincaré algebra of symmetries, no equivalent of the Flato-Fronsdal theorem seems to emerge due to the bilinears of the simpleton being at most quadratic in u, and therefore unable to encode the degrees of freedom of higher-spin fields coming from the bulk.[1] It was proposed that this issue should be addressed in Carrollian holography through the inclusion of sources encoding gravitational radiation reaching the boundary [94, 95], and which are absent in AdS/CFT if one imposes the usual reflective boundary conditions.

[1]Some recent progress in chiral higher-spin gravity [244, 245] indicates that there exists another framework in which one can translate most of the results of higher-spin holography into flat space. One can prove for instance the Flato-Fronsdal theorem, but one must be ready to accept some unconventional features as well. For instance, the gauge algebra relevant for flat self-dual higher-spin gravity is not based on the Poincaré algebra [197, 246], but on a different contraction of the AdS_4 isometry algebra. This last statement is not restricted to higher spins, since it is also the case for flat self-dual gravity [247].

Appendix A

Elements of Carrollian holography

In this section, we will review some facts about Carrollian geometry. We will mainly follow [220, 104, 231], and more information can be gathered from, e.g. [206, 229, 225]. We will define a Carrollian manifold as the triple (M, g, ξ), where g is a $(d+1)$-dimensional degenerate metric and ξ is a nowhere-vanishing vector field such that

$$g^\flat(\xi) = 0\,. \tag{A.0.1}$$

We will moreover assume that the manifold M can be split as the product of a null direction generated by ξ and a non-degenerate spatial sub-manifold $M \simeq \mathbb{R} \times \Sigma$, where the tangent space to the direction $\mathbb{R}$ is a one-dimensional vector space generated by ξ. In a coordinate chart $(u, \mathbf{x}^i)$, we can write for instance

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0 \times \mathrm{d}u^2 + \gamma_{ij}(\mathbf{x})\,\mathrm{d}\mathbf{x}^i\mathrm{d}\mathbf{x}^j\,, \quad \xi = \partial_u\,, \tag{A.0.2}$$

where $\gamma_{ij}(\mathbf{x})$ is a non-degenerate metric on the manifold Σ that only depends on $\mathbf{x}$, and the degenerate direction is spanned by $u \in \mathbb{R}$. We can introduce a torsion-free, metric-compatible[1] connection ∇. In our choice of coordinates, a simple choice is given by the Levi-Civita connection for the metric γ.

Infinitesimal Carroll symmetry is encoded in the Killing vectors of the metric g, i.e. the vectors $X \in \Gamma(TM)$ such that $\mathcal{L}_X g = 0$ (which may include ξ). With our choice of torsion-free metric-compatible connection, they include the Killing vectors of Σ

$$\nabla_{(i}X_{j)} = 0\,. \tag{A.0.3}$$

Conformal Carroll symmetry are the transformations that leave the data (g, ξ) invariant up to a rescaling by a power of a smooth function $\Omega \in \mathcal{C}^\infty(M, \mathbb{R}^*)$

$$g \to \Omega^2\, g\,, \quad \xi \to \Omega^{-1}\,\xi\,, \tag{A.0.4}$$

[1] Even with these two requirements, it is in general non-unique [248, 231].

or infinitesimally, they are vectors X such that there exists $\omega \in \mathcal{C}^\infty(M, \mathbb{R})$ verifying

$$\mathcal{L}_X g = 2\omega\, g\,, \quad \mathcal{L}_X \xi = -\omega\, \xi\,, \tag{A.0.5}$$

where we already specified that the metric g and the vector field ξ have weights two and minus one under conformal transformations respectively, in accordance with its relativistic parent (other choices exists as well, giving rise to the Newman-Unti family group of transformations [249, 220, 104]). With our choice of coordinates, if $X = X^i \partial_i$ is such a vector and the function ω takes the form

$$\omega = \tfrac{2}{d} \nabla \cdot X\,. \tag{A.0.6}$$

Moreover, the conformal Killing vectors of Σ are conformal Killing vectors of M

$$\nabla_{(i}\, X_{j)} = \tfrac{1}{d} \gamma_{ij} \nabla \cdot X\,, \tag{A.0.7}$$

where indices are lowered and raised thanks to γ, the non-degenerate part of the Carrollian metric.

In both cases, arbitrary infinitesimal shifts in the direction u leave the metric invariant, since $g_{uu} = g_{ui} = 0$ and γ_{ij} and ξ are independent of u. In general, (conformal) Carrollian diffeomorphisms are the ones that redefine space as a functions of space only, while redefining time as a function of space and time, thus they play a dual role to Galilean diffeomorphisms which transform space to a function of space and time and time to a function of time only. Therefore, vectors of the form $f(\mathbf{x})\xi$ are also trivially (conformal) Killing vectors. The conformal Carroll isometries are given by the semi-direct product of the conformal isometries of the spatial sub-manifold and the previous shifts in the direction of u, called 'super-translations' and parameterised by smooth real functions

$$\mathrm{CCarr}(M) = \mathrm{Conf}(\Sigma) \ltimes \mathcal{C}^\infty(\Sigma, \mathbb{R})\,. \tag{A.0.8}$$

This group may be further enhanced (by relaxing physical boundary conditions [99]) to $\mathrm{Diff}(\Sigma) \ltimes \mathcal{C}^\infty(\Sigma, \mathbb{R})$ to include transformations that modify the metric on the sphere to be anything smoothly related to the sphere, and not just the rescaling by a non-zero function. The new transformations are called *super-rotations*.

Appendix B

More on the magnetic simpleton

B.1 Weyl invariance of the magnetic simpleton

Let us verify in an independent way that the particular value of the linear term entering the definition of $\hat{\nabla}^2$ is the one needed to ensure that the action (3.3.4) is Weyl-invariant on a manifold with the geometry of $\mathbb{R} \times S^d$ (this proof is adapted from the one in [111], which we reproduce here for the geometry of $\mathscr{I}$).

Let us reformulate this action when coupled to a more general background

$$S_{\text{gen}} = \frac{1}{2}\int \mathrm{d}u\, \mathrm{d}^d\mathbf{x}\, e\left(\bar{\pi}v^\mu\partial_\mu\phi + \pi v^\mu\partial_\mu\bar{\phi} - g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi - \tfrac{d-1}{4\,d}\mathcal{R}[g]\bar{\phi}\phi\right), \tag{B.1.1}$$

which is specified by a degenerate metric $g_{\mu\nu}$ along the direction of v^μ and with non-vanishing density e, and $\mathcal{R}[g] = g^{\mu\nu}R_{\mu\nu}[g]$, where $R_{\mu\nu}[g]$ is the Ricci tensor of the metric $g_{\mu\nu}$, see [248, 111]. The clock form τ_μ dual to the vector v^μ verifies $v^\mu\tau_\mu = 1$ and $v^\mu\,\partial_{[\mu}\tau_{\nu]} = 0$, and the metric $g_{\mu\nu}$ has vanishing extrinsic curvature $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_v g_{\mu\nu} = 0$. In particular, the non-degenerate part of $g_{\mu\nu}$ can be the Euclidean metric on $\mathbb{R}^d$ or on the sphere S^d. In the following, we only allow transformations that stay within this class of 'flat' Carroll backgrounds. In particular, Weyl transformations must depend only on spatial coordinates.[1]

For instance, the background given by $g_{\mu\nu} = \mathrm{diag}(0, \gamma_{ij})$ with γ_{ij} the round metric on the sphere, $\tau_\mu = \delta^0_\mu$, $v^\mu = \delta^\mu_0$, and density $e = \sqrt{\det(g_{\mu\nu} + \tau_\mu\tau_\nu)}$ reproduces the action (3.3.4).

The various objects entering the action transform under Weyl rescalings with real parameter ω

$$\delta_\omega g_{\mu\nu} = 2\omega\, g_{\mu\nu}\,, \quad \delta_\omega v^\mu = -\omega\, v^\mu\,, \quad \delta_\omega e = (d+1)\omega\, e\,, \tag{B.1.2}$$

[1] A general argument for a wider class of metrics stable under arbitrary Weyl transformations can be found in [111] but involves additional fields playing the role of Lagrangian multipliers for the conditions listed above on the clock form and the extrinsic curvature.

and one can work out the transformation of the Ricci scalar under Weyl transformations

$$\delta_\omega R = -2\omega\, R - 2d\, \nabla^2 \omega\,, \tag{B.1.3}$$

with $\nabla^2\omega = e^{-1}\, \partial_\mu\, (e\, g^{\mu\nu}\, \partial_\nu \omega)$, and

$$\delta_\omega \phi = -\tfrac{d-1}{2} \omega\, \phi\,, \quad \delta_\omega \pi = -\tfrac{d+1}{2} \omega\, \pi\,, \tag{B.1.4}$$

for the fields.

Under a Weyl rescaling, the generalised action transforms as

$$\delta_\omega S_{\text{gen}} = \frac{1-d}{4} \int \mathrm{d}u\, \mathrm{d}^d\mathbf{x}\, e\left[(\bar\pi\phi + \bar\phi\pi) v^\mu \partial_\mu \omega - g^{\mu\nu} \partial_\mu \omega \partial_\nu (\bar\phi\phi) - \nabla^2 \omega \bar\phi \phi \right], \tag{B.1.5}$$

where the homogeneous terms cancel each other due to the total weight of the integrand under Weyl transformations being zero. The last two terms cancel each other upon integration by parts, while the first term is zero if and only if $v^\mu \partial_\mu \omega = 0$, i.e. in a ‘flat’ Carroll space-time with an adapted set of coordinates, the parameter ω does not depend on Carrollian time u.

Note that the value of the linear term in the case of the round sphere $-\frac{(d-1)^2}{4}$ does not correspond to the one giving rise to the Laplace-Yamabe operator on the round sphere S^d given by

$$\nabla^2_{\text{LY}} = \nabla^2 - \tfrac{d(d-2)}{4}\,, \tag{B.1.6}$$

which is the conformal completion of the Laplacian on the celestial sphere S^d. Instead, $\hat\nabla^2 - c^2\, {\partial_u}^2$ is the conformal completion of Laplacian which is conformally invariant on the whole cylinder $\mathscr{I} \simeq \mathbb{R} \times S^d$ (this is true both for $c \neq 0$ and $c = 0$).

For a more general class of metrics, Weyl rescalings with an arbitrary parameter are allowed, although this would require to modify the form of the action. In particular, in an adapted coordinate system where $v^\mu = \delta^\mu_0$, a metric of the form $g_{ij}(u, \mathbf{x}) = e^{2\omega(u,\mathbf{x})}\, \gamma_{ij}(\mathbf{x})$ is allowed since its extrinsic curvature is purely a trace.[2]

B.2 Conformal invariance of the magnetic simpleton

Here, we prove that eq. (3.3.4) is invariant under generalised BMS transformations. The actions of a super-translation $f(\mathbf{x})$ and a super-rotations $Y^i(\mathbf{x})\partial_i$ on the fields

[2] In the language of [206], these metrics have vanishing Carrollian shear and are of significant importance when studying BMS symmetry due to the presence of well-defined super-translation generators.

$\phi(u, \mathbf{x})$ and $\pi(u, \mathbf{x})$ are given by

$$\delta_{f,Y}\phi = \big(Y^i\,\partial_i + \tfrac{1}{d}\nabla\cdot Y(\Delta_- + u\,\partial_u) + f\partial_u\big)\,\phi\,, \tag{B.2.1a}$$

$$\begin{aligned}\delta_{f,Y}\pi = {} & \big(Y^i\partial_i + \tfrac{1}{d}\nabla\cdot Y(\Delta_+ + u\,\partial_u) + f\partial_u\big)\,\pi \\ & + \left(\partial^i f\partial_i + u\left[\nabla^2, Y^i\partial_i + \frac{\Delta_-}{d}\nabla\cdot Y\right] - \frac{u}{2d}\nabla\cdot Y\circ\hat{\nabla}^2\right)\phi\,,\end{aligned} \tag{B.2.1b}$$

where $\Delta_\pm = \frac{d\pm1}{2}$. In both cases, the terms in $u\,\partial_u$ coming from the action of conformal Killings are the effect of the shift $\Delta \to \Delta + u\,\partial_u$ already encountered in eq. (3.2.6). Note that the action is non-diagonal since in the transformation law $\delta\pi$ there is some piece proportional to ϕ.

B.3 The bulk simpleton is the flat singleton

Let us prove that the magnetic simpleton, although exotic-looking, is in fact the well-defined flat limit of the singleton, realised as a pair of scalars in AdS_D.

Let us start again with the ambient description of the singleton in $\mathbb{R}^{D-1,2}$ with Cartesian coordinates (x^a, w)

$$\big(\partial_a\partial^a - \partial_w{}^2\big)\,\Phi = 0\,, \quad (x^a\partial_a + w\partial_w + \Delta)\,\Phi = 0\,, \quad \Phi \simeq \Phi - \frac{x^2 - w^2}{R^2}\Psi\,, \tag{B.3.1}$$

where we restored the AdS radius R. The bulk value of the singleton is the evaluation of $\Phi(x, w)$ at $x^2 - w^2 = -R^2$. Let us pose $w = R\varsigma$, then in terms of ς we have

$$\left(\partial_a\partial^a - \frac{\partial_\varsigma{}^2}{R^2}\right)\Phi = 0\,, \quad (x^a\partial_a + \varsigma\partial_\varsigma + \Delta)\,\Phi = 0\,, \quad \Phi \simeq \Phi + \left(\varsigma^2 - \frac{x^2}{R^2}\right)\Psi\,. \tag{B.3.2}$$

It is obvious that the flat limit $R \to \infty$ of eqs. (B.3.2) reproduces eqs. (3.3.1) and therefore describe the bulk simpleton. The goal of this section is to find a realisation of the singleton in the bulk of AdS_D such that the flat limit takes us to the simpleton.

First, observe that one can always represent the singleton by a doublet of fields depending only on the coordinates x^a

$$\Phi(x, \varsigma) = \varphi_-(x) + \varsigma\,\varphi_+(x)\,, \tag{B.3.3}$$

where

$$(x^a\partial_a + \Delta_\pm)\,\varphi_\pm = 0\,, \quad \partial_a\partial^a\varphi_\pm = 0\,. \tag{B.3.4}$$

Indeed, by making a Taylor series expansion of the ambient field $\Phi(x, \varsigma)$ in the variable ς, the singleton reads

$$\Phi(x, \varsigma) = \sum_{n\geq0}\frac{\varsigma^n}{n!}\varphi_n(x)\,, \tag{B.3.5}$$

where we the functions $\varphi_n(x)$ are defined as $\partial_\varsigma{}^n\,\Phi(x,\varsigma)|_{\varsigma=0}$. Homogeneity dictates

$$(x^a\partial_a + \Delta_- + n)\,\varphi_n = 0\,. \tag{B.3.6}$$

Then, using the last relation of (B.3.2) we can trade powers of ς^2 for powers of $\frac{x^2}{R^2}$

$$\Phi(x,\varsigma) \simeq \sum_{n\geq 0}\frac{1}{(2n)!}\left(\frac{x^2}{R^2}\right)^n\varphi_{2n}(x) + \varsigma\sum_{n\geq 0}\frac{1}{(2n+1)!}\left(\frac{x^2}{R^2}\right)^n\varphi_{2n+1}(x)\,. \tag{B.3.7}$$

The fields $\varphi_{n\geq 2}(x)$ are actually not free: using the fact that Φ verifies the ambient wave equation, we deduce a relation among the fields $\varphi_n(x)$

$$\partial_a\partial^a\varphi_n(x) = \frac{1}{R^2}\varphi_{n+2}(x)\,. \tag{B.3.8}$$

Using this recursion relation, the ambient description of the singleton takes the form

$$\begin{aligned}\Phi(x,\varsigma) &= \sum_{n\geq 0}\frac{1}{(2n)!}x^{2n}\,\partial^{2n}\varphi_0(x) + \varsigma\sum_{n\geq 0}\frac{1}{(2n+1)!}x^{2n}\,\partial^{2n}\varphi_1(x)\\ &:= \varphi_-^{\text{new}}(x) + \varsigma\,\varphi_+^{\text{new}}(x)\,.\end{aligned} \tag{B.3.9}$$

Now, remark that $\varphi_\pm^{\text{new}}(x)$ have scaling dimensions $\Delta_\pm$, as a consequence of the homogeneity of φ_0 and φ_1 and the fact that the operator $x^2\,\partial^2$ has zero scaling dimension $[x^a\,\partial_a, x^2\,\partial^2] = 0$. In addition, $\varphi_\pm^{\text{new}}$ verify $\partial_a\partial^a\varphi_\pm^{\text{new}} = 0$. To see this, let us compute directly

$$\begin{aligned}&\partial_a\partial^a\varphi_-^{\text{new}} =\\ &\quad\sum_{n\geq 1}\frac{2n(d+2n)-8n^2-4n\Delta_-}{(2n)!}x^{2n-2}\partial^{2n}\varphi_0 + \sum_{n\geq 0}\frac{1}{(2n)!}x^{2n}\partial^{2n+2}\varphi_0\,,\end{aligned} \tag{B.3.10}$$

which vanishes precisely for $\Delta_- = \frac{d-1}{2}$. Similarly,

$$\begin{aligned}&\partial_a\partial^a\varphi_+^{\text{new}} =\\ &\quad\sum_{n\geq 1}\frac{2n(d+2n)-8n^2-4n\Delta_+}{(2n+1)!}x^{2n-2}\partial^{2n}\varphi_1 + \sum_{n\geq 0}\frac{1}{(2n+1)!}x^{2n}\partial^{2n+2}\varphi_1\,,\end{aligned} \tag{B.3.11}$$

which vanishes precisely for $\Delta_+ = \frac{d+1}{2}$.

All in all, we find that the singleton in the doublet representation verifies the same equations as the simpleton

$$\partial_a\partial^a\,\Phi = 0\,,\quad (x^a\partial_a + \varsigma\partial_\varsigma + \Delta)\Phi = 0\,,\quad \Phi \simeq \Phi + \varsigma^2\,\Psi\,. \tag{B.3.12}$$

The only difference between the singleton and the simpleton lies in the bulk realisation. The bulk of AdS space is the locus $\varsigma(x) = \sqrt{1+\frac{x^2}{R^2}}$ for the region $\varsigma > 0$, instead of $\varsigma = 1$ in the flat case.

The AdS metric in 'Minkowski-like coordinates' is the pullback of the metric η_{AB} on the curve $\varsigma(x)$, and reads

$$\mathrm{d}s^2_{\mathrm{AdS}} = -R^2 \mathrm{d}\varsigma(x)^2 + \mathrm{d}x^2 = g_{ab}(x)\,\mathrm{d}x^a\,\mathrm{d}x^b\,, \tag{B.3.13}$$

with

$$g_{ab}(x) = \eta_{ab} - \frac{x_a x_b}{R^2+x^2}\,, \quad g^{ab}(x) = \eta^{ab} + \frac{x^a x^b}{R^2}\,, \quad \det(g) = -\frac{R^2}{R^2+x^2}\,, \tag{B.3.14}$$

which indeed describes a space-time of constant negative curvature with radius R, i.e. $R_{abcd} = -\frac{1}{R^2}(g_{ac}g_{bd} - g_{bc}g_{ad})$.

Note that the determinant of the metric g_{ab} blows up when $x^2 = -R^2$, which corresponds to the boundary of AdS in this coordinate system, and that this metric covers only half of AdS space-time [250], and therefore only half of its conformal boundary. The other half is located at $\varsigma(x) = -\sqrt{1+\frac{x^2}{R^2}}$. When $R \to \infty$, the coordinate singularity at $x^2 = -R^2$ is pushed to infinity and disappears, and one recovers a full copy of Minkowski space.

One can verify that the equations $\partial_a \partial^a \varphi_\pm = 0$ are equivalent to

$$\left[\nabla^2 - \frac{1}{R^2}\Delta_\pm(\Delta_\pm - D + 1)\right]\varphi_\pm = 0\,, \tag{B.3.15}$$

where $\nabla^2\varphi = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,g^{ab}\,\partial_b\varphi\right)$ is the Laplace-Beltrami operator on AdS_D with metric g_{ab}, and we used the homogeneity relation. In this exotic realisation, the bulk singleton is the given of two harmonic scalar fields with weights $\Delta_\pm$.

Note that in (B.3.15), the two values of the scaling dimension $\Delta_\pm$ should not be interpreted as the two possible solutions of the mass-shell equation for a single scalar field of a given mass, but rather to the scaling dimensions of two distinct scalar fields with different values of their mass-squared $R^2\,m_\pm^2 = \Delta_\pm(\Delta_\pm - D + 1)$.

For the first scalar φ_- with weight Δ_-, there are two solutions to the mass-shell condition with scaling dimensions $\Delta = \Delta_- = \frac{d-1}{2}$ and $\Delta = \frac{d+3}{2}$. The homogeneity condition $(x^a\partial_a + \Delta_-)\,\varphi_- = 0$ tells us that only the first one, corresponding to the leading branch in an expansion close to the conformal boundary, is present. For the second scalar φ_+ with weight Δ_+, there is only one solution to the mass-shell equation $\Delta = \Delta_+ = \frac{d+1}{2}$, which is the one imposed by homogeneity $(x^a\partial_a + \Delta_+)\,\varphi_+ = 0$.

To conclude, let us present the actions of Lorentz transformations and translations on the AdS_D singleton represented by the doublet $\boldsymbol{\varphi} \sim \begin{pmatrix}\varphi_- \\ \varphi_+\end{pmatrix}$

$$J_{ab} \sim \begin{pmatrix} 2\,x_{[a}\,\partial_{b]} & 0 \\ 0 & 2\,x_{[a}\,\partial_{b]} \end{pmatrix}\,, \quad P_a \sim \begin{pmatrix} 0 & \frac{1}{R^2}\,(x^2\,\partial_a + x_a) \\ \partial_a & 0 \end{pmatrix}\,. \tag{B.3.16}$$

These expressions satisfy the AdS algebra and one can verify that the ideal corresponding to the AdS_D higher-spin algebra is factored out in this form

$$J_{[ab} \circ J_{cd]}\, \boldsymbol{\varphi} \sim 0\,, \tag{B.3.17a}$$

$$J_{[ab} \circ P_{c]}\, \boldsymbol{\varphi} \sim 0\,, \tag{B.3.17b}$$

$$\tfrac{1}{2} J_{ab} \circ J^{ba}\, \boldsymbol{\varphi} \sim -\tfrac{(D-3)(D-1)}{4}\, \boldsymbol{\varphi}\,, \tag{B.3.17c}$$

$$P_a \circ P^a\, \boldsymbol{\varphi} \sim -\tfrac{1}{R^2}\, \tfrac{D-3}{2}\, \boldsymbol{\varphi}\,, \tag{B.3.17d}$$

$$\big(J^c{}_{(a} \circ J_{b)c} + R^2\, P_{(a} \circ P_{b)}\big)\, \boldsymbol{\varphi} \sim -\tfrac{D-3}{2} \eta_{ab}\, \boldsymbol{\varphi}\,, \tag{B.3.17e}$$

$$\big(J_{ab} \circ P^b + P^b \circ J_{ab}\big)\, \boldsymbol{\varphi} \sim 0\,. \tag{B.3.17f}$$

We conclude that the singleton can be described by a doublet of homogeneous massless fields in AdS, admitting a non-degenerate[3] flat limit.

B.4 Boundary definition of the magnetic simpleton from ambient space

Usually, one finds the boundary value from ambient space by performing the projection along a null direction. Here, we start instead from boundary theory, and verified that the bulk definition has the same behaviour when transported to the boundary. If one wants to define the boundary magnetic simpleton following the usual procedure, we need a bit more gymnastics to recover our definition.

Let us perform the null reduction of eqs. (3.3.1) along $\varsigma^2 = 0$, given by identifying points up to rescaling $(x^a, \varsigma) \sim \lambda\,(x^a, \varsigma)$ by a positive number λ. The coordinate ς can be factored out by using a 2×2 matrix representation like before (and not forgetting to shift the scaling dimension by one unit in the ς direction), while the coordinates $x^a \in \mathbb{R}^{1,d+1}$ become ambient space coordinates for the celestial sphere S^d.

However, in addition to the homogeneity condition, the equation of motion that we need to pull back is not the quotienting condition $\Phi_\pm(x) \simeq \Phi_\pm(x) + x^2\, \Psi_\pm(x)$ that would allow to reduce $\Phi_\pm(x)$ to two arbitrary fields on the boundary, like in section 3.2.2. Instead, we have $\partial^2\, \Phi_\pm = 0$ in ambient space. This contrasts sharply with the expected answer, which is the given of two arbitrary fields on the celestial sphere. However, we can perform a simple transformation that will allow us to make contact with this result.

[3]Here, we refer to the usual lore that the flat limit of a singleton is a unitary, irreducible representation of the Poincaré group in which translations are realised trivially [71]. What we propose instead is a non-unitary, indecomposable representation in which translations are realised non-trivially.

By writing a Taylor expansion of $\Phi_\pm$ near $x^2 = 0$

$$\Phi_- = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}\Phi_-^{(n)}\,, \quad \Phi_+ = \sum_{n\geq 0} \frac{x^{2n}}{(2n+1)!}\Phi_+^{(n)}\,, \tag{B.4.1}$$

where each $\Phi_\pm^{(n)}$ has scaling dimension $\Delta_\pm + 2n$ and is free of x^2 terms (the latter can always be reabsorbed in the next n's). The fact that $\Phi_\pm$ verify the wave equation $\partial_a\partial^a\Phi_\pm = 0$ allows us to read a set of differential equations order by order in x^2

$$\Phi_\pm^{(n+1)} = \partial^2\,\Phi_\pm^{(n)}\,, \tag{B.4.2}$$

for $n \geq 0$, which means that all $\Phi_\pm^{(n)}$ with $n \geq 1$ are actually determined in terms of $\Phi_\pm^{(0)}$. Each $\Phi_\pm^{(n)}$ being itself free from x^2 terms, we can perform their pullback to the celestial sphere $S^d \ni \mathbf{x}$ and define an infinite collection of conformal scalars $\phi_\pm^{(n)}(\mathbf{x})$, all determined as a function of the first one $\phi_\pm^{(0)}(\mathbf{x})$ by differential relations. This matches with the expected on-shell boundary description of the simpleton, being parameterised by two arbitrary fields of the celestial sphere, see eq. (3.3.4).

B.5 Higher symmetries of the magnetic simpleton

The higher symmetries of the magnetic Carrollian scalar are given by the differential operators

$$\boldsymbol{D} = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}\,, \tag{B.5.1}$$

that commute weakly with the kinetic operator $\boldsymbol{K}$

$$\boldsymbol{K}\circ\boldsymbol{D} = \boldsymbol{D}^\dagger\circ\boldsymbol{K} \quad \Leftrightarrow \quad \begin{cases} \hat{\nabla}^2\circ M - \partial_u\circ P = M^\dagger\circ\hat{\nabla}^2 + P^\dagger\circ\partial_u \\ \hat{\nabla}^2\circ N - \partial_u\circ Q = -M^\dagger\circ\partial_u \\ \partial_u\circ M = N^\dagger\circ\hat{\nabla}^2 + Q^\dagger\circ\partial_u \\ \partial_u\circ N = -N^\dagger\circ\partial_u \end{cases}\,, \tag{B.5.2}$$

while quotienting by trivial isometries of the form

$$\boldsymbol{D} = \boldsymbol{D}'\circ\boldsymbol{K} = \begin{pmatrix} M'\circ\hat{\nabla}^2 + N'\circ\partial_u & -M'\circ\partial_u \\ P'\circ\hat{\nabla}^2 + Q'\circ\partial_u & -P'\circ\partial_u \end{pmatrix}\,. \tag{B.5.3}$$

Using the quotienting condition, we can always look for M, N, P and Q independent[4] of ∂_u. Indeed, if M or P possesses a part containing at least one ∂_u, we

[4]We may also choose a representative of $\boldsymbol{D}$ including powers of ∂_u, which will be better suited to represent super-translations, but it will prove to be more convenient to choose a representation where the differential operators are independent of ∂_u to compute symmetries.

may cancel it by using the N' or Q' symmetries, while if N or Q possesses a part written as $\tilde{N} \circ \partial_u$ or $\tilde{Q} \circ \partial_u$, we may push it to the column of M or P where it is transformed into $\tilde{N} \circ \hat{\nabla}^2$ or $\tilde{Q} \circ \hat{\nabla}^2$. We may then apply the same argument if $\tilde{N}$ or $\tilde{Q}$ contains a part proportional to ∂_u. Indeed, with this choice, the equations are equivalent to

$$\dot{N} = 0\,, \qquad N^\dagger = -N\,, \tag{B.5.4a}$$
$$\dot{Q} = \hat{\nabla}^2 \circ N\,, \qquad M^\dagger = Q\,, \tag{B.5.4b}$$
$$\dot{P} = \hat{\nabla}^2 \circ M - M^\dagger \circ \hat{\nabla}^2\,, \qquad P^\dagger = -P\,, \tag{B.5.4c}$$

and so the differential operator $\boldsymbol{D}$ reads

$$M = L_{-1} + i\,L_{+1} - i\,u\,K_{+1} \circ \hat{\nabla}^2\,, \tag{B.5.5a}$$
$$N = i\,K_{+1}\,, \tag{B.5.5b}$$
$$P = i\,K_{-1} + u\left[\hat{\nabla}^2, L_{-1}\right] + i\,u\left\{\hat{\nabla}^2, L_{+1}\right\} - i\,u^2\,\hat{\nabla}^2 \circ K_{+1} \circ \hat{\nabla}^2\,, \tag{B.5.5c}$$
$$Q = L_{-1} - i\,L_{+1} + i\,u\,\hat{\nabla}^2 \circ K_{+1}\,, \tag{B.5.5d}$$

parameterised by four arbitrary Hermitian operators on the celestial sphere, denoted here by $K_{\pm 1}$ and $L_{\pm 1}$, as was the case for the electric Carrollian scalar.

Moreover, they satisfy the same algebra $\mathcal{H}(S^d) \otimes \mathfrak{gl}(2,\mathbb{R})$ as for the electric scalar, which is not surprising since the space of solutions of the electric and magnetic scalars are equivalent on-shell. Consider a higher symmetry of the magnetic simpleton, that is a differential operator

$$\boldsymbol{D}\left(L_{-1}, L_{+1}, K_{-1}, K_{+1}\right)\,, \tag{B.5.6}$$

in the parameterisation of eqs. (B.5.5). The non-zero Lie brackets (time the imaginary unit) verify

$$i[\boldsymbol{D}(L,0,0,0)\,, \boldsymbol{D}(L',0,0,0)] = \boldsymbol{D}(i[L,L'],0,0,0)\,, \tag{B.5.7a}$$
$$i[\boldsymbol{D}(L,0,0,0)\,, \boldsymbol{D}(0,L',0,0)] = \boldsymbol{D}(0,i[L,L'],0,0)\,, \tag{B.5.7b}$$
$$i[\boldsymbol{D}(L,0,0,0)\,, \boldsymbol{D}(0,0,K',0)] = \boldsymbol{D}(0,0,i[L,K'],0)\,, \tag{B.5.7c}$$
$$i[\boldsymbol{D}(L,0,0,0)\,, \boldsymbol{D}(0,0,0,K')] = \boldsymbol{D}(0,0,0,i[L,K'])\,, \tag{B.5.7d}$$
$$i[\boldsymbol{D}(0,L,0,0)\,, \boldsymbol{D}(0,L',0,0)] = \boldsymbol{D}(-i[L,L'],0,0,0)\,, \tag{B.5.7e}$$
$$i[\boldsymbol{D}(0,L,0,0)\,, \boldsymbol{D}(0,0,K',0)] = \boldsymbol{D}(0,0,\{L,K'\},0)\,, \tag{B.5.7f}$$
$$i[\boldsymbol{D}(0,L,0,0)\,, \boldsymbol{D}(0,0,0,K')] = \boldsymbol{D}(0,0,0,-\{L,K'\})\,, \tag{B.5.7g}$$
$$i[\boldsymbol{D}(0,0,K,0)\,, \boldsymbol{D}(0,0,0,K')] = \boldsymbol{D}\left(\frac{-i}{2}[K,K'], \frac{1}{2}\{K,K'\},0,0\right)\,. \tag{B.5.7h}$$

are also the ones of the algebra $\mathcal{H}\left(S^d\right) \otimes \mathfrak{gl}(2,\mathbb{R})$ of eq. (3.2.9).

B.6 Expressions in various coordinate systems

In retarded Bondi coordinates

Alternatively, one can find a closed form expression that reproduces the expansion (3.3.4), solving the equations (3.3.3). In Bondi coordinates $(r, u, \mathbf{x})$ they read

$$\varphi_-(r,u,\mathbf{x}) = r^{-\Delta_-}\,\frac{[f(r,u)+g(r,u)]^{\kappa}+[f(r,u)+g(r,u)]^{-\kappa}}{2}\,\phi_-(\mathbf{x})\,, \tag{B.6.1a}$$

$$\varphi_+(r,u,\mathbf{x}) = r^{-\Delta_+}\,\frac{[f(r,u)+g(r,u)]^{\kappa}-[f(r,u)+g(r,u)]^{-\kappa}}{2\,g(r,u)\,\kappa}\,\phi_+(\mathbf{x})\,, \tag{B.6.1b}$$

with

$$f(r,u) := 1+\frac{u}{r}\,,\quad g(r,u) := \sqrt{f(r,u)^2-1}\,,\quad \kappa := \sqrt{\Delta_-^2-\nabla^2}\,. \tag{B.6.2}$$

Although one could worry about the possible non-analytic dependence in the differential operator ∇^2 or the retarded time u, the $1/r$ expansion yields exactly the expressions (3.3.4), which is well-defined in a $1/r$ expansion.

Near spatial infinity

Let us pick instead hyperbolic coordinates $(\eta, s, \mathbf{x})$, defined for $|t| \leq r$ by

$$\eta = \sqrt{r^2-t^2}\,,\quad s = \frac{t}{r}\,. \tag{B.6.3}$$

We have

$$\varphi_-(\eta,s,\mathbf{x}) = \frac{\left(\sqrt{1-s^2}\right)^{\Delta_-}}{\eta^{\Delta_-}}\,\frac{\left[s+\sqrt{s^2-1}\right]^{\kappa}+\left[s+\sqrt{s^2-1}\right]^{-\kappa}}{2}\,\phi_-(\mathbf{x})\,, \tag{B.6.4a}$$

$$\varphi_+(\eta,s,\mathbf{x}) = \frac{i\left(\sqrt{1-s^2}\right)^{\Delta_-}}{\eta^{\Delta_+}}\,\frac{\left[s+\sqrt{s^2-1}\right]^{\kappa}+\left[s+\sqrt{s^2-1}\right]^{-\kappa}}{2\,\kappa}\,\phi_+(\mathbf{x})\,. \tag{B.6.4b}$$

Note that when moving along the η-coordinate the fields are simply rescaled. In the limit $\eta \to \infty$, contrary to the limit $r \to \infty$ in Bondi coordinates, the boundary fields have a (fixed but non-trivial) s-dependence, on top of the arbitrary $\mathbf{x}$-dependence. This is because the Euler operator $x^a\,\partial_a$ takes the expression $r\,\partial_r + u\,\partial_u$ in Bondi coordinates, but $\eta\,\partial_\eta$ in hyperbolic coordinates.

Near i^0, corresponding to the limit $\eta \to \infty$ and $s \ll 1$, we have

$$\eta^{\Delta_-}\,\varphi_-(\eta,s,\mathbf{x}) \sim \left[\cos\left(\tfrac{\pi}{2}\kappa\right)+\kappa\,\sin\left(\tfrac{\pi}{2}\kappa\right)s+\mathcal{O}(s^2)\right]\phi_-(\mathbf{x})\,, \tag{B.6.5a}$$

$$\eta^{\Delta_+}\,\varphi_+(\eta,s,\mathbf{x}) \sim \left[-\kappa^{-1}\,\sin\left(\tfrac{\pi}{2}\kappa\right)+\cos\left(\tfrac{\pi}{2}\kappa\right)s+\mathcal{O}(s^2)\right]\phi_+(\mathbf{x})\,, \tag{B.6.5b}$$

where the pattern of trigonometric functions repeats at every group of two orders in the small-s expansion.

The functions φ_+ and φ_- do not have a definite parity under antipodal matching $(s, \mathbf{x}) \to (-s, -\mathbf{x})$ in even bulk dimension, but they do in odd bulk dimension. This is due to the scaling dimension Δ_- being a half-integer in even dimensions and an integer in odd ones. For instance for $d = 3$, the eigenvalue of κ on the subspace of spherical harmonics with principal number ℓ is $\ell + 1$, so that

$$\eta^{\Delta_-}\, \varphi_-(\eta, s, \mathbf{x}) \sim U_{\text{odd}}(\mathbf{x}) + U_{\text{even}}(\mathbf{x})\, s + \mathcal{O}(s^2)\,, \tag{B.6.6a}$$

$$\eta^{\Delta_+}\, \varphi_+(\eta, s, \mathbf{x}) \sim V_{\text{even}}(\mathbf{x}) + V_{\text{odd}}(\mathbf{x})\, s + \mathcal{O}(s^2)\,, \tag{B.6.6b}$$

and the pattern repeats at every order in s. Therefore, $\varphi_-(\eta, s, \mathbf{x})$ is odd under antipodal matching while $\varphi_+(\eta, s, \mathbf{x})$ is even.

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