TOPOLOGICAL DIFFERENTIAL FIELDS AND DIMENSION FUNCTIONS.

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Abstract. We construct a fibered dimension function in some topological differential fields.

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1. Introduction

Let $\mathcal{K}$ be a topological $\mathcal{L}$-field. We are going to consider two kinds of expansions $< K, D >$, where D is a derivation on K. One where the operator D has a priori no interactions with the topology on K and the other one where D induces a continuous map from K to K.

In the first case, we will work in the setting we developed in [6]. Assume $\mathcal{K}$ is a model of a universal $\mathcal{L}$-theory T which has a model completion T_c. Under certain hypothesis on T_c, we showed that the expansion of T to the $\mathcal{L} \cup \{D\}$-theory T_D consisting of T together with the axioms expressing that D is a derivation, admits a model-completion $T^*_{c,D}$ which we axiomatized. To the theory $T_D \cup T_c$, we added a scheme of axioms (DL), which expressess that each differential polynomial has a zero close to a zero of its associated algebraic polynomial. This scheme (DL) is related to the axiom scheme (UC) introduced by M. Tressl in the framework of large fields, and also to the axiomatization, due to M. Singer, of the theory of closed ordered differential fields (CODF).

We further observed that whenever T_c has the non-independence property (NIP), then $T^*_{c,D}$ has NIP. This implies for instance that the definable subsets in a model of $T^*_{c,D}$ have a VC-dimension.

Here, using former results of L. van den Dries, we will show the existence of a fibered dimension function in models of $T^*_{c,D}$. In [17], L. van den Dries showed, under a certain condition: *the algebraic (or differential) boundedness property*, the existence of a fibered dimension function on definable sets.

In that framework, we show on one hand that an analog of the algebraic boundedness property, the equational boundedness property, holds in the presence of (DL) and on the other hand that the equational boundedness property plus a quantifier elimination result suffice to show the existence of a fibered dimension function on the definable subsets.

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This result gives for the theory of closed ordered fields (CODF) another proof that definable subsets in models of that theory can be endowed with a fibered dimension function (see [1]). There the authors first proved a cell decomposition theorem, introducing the notion of δ-cells and associated with it a fibered dimension function (Theorem 5.19 [1]).

In the second case, we will show that a class of differential valued fields, the D-henselian fields (see section 4) introduced by T. Scanlon ([15]), has the equational boundedness property. We first show that this class of differential topological fields can be equipped with a topological system, as introduced by L. van den Dries ([17]). Then we will use the result of T. Scanlon that these D-henselian fields admit quantifier elimination ([15], [16]), revisited by N. Guzy ([5]) who dealt with a one-sorted language, more convenient to our purpose. Note that these D-valued fields do have the NIP property.

2. Fibered dimension function and equationnal boundedness.

We will make the assumption throughout the paper that $\mathcal{L}$ is a first-order language containing the language of rings $\mathcal{L}_{\text{rings}} := \{+, -, ., 0, 1\}$.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure. We will denote by $Def(\mathcal{A})$ the set of (all) definable subsets (with parameters) in the cartesian products A^n of A, $n \in \omega$. For $S \subseteq A^m$ and σ a permutation of $\{1, \cdots, m\}$, we denote by S^σ the set $\{(a_{\sigma(1)}, \cdots, a_{\sigma(m)}) \in A^m : (a_1, \cdots, a_m) \in S\}$.

L. van den Dries introduced a dimension function on $Def(A)$ (and more generally on a Tarski system [17]) satisfying the axioms below. Recently it was revisited by C. Ealy and A. Onshuus who call such dimension, a fibered dimension function ([4] section 6).

Definition 2.1. ([17], [4]) A *fibered dimension function* $d : Def(\mathcal{A}) \to \{-\infty\} \cup \mathcal{O}n$ satisfies the following axioms. Let $S, S_1, S_2, T \in Def(\mathcal{A})$.

1. $d(S) = -\infty$ iff $S = \emptyset$, $d(\{a\}) = 0$, for each $a \in A$.
2. $d(S_1 \cup S_2) = max\{d(S_1), d(S_2)\}$.
3. Let $S \in A^m$, then for any permutation σ of $\{1, \cdots, m\}$, we have $d(S^\sigma) = d(S)$.
4. Let $S \subseteq A^{n+m}$, $S_{\bar{x}} := \{\bar{y} \in A^m : (\bar{x}, \bar{y}) \in S\}$ and $S(\gamma) := \{\bar{x} \in A^n : d(S_{\bar{x}}) = \gamma\}$, $\gamma \in \mathcal{O}n$. Then, $S(\gamma) \in Def(\mathcal{A})$ and $d(\{(\bar{x}, \bar{y}) \in S : \bar{x} \in S(\gamma)\} = d(S(\gamma)) + \gamma$.

If one adds that $d(A) = 1$, one obtains a dimension function taking its values in $\{-\infty\} \cup \mathbb{N}$ and one can relax the condition 4 by asking it only for $m = 1$ (see [17] Proposition 1.4).

In [4], in the setting of rosy theories, they connected the thorn rank and van den Dries fibered dimension, under the assumption that the dimension of a set is zero if and if it is finite (see Proposition 6.7 and Corollary 6.9 in [4]).

Note that there are other dimension functions on definable sets which are in general ordinal valued and often not fibered. For instance, since the theory of differentially closed fields is ω-stable, every definable subset in a model of DCF_0 has a Morley rank. However, it has been shown by E. Hrushovski and T. Scanlon that there are definable sets where the Lascar rank and Morley rank do not coincide (Corollary 2.8 in [7]). (But it does coincide for definable sets without parameters (see [7] Question 2.9.))

In ([17]), L. van den Dries first showed for *algebraically bounded* structures that their algebraic dimension is a fibered dimension function. Then, he considered expansions with a derivation and adapted his algebraic setting to a differential one (see 2.25 in [17]).

More generally we will consider terms in a language $\mathcal{L}$ containing the ring language and make analogous definitions, provided that non trivial terms (modulo the theory) induce non-trivial functions on models of that theory. This is why we will use the notation *t-dimension*.

Let $\mathcal{M}$ be a $\mathcal{L}$-structure, model of a $\mathcal{L}$-theory T extending the theory of domains i.e. commutative rings without zero-divisors. We will assume that M is infinite.

Let $\mathcal{L}_M$ be the language $\mathcal{L}$ expanded by new constants for elements of M.

Definition 2.2. Let $\mathcal{M} \models T$ and let $\mathcal{M}\{x_1, \cdots, x_n\}$ be the set of all $\mathcal{L}_M$-terms in free variables $x_1, \cdots, x_n$ up to equivalence $\sim$ in T. (Namely, $t_1 \sim t_2$ if $T \models \forall \bar{x}\, t_1(\bar{x}) = t_2(\bar{x})$.) We will call the following hypothesis on $\mathcal{M}$ *assumption* (C): for any natural number n and any term $t(x_1, \cdots, x_n)$ if $t \not\sim 0$, then the corresponding function t on M^n is non trivial.

Definition 2.3. ([17]) Let $\mathcal{M}$ be a model of T, let S be a subset of M^n and denote by $M\{S\}$ the set of functions on S induced by $\mathcal{M}\{x_1, \cdots, x_n\}$.

(1) The elements $f_1, \ldots, f_k \in M\{S\}$ are called t-independent over M if for any $f \in M\{y_1, \ldots, y_k\}$ with $f \neq 0$ we have $f(f_1, \ldots, f_k) \neq 0$, as an element of $M\{S\}$.
(2) The t-dimension of S, denoted $t\text{-dim}_M(S)$ is the maximal number of functions in $M\{S\}$ that are t-independent over M.
(3) By convention we put $t\text{-dim}(\emptyset) = -\infty$.
(4) We will say that a (definable) subset S of M^n is *small* if t-$dim(S) = 0$. The small subsets of M^n form an ideal in the power set $\mathcal{P}(M^n)$.

Remark 2.4. (1) Let $\sigma \in Aut(\mathcal{M})$ and assume that $\sigma(S) = S$, then t-$dim(S)$=t-$dim(S^\sigma)$.
(2) Assumption (C) implies that t-$dim(M^n) \geq n$. (Take the projections functions.)
(3) Let $S \subseteq M$, then for any $f \in M\{S\} - \{0\}$, x and $f(x)$ are not t-independent. (Consider $t(y_1, y_2) := f(y_1) - y_2$; since $t \neq 0$ and $t(x, f(x)) = 0$, x and $f(x)$ are not t-independent.)

The dimension t-dim satisfies the first three axioms of a fibered dimension function (see Definition 3.1). We will introduce additional hypothesis in order to obtain a fibered dimension.

First, following [17], we introduce another notion of dimension via a notion of independence which generalizes the usual notions of transcendence degree and algebraic independence (see Section 2 of [17]).

Definition 2.5. Let $\mathcal{M}^*$ be a $|M|^+$-saturated elementary extension of $\mathcal{M}$ and let $\bar{a} := (a_1, \cdots, a_n) \in (M^*)^n$.

Let cl be the following operator on M^*: $b \in cl(\bar{a})$ if there exists a non zero term $t(x_1, \cdots, x_k) \in \mathcal{M}\{x_1, \cdots, x_k\}$ such that $t(b, \bar{a}) = 0$ and $t(x, \bar{a})$ non-trivial on M^*. More generally let $A \subseteq M^*$, then $b \in cl(A)$ if there exists a k-tuple $\bar{c} \subseteq A$ and a non zero term $t(x_1, \cdots, x_k) \in \mathcal{M}\{x_1, \cdots, x_k\}$ such that $t(b, \bar{c}) = 0$ and $t(x, \bar{c})$ non-trivial on M^*.

We say that cl is a *closure* operator (or transitive) whenever $cl(cl) = cl$. A subset $A \subseteq M^*$ is *independent* if $\forall a \in A$, $a \notin cl(A - \{a\})$.

Note that this operator has the following properties.

(1) $A \subseteq cl(A)$,
(2) if $A \subseteq B$, then $cl(A) \subseteq cl(B)$.
(3) If $a \in cl(A)$, then there exists a finite subset A_0 of A such that $a \in cl(A_0)$.

Remark 2.6. In the case where $\mathcal{L} = \mathcal{L}_{rings}$ or $\mathcal{L} = \mathcal{L}_{rings} \cup \{D\}$, where D is a unary symbol for a derivation, then cl is a closure operator.

Lemma 2.7. *Let $b_1, \cdots, b_k \in M^*$. We have the following equivalence: $b_1, \cdots, b_k$ are independent over M iff for any non zero term $t(x_1, \cdots, x_k) \in \mathcal{M}\{x_1, \cdots, x_k\}$, $\mathcal{M}^* \models t(b_1, \cdots, b_k) \neq 0$.*

Proof: Set $\bar{b} := (b_1, \cdots, b_k)$.

($\leftarrow$) This is immediate.

($\rightarrow$) By induction on k, we will show that this contradicts the fact that $\bar{b}$ is independent.

If $k = 1$, then, since t is non-trivial on M, it implies that $b_1 \in cl(\emptyset)$, contradiction.

So, assume that $k \geq 2$ and write $\bar{b} := (\tilde{b}, b_k)$. Let t be a non-zero term and assume that $t(\tilde{b}, b_k) = 0$. Since $b_k \notin cl(\tilde{b})$, this implies that $t(\tilde{b}, x)$ is trivial on M^*. Since t is non zero, there exists a tuple $(m_1, \cdots, m_k) \subseteq M$ such that $t(\bar{m}) \neq 0$. But $t(\tilde{b}, m_k) = 0$ which contradicts the induction hypothesis. (A subtuple of an independent tuple is independent too.) □

Lemma 2.8. *Assume that cl is a closure operator, then $(\mathcal{M}^*, cl)$ is a pregeometry.*

Proof: Let us check the axioms of a pregeometry (see for instance Definition 8.1.1 in [10]). In addition to properties (1) up to (3) stated above, it remains to check that cl has the exchange property.

Let $u \in cl(A, b) - cl(A)$ and let us show that $b \in cl(A, u)$. By the finite character of cl, we may assume that $u \in cl(\bar{a}, b)$ with $\bar{a} \subseteq A$ a finite tuple of minimal length. Moreover, since cl is a closure operator, we may assume that $\bar{a}$ is independent.

So, there exists a non zero term $t(x_0, \cdots, x_k, x_{k+1}) \in \mathcal{M}\{x_0, \cdots, x_{k+1}\}$ such that $t(u, \bar{a}, b) = 0$ and $t(x, \bar{a}, b)$ non-trivial on M^*. Either $t(u, \bar{a}, y)$ is non-trivial on M^* and so $b \in cl(A, u)$, or $\forall n \in M^*$ $t(u, \bar{a}, n) = 0$. Since $u \notin cl(A)$, for all $m \in M$, $t(x, \bar{a}, m)$ is trivial on M^*, in particular for all $m_1, m_2 \in M$, $t(m_1, \bar{a}, m_2) = 0$. Moreover, since t is a non-zero term, $\bar{a}$ has non-zero length.

By the preceding Lemma, this contradicts contradicts the fact that $\bar{a}$ is independent over M. □

So, under the assumption of the Lemma, we can associate with cl a notion of dimension ([8] 3.6), that we will denote by dim.

First we localize this notion of closure as follows. Let A be a subset of M^*, then $cl_A(\bar{a}) := cl(\bar{a} \cup A)$. Relative to this localized closure operator cl_A will correspond a notion of independence over A.

Definition 2.9. Let $\mathcal{M}^*$ be $|M|^+$-saturated elementary extension of $\mathcal{M}$ and let $\bar{a} := (a_1, \cdots, a_n) \in (M^*)^n$ and B a subset of $(M^*)^n$.

Then $dim(\bar{a}) := dim(M\langle\bar{a}\rangle)$ is the maximal number of elements $b_1, \cdots, b_k \in M\langle\bar{a}\rangle$ which are independent (relative to cl) and $dim(M\langle B\rangle) = max\{dim(M\langle\bar{a}\rangle) : \ \bar{a} \in B\}$.

Let $A \subseteq M^*$. We will denote by $dim(\cdot/A)$ the dimension relative to cl_A.

An element $\bar{a} \in B$ is called *generic* if $dim(\bar{a})$=$dim(B)$ and $\bar{a}$ is generic over A if $dim(\bar{a}/A) = dim(B/A)$.

Remark 2.10. We have the usual properties of the dimension. Let $\bar{a},\ \bar{b} \in (M^*)^n$ and A a subset of M^*. Namely that $dim(\bar{a}\bar{b}/A) = dim(\bar{a}/A \cup \bar{b}) + dim(\bar{b}/A)$.

From that relation, one deduces that $dim(\bar{a}/A) = dim(\bar{a}/A \cup \bar{b})$ iff $dim(\bar{b}/A) = dim(\bar{b}/A \cup \bar{a})$. Therefore $\bar{a}$ is generic over $A \cup \bar{b}$ iff $\bar{b}$ is generic over $A \cup \bar{a}$.

Let $\mathcal{L}_S$ be the expansion of the language $\mathcal{L}$ by a new unary predicate S; this predicate will be interpreted by a subset S of M^n.

Lemma 2.11. *Let $\langle\mathcal{M}^*, S^*\rangle$ be a $|M|^+$-saturated elementary $\mathcal{L}_S$-extension of $\langle\mathcal{M}, S\rangle$, where $S \subseteq M^n$ and assume that cl is a closure operator. Then,*

(1) *t-$dim_M(S)$=$dim(M\langle S^*\rangle)$.*
(2) *t-$dim_M(M) = 1$.*

Proof: (1) Suppose that t-$\dim_M(S) \geq m$, then there exist $f_1, \cdots, f_m \in M\{S\}-\{0\}$, $t(f_1, \cdots, f_m) \neq 0$ for every non zero term $t(x_1, \cdots, x_m) \in M\{x_1, \cdots, x_m\}$. For each term choose an element $s(t) \in S$ such that $t(f_1(s(t)), \cdots, f_m(s(t))) \neq 0$. Since M is a domain, this also holds for any finite number of terms by taking their product. Then by saturation there exists an element $u \in S^*$ such that the m-tuple $(f_1(u), \cdots, f_m(u))$ of elements in S^* has the property that for every non zero term $t(x_1, \cdots, x_m) \in M\{x_1, \cdots, x_m\}$, we have $t(f_1(u), \cdots, f_m(u)) \neq 0$. So, $dim(M\langle f_1(u), \cdots, f_m(u)\rangle) \geq m$. Therefore $dim(M\langle S^*\rangle) \geq m$.

Conversely, suppose that $dim(M\langle S^*\rangle) \geq m$, then there exists an element $\bar{u} \in S^*$ with $\bar{u} := (u_1, \cdots u_n)$ such that $M\langle\bar{u}\rangle$ contains m independent elements. Namely, there exist $f_1(\bar{u}), \cdots, f_m(\bar{u})$ independent over M, where $f_1, \cdots f_m$ are $\mathcal{L}_M$-terms. Namely, for every non zero term $t \in M\{x_1, \cdots, x_m\}$, we have $t(f_1(\bar{u}), \cdots, f_m(\bar{u})) \neq 0$.

So, $\mathcal{M}^* \models \exists x_1 \cdots \exists x_n \bigwedge_{i=1}^n S^*((x_1, \cdots, x_n)) \,\&\, t(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)) \neq 0$. Since $\mathcal{M}^*$ is an elementary extension of $\mathcal{M}$, we get that for every non zero $\mathcal{L}_M$-term $t(x_1, \cdots, x_m)$, there exists an element in S such that in M, the function $t(f_1, \cdots, f_m)$ is non zero, which means that t-$\dim_M(S) \geq m$.

(2) We will show that $dim(M\langle M^*\rangle) = 1$. By Remark 3.4 (3), it suffices to show that t-$dim(M) < 2$. So, let $m^* \in M^*$ and let $t_1(m^*)$ belong to the $\mathcal{L}$-substructure $M\langle m^*\rangle$, where $t_1(x) \in \mathcal{M}\{x\} - \{x\}$. Then, m^* and $t_1(m^*)$ are dependent since if $t(x, y) := t_1(x) - y$, then $t(m^*, t_1(m^*)) = 0$ and $t(x, y) \neq 0$. Since cl satisfies the exchange and $cl(cl) = cl$, we have that any two elements of $M\langle m^*\rangle$ are dependent. □

Definition 2.12. An $\mathcal{L}$-structure $\mathcal{M}$ is called *equationally bounded* if for each definable set $S \in M^{m+1}$ such that for every $\bar{a} \in M^m$, $S_{\bar{a}}$ is small, there exist finitely many terms $f_1, \dots, f_r \in M\{x_1, \dots, x_m, y\}$ such that for every $\bar{a} \in M^m$, there exists $1 \le i \le r$ with $f_i(\bar{a}, y) \neq 0$ and $S_{\bar{a}} \subseteq \{b \in M : \ f_i(\bar{a}, b) = 0\}$.

In the particular case where the reduct of $\mathcal{L}$ to its non relational symbols is the language of rings, an equationally bounded $\mathcal{L}$-structure is called *algebraically bounded* and in that case small subsets of M are just the finite subsets. When the reduct of $\mathcal{L}$ to its non relational symbols is the language of differential rings, an equationally bounded $\mathcal{L}$-structure is called *differentially bounded.* (See [17]).

Proposition 2.13. *Let $\mathcal{M}$ be a model of T satisfying assumption (C) and assume that the operator cl is a closure operator. Then, if $\mathcal{M}$ is an equationally bounded $\mathcal{L}$-structure, the function t-dim on $Def(M)$, defines a fibered dimension function with values in $\{-\infty\} \cup \omega$.*

Proof. This is analogous to the proof of 2.7 in [17].

The first three axioms are easily checked and t-$dim(M) = 1$ (see Remark 2). So to check the fourth one, it suffices to consider the case of subsets included in M^{n+1} (namely the case where $m = 1$).

Let $S \subseteq M^{n+1}$ be definable (with parameters) in $\mathcal{M}$. For $i \in \{0,\ 1\}$, let $S(i) := \{\bar{x} \in M^n : t - dim_M(S_{\bar{x}}) = i\}$.

Since $\mathcal{M}$ is equationally bounded, there exist non-trivial terms $f_1, \dots, f_k$ in $M\{x_1, \dots, x_n, y\}$ such that if $S_{\bar{x}}$ is small ($\bar{x} \in M^n$) then $S_{\bar{x}} \subseteq \{y \in M : \ f_i(\bar{x}, y) = 0\}$ for some $i = i(\bar{x}) \in \{1, \dots, k\}$ with $f_i(\bar{x}, Y) \neq 0$. Hence we have that

$$S(0) = \bigcup_{i=1}^{k} \{\bar{x} \in M^n : \ \emptyset \neq S_{\bar{x}}\} \subseteq \{y \in M : \ f_i(\bar{x}, y) = 0 \text{ with } f_i(\bar{x}, Y) \neq 0\}\}.$$

Therefore we deduce easily that $S(0)$ and $S(1)$ are definable (indeed $\bar{x} \in S(0)$ iff $\exists y \ (\bar{x}, y) \in S \ \& \ \bigvee_i \ f_i(\bar{x}, y) = 0$ and $S(1) = M^n - S(0)$. It remains to show that

t-$dim_M\{(\bar{x}, y) \in S : \ \bar{x} \in S(0)\}$=t-$dim_M(S(0))$,

t-$dim_M\{(\bar{x}, y) \in S : \ \bar{x} \in S(1)\}$=t-$dim_M(S(1)) + 1$.

Now we use Lemma 2.11. Let us consider an elementary $|M|^+$-saturated extension $\langle M^*, S^* \rangle$ of $\langle M, S \rangle$. Let $(\bar{a}, b) \in S^*$.

If $\bar{a} \in S(0)^*$, then $f_i(\bar{a}, b) = 0$ for some i with $f_i(\bar{a}, y) \neq 0$. So $b \in cl(\bar{a})$ and so $\dim(M\langle \bar{a}, b \rangle)$ =$\dim(M\langle \bar{a} \rangle)$.

If $\bar{a} \in S(1)^*$, then $S^*_{\bar{a}}$ is not small and so by equationnal boundedness, for any finite subset of non-zero terms $t_1(\bar{x}, y), \cdots, t_k(\bar{x}, y)$, there exists an element $c \in S^*_{\bar{a}}$ such that $\bigwedge_{i=1}^k t_i(\bar{a}, c) \neq 0$. Therefore, by saturation of M^*, there exists $b \in S^*_{\bar{a}}$ such that $b \notin cl(\bar{a})$. So, $\dim(M\langle \bar{a}, b \rangle)$ =$\dim(M\langle \bar{a} \rangle) + 1$. □

3. Topological differential fields and dimension functions.

We will now apply the results of the previous section to topological structures. We will deal with two kinds of topological structures.

In [17], L. van den Dries defined *a topological system* on $\mathcal{M}$ (see [17] section 2, 2.11), as follows. It consists of a topology τ_n on each M^n, $n \ge 1$ such that the $\mathcal{L}_M$-terms induce continuous maps, the singletons $\{m\}$, $m \in M$, are closed and certain subsets of the relations and their complements are open. Namely, for each n-ary relation R and any subsequence $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq k \leq n$, both sets:

$\{(a_{i_1}, \cdots, a_{i_k}) \in M^k : \mathcal{M} \models R(\frown a_{i_1}^\frown \cdots \frown a_{i_k}^\frown) \;\&\; a_{i_1} \neq 0 \;\&\cdots\&a_{i_k} \neq 0\}$,

$\{(a_{i_1}, \cdots, a_{i_k}) \in M^k : \mathcal{M} \models \neg R(\frown a_{i_1}^\frown \cdots \frown a_{i_k}^\frown) \;\&\; a_{i_1} \neq 0 \;\&\cdots\&a_{i_k} \neq 0\}$

are open in M^k, where $(\frown a_{i_1}^\frown \cdots \frown a_{i_k}^\frown)$ is the element of M^n, whose $i_1^{th} \cdots, i_k^{th}$ coordinates are $a_{i_1}, \cdots, a_{i_k}$ and the other coordinates are zero.

In [6], we defined a *topological $\mathcal{L}$-structure* $\mathcal{M} := \langle M, \tau \rangle$ to be a first-order $\mathcal{L}$-structure with a Hausdorff topology τ on M such that every n-ary function symbol f of $\mathcal{L}$ is interpreted by a continuous function M^n to M, and every m-ary relation symbol R of $\mathcal{L}$ and its complement is interpreted by the union of an open subset of M^m and an algebraic set (M^n and M^m are endowed with the product topology). We will denote by $\mathcal{V}$ (respectively $\mathcal{V}_n$) a basis of neighbourhoods of 0 (respectively of $\bar{0}$ in M^n).

Let $\mathcal{K}$ be an $\mathcal{L}\cup\{^{-1}\}$-structure such that its restriction to $\mathcal{L}_{\text{rings}}\cup\{^{-1}\}$ is field, $\langle K, \tau \rangle$ is a topological $\mathcal{L}$-structure and the inverse function $^{-1}$ is continuous on $K \setminus \{0\}$. We call such structure a *topological $\mathcal{L}$-field*. We assume our fields to be of characteristic 0.

We say that K is a *V-field*, where $V \in \mathcal{V}$, if any element of K can be written as $x.y^{-1}$, with $x,\ y \in V$ and $y \neq 0$.

We consider expansions of topological $\mathcal{L}$-fields to $\mathcal{L} \cup \{^{-1}, D\}$-structures, where D is a new unary function symbol for a derivation:

$$\forall a \forall b \;\; D(a+b) = D(a) + D(b), \forall a \forall b \quad D(a.b) = a.D(b) + D(a).b.$$

We shall denote $\mathcal{L} \cup \{^{-1}, D\}$ by $\mathcal{L}_D$ and the i-th iterate $D^i(x)$ of the derivation by $x^{(i)}$. We will call these expansions $\mathcal{K} := \langle K, \tau, D \rangle$ *differential topological $\mathcal{L}$-fields*, but we *do not* assume that this new function D is continuous. Let C_K denote the subfield of K of elements with zero derivative (i.e. the constant elements). Note that this is an infinite small subset of K.

Notation 3.1. Let $K\{X_1, \cdots, X_n\}$ be the ring of differential polynomials over K in n differential indeterminates $X_1, \cdots, X_n$ over K, namely it is the ordinary polynomial ring in indeterminates $X_i^{(j)}$, $1 \leq i \leq n$, $j \in \omega$, with by convention $X_i^{(0)} := X_i$. One can extend the derivation D of K to this ring by setting $D(X_i) := X_i^{(1)}$ and using the additivity and the Leibniz rule.

Set $\mathbf{X} := X_1, \cdots, X_n$ and for $\ell := (\ell_1, \cdots, \ell_n) \in \mathbb{N}^n$, let $\mathbf{X}^\ell := X_1^{\ell_1} \cdots X_n^{\ell_n}$ and $\partial_\ell := \frac{\partial^{\ell_1}}{\partial X_1} \cdots \frac{\partial^{\ell_n}}{\partial X_n}$. We will use the lexicographic ordering on $\mathbb{N}^n$.

Let $f(\mathbf{X}) \in K\{\mathbf{X}\} - K$ and suppose that f is of order m, then we write $f(\mathbf{X}) = f^*(X_1, \ldots, X_1^{(m)}, \cdots, X_n, \ldots, X_n^{(m)})$ for some ordinary polynomial $f^*(X_1, \cdots, X_{n.(m+1)})$ in $K[X_1, \cdots, X_{n.(m+1)}]$. We will make the following abuse of notation: if $b \in K^n$, then $f^*(b)$ means that we evaluate the polynomial f^* at the tuple $(b_1, \cdots, b_1^{(m)}, \cdots, b_n, \cdots, b_n^{(m)})$.

If $n = 1$, recall that the separant s_f of f is defined as $s_f := \frac{\partial f}{\partial X_1^{(m)}}$.

Consider now any infinitely differentiable function $g(\mathbf{X})$; we will use the Taylor expansion of g:

$$g(\mathbf{X}+\mathbf{Y}) = g(\mathbf{X}) + \sum_{\ell\neq 0} g_\ell(\mathbf{X}).\mathbf{Y}^\ell + R(\mathbf{Y}),$$

where $\ell !.g_\ell := \partial_\ell g$, and $R(\mathbf{Y})$ is a remainder.

Notation 3.2. Let $\phi(x_1,\cdots,x_n)$ be an open $\mathcal{L}_D$-formula, for each x_i, $1\leq i\leq n$, let m_i be the maximal natural number m such that $x_i^{(m)}$ occurs in an atomic subformula. Then, we denote by $\phi^*((x_{i,j})_{i=1,j=0}^{n,m_i})$ the formula we obtain from ϕ by replacing each $x_i^{(j)}$ by $x_{i,j}$. Let $N := \sum_{i=1}^n m_i + n$ and if S is the subset of K^n defined by ϕ, we denote by S^{alg} the subset of K^N definable by ϕ^*.

Note that given a definable set $S := \phi(K)$, the *operation* of taking S^{alg} depends on ϕ (so the notation is misleading). However, if $A\subseteq B$ are two definable subsets of K^n, then by possibly taking the direct product with a power of K, one may assume that $A^{alg}\subseteq B^{alg}$.

Definition 3.1. A definable subset S has a non-empty $*$-interior if the corresponding definable set S^{alg} has a non-empty interior.

Definition 3.2. (See [6]) Let $\mathcal{K} := \langle K,\mathcal{V},D\rangle$ be a differential topological $\mathcal{L}$-field and assume that K is a V-field for some $V\in\mathcal{V}$. Let $\mathcal{V}_0 := \bigcup_{n\geqslant 1}\mathcal{V}_{0,n+1}$ where $\mathcal{V}_{0,n+1}\subseteq\mathcal{V}_{n+1}$. Then we say that $\mathcal{K}$ satisfies $(DL)_{\mathcal{V}_0}$ if for every $n\geqslant 1$, for every differential polynomial $f(X) = f^*(X,X^{(1)},\dots,X^{(n)})$ belonging to $V\{X\}$ and for every $W\in\mathcal{V}_{0,n+1}$, the following implication holds:

$(\exists\alpha_0,\dots,\alpha_n\in V)(f^*(\alpha_0,\dots,\alpha_n)=0\wedge s_f^*(\alpha_0,\dots,\alpha_n)\neq 0)\Rightarrow$
$\Big((\exists z)\big(f(z)=0\wedge s_f(z)\neq 0\wedge (z^{(0)}-\alpha_0,\dots,z^{(n)}-\alpha_n)\in W\big)\Big)$.

When each $\mathcal{V}_{0,n+1}=\mathcal{V}_{n+1}$, we shall not put any subscript to (DL).

In [6], we showed transfer results on the existence of model-completions for theories of topological $\mathcal{L}$-fields to theories of differential topological $\mathcal{L}$-fields, assuming the topology is first-order definable. We follow the treatment of topological structures as used by A. Pillay in [12]. Let us recall the setting.

Definition 3.3. A topological $\mathcal{L}$-field K satisfies *Hypothesis* (D) if there is a formula $\phi(x,\bar{y})$ such that the set of subsets of the form $\phi(K,\bar{a}) := \{x\in K:\ K\models\phi(x,\bar{a})\}$ with $\bar{a}\subseteq K$ can be chosen as a basis $\mathcal{V}$ of neighbourhoods of 0 in K.

From now on, we will assume in addition that $\mathcal{L}$ is the language $\mathcal{L}_{\text{rings}}\cup\{R_i; i\in I\}\cup\{c_j; j\in J\}$ where the c_j's are constants and the R_i are n_i-ary predicates.

Let T be a universal $\mathcal{L}\cup\{^{-1}\}$-theory of fields of characteristic zero, assume T admits a model-completion T_c and that the class $\mathcal{M}(T_c)$ of all the models of T_c satisfies Hypothesis (D). So, we have on the elements of $\mathcal{M}(T_c)$ a first-order definable topology and we denote the corresponding class $\mathcal{M}(T_c)_{top}$.

Let T_D (respectively $T_{c,D}$) be the $\mathcal{L}_D$-theory T (respectively T_c) together with the axioms stating that D is a derivation. In [6], we showed that under the assumption that the elements of $\mathcal{M}(T_c)_{top}$ satisfies the so-called *Hypothesis* (I) $(\star)$, that any element of $\mathcal{M}(T_c)_{top}$ embeds in an element of $\mathcal{M}(T_c)_{top}$ satisfying the scheme (DL). The assumption $(\star)$ is the analog in our topological setting of the notion of *large fields* (see [13]).

Note that since the models of T satisfies Hypothesis (D), the scheme of axioms (DL) can be expressed in a first-order way. Let $T^*_{c,D}$ be the $\mathcal{L}_D$-theory consisting of $T_{c,D}$ together with the scheme (DL).

Theorem 3.4. ([6] Theorem 4.1.) Under the above hypotheses on T and T_c, we have that the theory $T^*_{c,D}$ is the model-completion of T_D. $\square$

Corollary 3.5. ([6].) The theory $T^*_{c,D}$ admits quantifier elimination.

Proof: The result follows from the fact that it is the model-completion of a universal theory ([2] Proposition 3.5.19). $\square$

Corollary 3.6. ([6] Corollary 4.3.) Assume that both theories T_c and $T^*_{c,D}$ are complete and that T_c has the non-independence property (NIP). Then the theory $T^*_{c,D}$ has NIP.

Lemma 3.7. *Let $\mathcal{K}$ be a differential topological $\mathcal{L}$-field which satisfies the scheme (DL). Let S be a definable subset of some K^k either with non empty interior or such S^{alg} has non empty interior, then S is not included in a zero-set of a non trivial (differential) polynomial and so, in particular, S is not small.*

Proof: Let O be a non empty open subset of K^k, let $f(\mathbf{X})$ be a differential polynomial of order N in k differential indeterminates. Let $b \in O \subseteq S$ and assume that $f(b) = 0$. We will find an element in O close to b on which f doesn't vanish. Let $f^*(\mathbf{X})$ be the corresponding ordinary polynomial. Write $f^*(b + \mathbf{Z}) = f(b) + \sum_{\ell \neq 0} f^*_\ell(b).\mathbf{Z}^\ell$, where $\ell! f^*_\ell = \partial_\ell f^*$ (see Definition 3.1) and the sum is over tuples ℓ of length less than or equal to $N.k$.

Let T be the set of tuples ℓ such that $\partial_\ell f^*(b) \neq 0$. Since f is non-zero, this set is non-empty. Let $\tilde{\ell}$ be the minimum of T in the lexicographic ordering; $\tilde{\ell} := (\ell_{n_1}, \cdots, \ell_{n_m})$ with $1 \leq n_1 < \cdots < n_m \leq N.k$. Denote the i^{th} component of ℓ by $\ell(i)$. For the indices $i \notin \{n_1, \cdots, n_m\}$, set $Z_i = 0$ and denote the obtained m-tuple by $\mathbf{Y}$. Let T_0 the subset of $N.k$-tuples ℓ in $T - \{\tilde{\ell}\}$ such that $\ell(i) \neq 0$ iff $i \in \{n_1, \cdots, n_m\}$. Note that for $\ell \in T_0$, i is minimal such that $\ell(i) - \ell_{n_i}$ is non zero, than it is strictly positive. Let $d := max_{\ell \in T_0}\{0, \ell_{n_i} - \ell(i)\} + 1$.

Write $f^*(b + \mathbf{Y}) = f^*(b) + g^*(\mathbf{Y})$ where $g^*(\mathbf{Y}) = \partial_{\tilde{\ell}} f(b).Y_{n_1}^{\ell_{n_1}} \cdots Y_{n_m}^{\ell_{n_m}}.h^*(\mathbf{Y})$ and $h^*(\mathbf{Y})$ is the rational function: $1 + \sum_{\ell \in T_0} \frac{\partial_\ell f(b)}{\partial_{\tilde{\ell}} f(b)}.Y_{n_1}^{\ell(1)-\ell_{n_1}} \cdots Y_{n_m}^{\ell(m)-\ell_{n_m}}$.

First given a neighbourhood W_0 of zero, there exists a neighbourhood W_1 of 0 such that $0 \notin 1 + W_1$ and if $\epsilon \in W_1$, then for all $\ell \in T_0$, $\epsilon.\frac{\partial_\ell f(b)}{\partial_{\tilde{\ell}} f(b)} \in W_0$, and such that the sum (respectively the product) of $|T_0|$ (respectively $\leq c$) elements of W_1 is in W_0, where c is a natural number that we will determine later. Then evaluate $h^*(\mathbf{Y})$ for $Y_{n_i} := \epsilon^{d^{m+1-i}}$ so $Y_{n_1}^{\ell(1)-\ell_{n_1}} \cdots Y_{n_m}^{\ell(m)-\ell_{n_m}} := \epsilon^{q_\ell(d)}$ where $q_\ell(d)$ is a polynomial of the form $\sum_{i=1}^m d^{m+1-i}.(\ell(i) - \ell_{n_i})$ and so $q_\ell(d) \geq 1$. Choose $c := max_{\ell \in T_0} q_\ell(d)$. The choice of W_1 implies that $\sum_{\ell\in T_0}\frac{\partial_\ell f(b)}{\partial_{\bar{\ell}} f(b)}.\epsilon^{q_\ell(d)}$ belongs to W_1. For $j=1,\cdots,k$, let $a_j(n)=0$ if $j.(N-1)\leq n<j.N$ and $a_j=\epsilon^{m+i-1}$ for $n\in\{n_1,\cdots,n_m\}$. Let $a:=(a_1,\cdots,a_k)$, we have $h^*(a)\neq 0$.

Therefore applying k times the scheme (DL) to $X^{(N+1)}$=0 and its algebraic solutions $(a_j,0)$, we can find elements $a'_j\sim a_j$, $1\leq j\leq k$, so that for $a'=(a'_1,\cdots,a'_k)$, we have $b+a'\in O$, with $f(b+a')\neq 0$. □

Lemma 3.8. *Suppose we have a topological system on the $\mathcal{L}_D$-structure $\mathcal{A}$. Then, no non empty open subset in some cartesian power of A is small.*

Proof: Let O be a non empty open subset of A^k, let $f(\mathbf{Y})$ be a nonzero differential polynomial in k differential indeterminates. Let $b\in O$ and assume that $f(b)=0$. Set $g(\mathbf{Y}):=f(b+\mathbf{Y})=f(b)+\sum_\ell\frac{\partial^{\ell_1}f}{\partial Y_1}\cdots\frac{\partial^{\ell_n}f}{\partial Y_n}(b).Y_1^{l_1}\cdots Y_k^{l_k}$, where $\ell=(l_1,\cdots,l_k)$.

Let L be the set of tuples ℓ such that $\partial_\ell f(b)=\frac{\partial^{\ell_1}f}{\partial Y_1}\cdots\frac{\partial^{\ell_n}f}{\partial Y_n}(b)\neq 0$. Since f is non-zero, this set is non-empty.

Let L be the set of tuples ℓ such that $\partial_\ell f(b)\neq 0$. Since f is non-zero, this set is non-empty. Let ℓ_0 be the minimum of T in the lexicographic ordering and suppose it is of the form $(\ell_1,\cdots,\ell_m)$ with $m\leq N.k$. For the indices $i\notin\{n_1,\cdots,n_m\}$, set $Y_i=0$ and denote the obtained tuple by $\mathbf{Y}$. Let T_0 the subset of tuples ℓ in $T-\{\ell_0\}$ with the same support as ℓ_0. We denote the i^{th} component of ℓ by $\ell(i)$. Note that for $\ell\in T_0$, if i is minimal such that $\ell(i)-\ell_{n_i}$ is non zero, than it is strictly positive.

Write $f(b+\mathbf{Y})=f(b)+g(\mathbf{Y})$ where $g(\mathbf{Y})=Y_{n_1}^{\ell_{n_1}}\cdots Y_{n_m}^{\ell_{n_m}}.h^*(\mathbf{Y})$, where $h(\mathbf{Y})$ is the rational function:

$$h(\mathbf{Y}):=\partial_{\ell_0}f(b)+\sum_{\ell\in T_0}\partial_\ell f(b).Y_{n_1}^{\ell(1)-\ell_{n_1}}\cdots Y_{n_m}^{\ell(m)-\ell_{n_m}}.$$

Since h is continuous away from 0, there exists $a\sim 0$ such that $b+a\in O$, $h(a)\neq 0$ and so $g(a)\neq 0$ and $f(b+a)\neq 0$. (One way to check this is to evaluate $h(\mathbf{Y})$ at an element of the form $Y_{n_i}:=\epsilon^{n^{m+1-i}}$ and so $Y_{n_1}^{\ell(1)-\ell_{n_1}}\cdots Y_{n_m}^{\ell(m)-\ell_{n_m}}=\epsilon^{q(n)}$ where $q(n)$ is a polynomial of the form $\sum_{i=1}^m n^{m+1-i}.(\ell(i)-\ell_{n_i})$ and so the coefficient of its term of highest degree is positive.)

□

Remark 3.9. In our setting of differential topological $\mathcal{L}$-fields $\mathcal{K}=(K,D)$, the derivation D on $\mathcal{K}$ is always non trivial, i.e. $D(K)\neq 0$. So nonzero differential polynomials $f(\mathbf{X})$ in $K\{\mathbf{X}\}$, with $\mathbf{X}=(X_1,\ldots,X_m)$, define differential polynomial functions $K^m\to K$ that are not identically zero ([9]). Note that if $\mathcal{K}$ satisfies the scheme (DL), then whenever f^* is non trivial, f is non trivial ([6] Corollary 3.15).

If $\mathcal{K}$ is a differential topological $\mathcal{L}$-field satisfying the scheme (DL), then the subfield C_K of constants is a dense small subfield of K ([6] Corollary 3.13).

Proposition 3.10. *Let $\mathcal{K}$ be a differential topological $\mathcal{L}$-field such that*

(1) *for no small definable non empty subset S, S^{alg} is open,*

(2) *$\mathcal{K}$ has quantifier elimination in $\mathcal{L}_D$.*

Then $\mathcal{K}$ is equationally bounded.

Proof. Let S be a definable subset of K^{m+1}. Since $\mathcal{K}$ admits quantifier elimination, S is a finite union of subsets defined by $\mathcal{L}_D$-formulas of the following form:

$$\bigwedge_{i=1}^{k} f_i(\bar{x}, y) = 0 \wedge \phi(\bar{x}, y),$$

where $\phi^*(\bar{x}, \bar{y})$ defines a non-empty open subset of a cartesian product of K.

Hence for $\bar{a} \in K^m$, the definable subset $S_{\bar{a}}$ of K is defined by

$$\bigwedge_{i=1}^{k} f_i(\bar{a}, y) = 0 \wedge \phi(\bar{a}, y).$$

Suppose that $S_{\bar{a}}$ is small. Let us show that for some i, $f_i(a, Y)$ is not identically zero. Suppose otherwise, so $S_{\bar{a}} = \{y \in K : \mathcal{K} \models \phi(\bar{a}, y)\}$ and so the induced set $S_{\bar{a}}^{alg}$ is open in some cartesian product of K. Therefore by hypothesis, $S_{\bar{a}}$ is empty.

Thus, not all $f_1(\bar{a}, Y), \ldots, f_k(\bar{a}, Y)$ are identically zero, say $f_i(\bar{a}, Y) \neq 0$ and $S_{\bar{a}} \subseteq \{y \in K : f_i(\bar{a}, y) = 0\}$. Since it only depends on S and not on $\bar{a}$, we have that $\mathcal{K}$ is equationally bounded. □

Corollary 3.11. *Let $\mathcal{K}$ be a differential topological $\mathcal{L}$-field satisfying the scheme (DL). Suppose that $\mathcal{K}$ has quantifier elimination in $\mathcal{L}_D$. Then, the function t-dim defines a fibered dimension function on $\mathcal{K}$.*

Proof. It suffices to apply Proposition 2.13, together with Lemmas 3.7 and Proposition 3.10. □

In particular in any model of $T^*_{c,D}$, the function t-*dim* defines a fibered dimension function.

Proposition 3.12. (See [17]).

Let $\mathcal{K}$ be a $\mathcal{L}_D$-structure endowed with a topological system such that

(1) *no non empty open subset is small,*
(2) *$\mathcal{K}$ has quantifier elimination in $\mathcal{L}_D$,*

Then $\mathcal{K}$ is equationally bounded and the function t-dim defines a fibered dimension function on $\mathcal{K}$.

Proof: See page 203 in [17]. It suffices to apply Proposition 2.13, together with Lemmas 3.8 and the analog of Proposition 3.10 for topological systems. □

Proposition 3.13. *Let $\mathcal{K}$ be either a differential topological $\mathcal{L}$-field satisfying the scheme (DL) or a $\mathcal{L}_D$-structure with a topological system. Suppose that $\mathcal{K}$ has quantifier elimination in $\mathcal{L}_D$. Let X be a definable subset of K^n, then t-dim$(X) \geq k$ if and only if there exists a projection π_k of X in K^k with non empty $*$-interior.*

Proof: $(\rightarrow)$ Let $\bar{a} := (a_1, \cdots, a_n)$ be a generic point of X and let us assume that $a_1, \cdots, a_k$, $1 \leq k \leq n$, are independent and let Y be the projection of X along the first k coordinates. Set $\bar{a}_k := (a_1, \cdots, a_k)$. Note that Y is definable by a quantifier free formula $\phi(x) := \bigvee_j \bigwedge_{i \in I_j} t_{ij}(x) = 0 \ \& \ \theta_j(x)$, where $\theta_j^*(K)$ is an open subset in some cartesian product of K. We will proceed by induction on k.

Case: $k = 1$. There exists an index j with $I_j = \emptyset$, otherwise a_1 would belong to $cl(\emptyset)$.

Case: $k = \ell + 1$. Either there exists an index j with $I_j = \emptyset$.

Then, either X has a generic point $\bar{a}_k$ belonging to $\{\bar{x} \in K^k : \mathcal{K} \models \theta_j(\bar{x})\}$. So $\theta_j(K)$ is a non empty subset of Y and $\theta_j^*(K)$ is an open subset of a cartesian product of K, or all the generic points $\bar{a}$ of X we have for some j that $\bigwedge_{i \in I_j} t_{ij}(\bar{a}_k) = 0$ and so $\forall x \ \bigwedge t_i(\bar{a}_\ell, x) = 0$. Since the dimension is fibered t-$dim(Y(1)) = \ell$ and so by induction hypothesis $Y(1)$ has non empty $*$-interior and so Y has non empty $*$-interior since we put the product topology on cartesian products of K.

($\leftarrow$) W.l.o.g., we may assume that the projection with non empty $*$-interior is onto the first k coordinates and Y be the image of X under that projection. Again, Y is definable by a quantifier free formula $\phi(x) := \bigvee_j \bigwedge_{i \in I_j} t_{ij}(x) = 0 \ \& \ \theta_j(x)$, where $\theta_j^*(K)$ is an open subset. Assume that each t_{ij} is non trivial. So there is an index j such that $I_j = \emptyset$ and $\theta_j(K)$ is included in Y. We proceed by induction on k. By Lemma 3.7, Y is not small and so it cannot be of dimension 0. Suppose $k = \ell + 1$. The set $\{\bar{x}_\ell : \text{t-}dim(Y_{\bar{x}_\ell}) = 1\}$ has also non-empty $*$-interior and so by induction hypothesis, it has dimension ℓ. □

Corollary 3.14. *Let A, $B \in Def(\mathcal{K})$. Assume that $A \subseteq B$ and $A^{alg} \subseteq B^{alg}$, then t-$dim(A) <$t-$dim(B) = k$ iff there exists a projection π_k in K^k such that the relative $*$-interior of $\pi_k(A)$ in $\pi_k(B)$ is empty.*

Proof: ($\rightarrow$) Assume that t-$dim(B) = k$, then there exists a projection π_k of B in K^k such that $\pi_k(B)$ has non empty $*$-interior. Suppose that $\pi_k(A)$ in K^k has non-empty $*$-interior (in $\pi_k(B)$). Then by the above proposition, its dimension is greater than or equal to k, a contradiction.

($\leftarrow$) Suppose that the relative $*$-interior of $\pi_k(A)$ in $\pi_k(B)$ is empty. Then by the preceding proposition, t-$dim(\pi_k(A)) <$t-$dim(\pi_k(B))$. Since t-dim is fibered, we have that t-$dim(A)$=t-$dim(\pi_k(A)) + sup_{\bar{x} \in \pi_k(A)}$ t-$dim(A_{\bar{x}})$. Since $A \subseteq B$, for $\bar{x} \in \pi_k(A)$, $A_{\bar{x}} \subseteq B_{\bar{x}}$ and so t-$dim(A_{\bar{x}}) \leq$ t-$dim(B_{\bar{x}})$. Therefore, t-$dim(A) <$t-$dim(B)$. □

Proposition 3.15. *Let $\mathcal{K}$ be either a differential topological $\mathcal{L}$-field satisfying the scheme (DL) or a $\mathcal{L}_D$-structure with a topological system. Suppose that $\mathcal{K}$ has quantifier elimination in $\mathcal{L}_D$. Let X be a definable subset of K^n and let f be a definable bijection. Then t-$dim(X) \geq k$ if and only if t-$dim(f(X)) \geq k$.*

Proof: Let $\phi(\bar{x}, \bar{y})$ be a formula such that $f(\bar{x}) = \bar{y}$ iff $\phi(\bar{x}, \bar{y})$. Denote by $G(f)$ the graph of f.

By quantifier elimination, $\phi(\bar{x}, \bar{y})$ is equivalent to $\bigvee_j \phi_j(\bar{x}, \bar{y})$ with $\phi_j(\bar{x}, \bar{y}) := \bigwedge_{i \in I_j} t_{ij}(\bar{x}, \bar{y}) = 0 \ \& \ \theta_j(\bar{x}, \bar{y})$, where $\theta_j^*(\bar{x}, \bar{y})\}$ defines a non-empty subset of some cartesian product of K.

So, for every j the formula $\phi_j(\bar{x}, \bar{y})$ is of the form $\bigwedge_{i \in I_j} t_{ij}(\bar{x}, \bar{y}) = 0$, where there exists i such that t_{ij} is a non-trivial term which has only finitely many solutions for each $\bar{x}$. We show that t-$dim(X)$=t-$dim(G(f))$ by induction on n. Set $\bar{y} :=$

$(\bar{y}_1, y_n)$. Since t-dim is a fibered dimension and since f is a function, t-$dim(G(f))$=t-$dim(\{(\bar{x}, \bar{y}_1)$: t-$dim(G(f)_{\bar{x},\bar{y}_1} = 0\})$. This set is definable. Indeed, $\{(\bar{x}, \bar{y}_1)$: t-$dim(G(f)_{\bar{x},\bar{y}_1} = 0\})$=$\{(\bar{x}, \bar{y}_1) : \exists y_n \bigvee_j \bigwedge_{i \in I_j} t_{ij}(\bar{x}, \bar{y}) = 0\})$; let us denote this set by X_1. Set $\bar{y}_1 := (\bar{y}_2, y_{n-1})$ Again the t-dimension of X_1 is equal to the t-dimension of $\{(\bar{x}, \bar{y}_2)$:t-$dim((X_1)_{\bar{x},\bar{y}_2}) = 0\})$. Iterating, we obtain that t-$dim(X)$=t-$dim(G(f))$. $\square$

4. A dimension function on valued D-fields

In this section, we will consider a class of topological differential fields where the derivation D is continuous.

Recall that a valued field K is a field together with a valuation map $v : K \to \Gamma \cup \{+\infty\}$, where $\Gamma := v(K^{\times})$ is a totally ordered abelian group (the value group). The valuation ring $\mathcal{O}_K$ is defined as $\{x \in K : v(x) \geqslant 0\}$ and the residue field of K will be denoted by $k_K := \mathcal{O}_K/\mathcal{M}_K$, where $M_K := \{x \in K : v(x) > 0\}$ is the maximal ideal of $\mathcal{O}_K$. *We will assume that k_K is a field of characteristic* 0. The residue map will be denoted by $\pi : \mathcal{O}_K \longmapsto k_K$. We will extend it on K by sending $K - \mathcal{O}_K$ to $+\infty$.

We will think of a valued field as the three-sorted structure $\langle K, \Gamma, k_K\rangle$ where the sorts are connected by the functions $\pi : K \to k_K \cup \{+\infty\}$ and $v : K \to \Gamma \cup \{+\infty\}$.

We will consider expansion of valued fields with a continuous derivation D, namely

$$\forall x \in K \quad v(D(x)) \geqslant v(x).$$

This implies that the derivation D on K induces a derivation, also denoted by D, on the residue field k_K as follows:

$$\forall x \in K \quad D(\pi(x)) = \pi(D(x)).$$

This class of differential valued fields was investigated by T. Scanlon ([15]). (In fact he placed himself in the more general context of valued D-fields and the present setting corresponds to the case $e = 0$.) He considered the following problem: given a differential residue field theory $Th(\mathbf{k})$ and a totally ordered abelian group theory $Th(\mathbf{\Gamma})$, both admitting quantifier elimination in suitable languages (respectively $\mathcal{L}_r$ and $\mathcal{L}_g$), under which additional conditions does the corresponding three-sorted theory admits quantifier elimination in a reasonable language? He called the corresponding class of valued differential fields: $(\mathbf{k}, \mathbf{G})$-D-henselian fields (see [15], [16]). For the present application, it is more convenient not to use the language in which T. Scanlon proved his result, but a variation of the language introduced by F. Delon in her thesis [3], for valued fields. This version of the quantifier elimination result for the $(\mathbf{k}, \mathbf{G})$-D-henselian fields is due to N. Guzy ([5]).

Using this quantifier elimination, we will show that this class of differential valued fields is equationally bounded and from that we will deduce, as in [17], that they can be endowed with a fibered dimension function.

Let us recall some terminology and the results that we need which appeared in [15], [16] and [5].

Definition 4.1. (See Definition 7.3 in [15]).

(1) A differential valued field K *has enough constants* if it satisfies $v(K^\times) = v(C_K^\times)$
(2) A differential valued field K satisfies D-Hensel's Lemma if:
if $P \in \mathcal{O}_K\{X\}$ is a differential polynomial over $\mathcal{O}_K$, $a \in \mathcal{O}_K$ and $v(P(a)) > 0 = v(\frac{\partial}{\partial X_i}P(a))$ for some i, then there is some $b \in K$ with $P(b) = 0$ and $v(a-b) \geqslant v(P(a))$.

From now on, we further assume that $\langle k_K, D\rangle$ is a differential field which admits quantifier elimination in an expansion $\mathcal{L}_r$ of the language of differential rings and that $\langle \Gamma, +, 0, \leqslant\rangle$ is an abelian totally ordered group which admits quantifier elimination in a language $\mathcal{L}_g$ extending the language of ordered abelian groups. We denote by $\mathcal{L}_{rings,D}$ the language of differential rings. Then we may define the notion of $(\mathbf{k}, \mathbf{G})$-D-henselian field.

Definition 4.2. A three-sorted $(\mathcal{L}_{rings,D}, \mathcal{L}_r, \mathcal{L}_g)$-structure $\langle K, k_K\Gamma\rangle$ is $(\mathbf{k}, \mathbf{G})$-D-henselian if the following are satisfied:
(1) $\langle k_K, D\rangle$ is a model of $\mathrm{Th}(\mathbf{k})$,
(2) the maps $v : K^\times \to \Gamma$ and $\pi : \mathcal{O}_K \to k_K$ are surjective,
(3) $v(K^\times) \neq \{0\}$ is a model of $\mathrm{Th}(\mathbf{G})$,
(4) $\langle K, D, v\rangle$ has enough constants and satisfies D-Hensel's Lemma.

We will need the fact that if $\langle K, k_K\Gamma\rangle$ is is $(\mathbf{k}, \mathbf{G})$-D-henselian, then K and $\langle k_K, D\rangle$ are linearly D-closed (see [15] 5.3, [16] Remark 6.2).

Let $\mathcal{L}_1$ be the three-sorted language $(\mathcal{L}_{rings,D}, \mathcal{L}_g, \mathcal{L}_r)$. Then the class of $(\mathbf{k}, \mathbf{G})$-D-henselian valued fields is an $\mathcal{L}_1$-elementary class.

Now we introduce as in Section 2 of [5], the language of Delon for valued fields, denoted by $\mathcal{L}_2$ (see Chapter 2 in [3]).

First let $P_n(x)$ be the formula: $\exists y \neq 0 \quad (y^n = x)$; so $P_n(K)$ is the subset of non-zero n^{th} powers of K.

We describe $\mathcal{L}_2$ as the language obtained by adding to $\mathcal{L}_1$ the following set of new predicates:

$$\Big\{F_{\phi,n_1,\ldots,n_r}; \phi \;\; \mathcal{L}_r\text{-formula with } r+s \text{ variables, } n_1,\ldots,n_r \in \mathbb{N}\setminus\{0,1\}\Big\}.$$

These predicates will be interpreted in the following way:

$$\begin{aligned}\forall x_1,\ldots,x_r \in K\, \forall \eta_1,\ldots,\eta_s \in k_K \Big\{ & F_{\phi,n_1,\ldots,n_r}(x_1,\ldots,x_r,\eta_1,\ldots,\eta_s) \iff \\ & \exists z_1,\ldots,z_r \in K \;\; [\bigwedge_{i=1}^r v(z_i) = 0 \wedge \phi(\pi(z_1),\ldots,\pi(z_r),\eta_1,\ldots,\eta_s) \\ & \wedge \bigwedge_{i=1}^r P_{n_i}(x_i z_i)]\Big\}.\end{aligned}$$

Now we can naturally consider the $\mathcal{L}_1$-structures of differential valued fields described above as $\mathcal{L}_2$-structures, since the interpretation of *Delon predicates* are $\mathcal{L}_1$-definable.

In that language, N. Guzy proved the following quantifier elimination result ([5]).

Theorem 4.3. (Theorem 3.13 of [5]). *The $\mathcal{L}_2$-theory of* $(\mathbf{k},\mathbf{G})$*-D-henselian fields eliminates quantifiers.*

In order to show that $(\mathbf{k},\mathbf{G})$-D-henselian fields can be equipped with a topological system, we need to slightly modify the language in a way analogous to the one done by L. van den Dries for proving that Henselian fields of equal characteristic 0 are algebraically bounded ([17] section 3); this language will be denoted by $\mathcal{L}_{\mathrm{vdD}}$.

Definition 4.4. Let $\langle K, D, v\rangle$ be a differential valued field. The language $\mathcal{L}_{\mathrm{vdD}}$ is an expansion of the language of differential rings by a lot of extra predicates (with their interpretations in K) defined as follows:

(1) for any set $S \subseteq \Gamma^m$ $(m > 1)$, an $(m+1)$-ary predicate V_S is interpreted in K as

$$V_S(a_1,\dots,a_m,b) \iff a_1,\dots,a_m,b \in K^\times \wedge (v(\frac{a_1}{b}),\dots,v(\frac{a_m}{b})) \in S;$$

(2) for any set $U \subseteq k^{r+s}$ $(r,s \geqslant 0)$ and any positive integers $n_1,\dots,n_r$ an $(r+s+1)$-ary predicate $P_{U,n_1,\dots,n_r}$ is interpreted in K by:

$$P_{U,n_1,\dots,n_r}(a_1,\dots,a_r,b_1,\dots,b_s,c) \iff c \neq 0, v(b_1),\dots,v(b_s) \geqslant v(c)$$

$$\wedge \exists z_1,\dots,z_r\,[\bigwedge_{i=1}^r v(z_i) \geqslant 0 \wedge (\pi(z_1),\dots,\pi(z_r),\pi(\frac{b_1}{c}),\dots,\pi(\frac{b_s}{c})) \in U \wedge \bigwedge_{i=1}^r P_n(\frac{a_i z_i}{c})].$$

Remark 4.5 (See Section 3 in [17]). (1) *The set $\Gamma_- := \{\gamma \in \Gamma : \gamma \leqslant 0\}$ corresponds via (1) to the following binary predicate (a divisibility relation) on $(K^\times)^2$:*

$$a|b \iff v(a) \leqslant v(b) \quad (a,b \in K^\times).$$

This divisibility relation $|$ determines the valuation v.

(2) *By taking $r = 0$ in item (2) in Definition above, the inverse image in $\mathcal{O}_K^s$ of any subset of k_K^s is definable by an atomic $\mathcal{L}_{vdD}$-formula.*

Now we can state our quantifier elimination theorem in $\mathcal{L}_{\mathrm{vdD}}$.

Theorem 4.6. *Let K be a* $(\mathbf{k},\mathbf{G})$*-D-henselian field. Then K admits quantifier elimination in $\mathcal{L}_{vdD}$.*

Proof. By using the Appendix in [5], we may translate an $\mathcal{L}_{\mathrm{vdD}}$-formula $\varphi(x_1,\dots,x_n)$ to an $(\mathcal{L}_{rings,D},\mathcal{L}_g,\mathcal{L}_r)$-formula $\tilde{\varphi}(x_1,\dots,x_n)$ for suitable language $\mathcal{L}_g$, $\mathcal{L}_r$ such that k_K (respectively $v(K^\times)$ admits quantifier elimination in $\mathcal{L}_r$ (respectively $\mathcal{L}_g$). Then we apply Theorem 4.3 to get a quantifier-free $\mathcal{L}_2$-formula $\tilde{\theta}(x_1,\dots,x_n)$ equivalent to $\tilde{\varphi}$ in $Th(K)$. Now we use again the following fact (see Appendix in [5]):

$$K \models F_{\phi,n_1,\dots,n_r}(x_1,\dots,x_r,\eta_1,\dots,\eta_s) \iff P_{U,n_1,\dots,n_r}(x_1,\dots,x_r,e_1,\dots,e_s,1),$$

with U is the set $\phi(z_1,\dots,z_r,w_1,\dots,w_s) \wedge \bigwedge_{i=1}^r z_i \neq 0$, η_j are free $\mathcal{L}_r$-terms and e_i are the corresponding free $\mathcal{L}_{\mathrm{vdD}}$-terms by using the fact k_K can be embedded in K. $\square$

Lemma 4.7. *A* $(\mathbf{k},\mathbf{G})$*-D-henselian field considered as an $\mathcal{L}_{vdD}$-structure can be equipped with a topological system.*

Proof. The valuation topology is Hausdorff, the derivation D on K is continuous and so, K is a topological differential field. So it remains to check the third condition on the new predicates introduced in $\mathcal{L}_{\mathrm{vdD}}$.

For the predicates V_S in (1), it is immediate since they define clopen subsets of $(K^\times)^m$.

We will simply check that $P_{U,\vec{n}}$ defines an open subset of K^{r+s+1}, for the other subsets of these relations and their complements, the argument is similar.

Assume that $(\vec{a},\vec{b},c) \in P_{U,\vec{n}}$. Then we have $c \neq 0$, $v(\vec{b}) \geqslant v(c)$ and

$$K \models \exists z_1,\dots,z_r\,[\bigwedge_{i=1}^r v(z_i) \geqslant 0 \wedge (\pi(z_1),\dots,\pi(z_r),\pi(\frac{b_1}{c}),\dots,\pi(\frac{b_s}{c})) \in U \wedge \bigwedge_{i=1}^r P_n(\frac{a_i z_i}{c})].$$

Let us consider the following open balls in K (of center l and radius $v(k)$).

$$B_k(l) := \{x \in K : \, v(x-l) > v(k);\ k \neq 0, l \in K\}.$$

We know by Hensel's Lemma that P_n is open, so there exists open balls $B_{k_i}(a_i z_i/c)$ in P_n for some $k_i \in K^\times$. Moreover for any $b'_i \in B_{b_i}(b_i)$ and $c' \in B_c(c)$, we get

$$(\pi(z_1),\dots,\pi(z_r),\pi(\frac{b'_1}{c'}),\dots,\pi(\frac{b'_s}{c'})) \in U.$$

Now if $v(k_i) + v(c) - v(z_i) < v(a_i - a'_i)$ for some a'_i in K then

$$K \models \exists z'_1,\dots,z'_r\,[\bigwedge_{i=1}^r v(z'_i) \geqslant 0 \wedge (\pi(z'_1),\dots,\pi(z'_r),\pi(\frac{b'_1}{c'}),\dots,\pi(\frac{b'_s}{c'})) \in U \wedge \bigwedge_{i=1}^r P_n(\frac{a'_i z'_i}{c'})].$$

It suffices to choose $z'_i := z_i c'/c$. □

Lemma 4.8. *Let K be a $(\mathbf{k},\mathbf{G})$-D-henselian field, and let S be a non-empty open ball of zero in K. Then no non-zero differential polynomial with coefficients in K vanishes on S.*

Proof. This follows from Lemma 3.8, but we will give an alternative proof below. Assume that $\{x \in K : \, v(x-a) > v(b),\ b \neq 0\}$ and let P be in $K\{X\}$ of order $n \geqslant 1$ (the case $n = 0$ is trivial). If $P(a) \neq 0$ then we are done. So assume that $P(a) = 0$ and consider the differential polynomial $Q(X) := P(X+a)$. Then $Q(0) = 0$ and Q is non identically zero. Therefore we may assume $a = 0$. Moreover w.l.o.g., we may assume that all the coefficients in P are of valuation bigger than $v(b)$. Let us consider the Taylor expansion of P at 0

$$Q(X) = \sum_{i=0}^n \frac{\partial Q}{\partial X^{(i)}}(0) X^{(i)} + R(X),$$

with $R(X)$ contains only differential monomials of order at least 2 or $R(X) = 0$ if $n = 1$. So if $c \in K$ has valuation bigger than $v(b)$ then $v(R(b)) \geqslant v(b)$.

Let $a_j := \frac{\partial Q}{\partial X^{(j)}}(0) \in K$ and let $\alpha \in K\backslash\{0\}$ be such that $v(\alpha) = \min_j\{v(a_j)\} \geqslant v(b)$. Let us consider the following linear differential polynomial $L(X)$

$$L(X) := \alpha + \sum_{i=0}^n a_i X^{(i)}.$$

By Lemma 8.3 in [15], we know that K is linearly closed. So we have a solution z of $L(X)$ with $v(z) = v(\alpha)$. Therefore we get $v(P(z)) = v(\alpha)$; and $P(z) \neq 0$ with $v(z) \in S$. □

Proposition 4.9. *A* $(\mathbf{k}, \mathbf{G})$*-*D *henselian field* K *is equationally bounded.*

Proof. By Lemma 4.7 the $\mathcal{L}_{\mathrm{vdD}}$-structure K is a topological system. So by using Lemma above and Theorem 4.6, the result follows from Proposition 3.12. □

Theorem 4.10. *If* K *is a* $(\mathbf{k}, \mathbf{G})$*-*D*-henselian valued* D*-field then* K *considered as an* $\mathcal{L}_{vdD}$*-structure has a unique fibered dimension function on* $Def(K)$*.*

Proof. By Proposition 3.12, we get that the t-*dim* is a fibered dimension.

Now if S is an open subset of K then we get that S has t-*dim* equal to 1 by Lemma 4.8. So we need to show for any fibered dimension function d, $d(S) = 1$. We follow the argument in Theorem 3.5 of [17]. We translate S and multiply by a non zero constant element such that $S \supseteq \mathcal{O}_K$. Hence $K = S \cup S^{-1}$, which by $d(K) = 1$ implies that $d(S) = 1$. □

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