QUANTIFIER ELIMINATION IN VALUED ORE MODULES

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Abstract. We consider valued fields with a distinguished isometry or contractive derivation as valued modules over the Ore ring of difference operators. Under certain assumptions on the residue field, we prove quantifier elimination first in the pure module language, then in that language augmented with a chain of additive subgroups, and finally in a two-sorted language with a valuation map. We apply quantifier elimination to prove that these structures do not have the independence property.

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1. Introduction.

The model theory of Witt vectors with the Witt Frobenius as a distinguished automorphism has been investigated in [5], [27], [6]. The results are of Ax-Kochen-Ershov type. For instance:

Let $K = W(F)$ be the field of Witt vectors with coefficients in F, where F is a p-closed field of characteristic p, let v be the p-adic valuation, and let σ be the Witt Frobenius (see below). Then $Th(K, v, \sigma)$ is axiomatized by:

1. (K, v) is a valued field of characteristic 0 and σ is an isometry, i.e. $v(\sigma(x)) = v(x)$, inducing the ordinary Frobenius $x \mapsto x^p$ on the residue field F;
2. a suitable analog of Hensel's lemma holds for polynomials involving σ;
3. the residue field satisfies $Th(F)$,
4. the value group is a $\mathbb{Z}$-group with least element $v(p)$.

In this paper we will consider the theory of these fields, and other valued fields with an isometry, in weaker formalisms of (valued) modules. This amounts essentially to restrict ourselves to study *linear* equations of the form $c_n\sigma^n(X)+\ldots+c_1\sigma(X)+c_0X = b$, where $c_i, b \in K$. Ore was the first to study systematically these equations in the case of the usual Frobenius map $x \mapsto x^p$. There is a well established analogy between differential fields and fields with a distinguished automorphism, also called difference fields in the literature. It turns out that if a valued field has a derivation ∂ which is contractive, i.e. such that $v(\partial x) \geq v(x)$, then much of our results hold as well for these structures. We will treat both cases simultaneously in a suitable formalism. In all main results, a linear form of an analog of Hensel's lemma (alluded to above) plays a crucial role and corresponds to divisibility of the corresponding module. The main results, which apply also in positive characteristic, consist of axiomatizations

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and quantifier elimination in appropriate languages (see sections 4 and 5), with an application to the absence of the independence property. They were sketched in [7].

Some of the basics also hold when σ is just an endomorphism (e.g. the Hensel property) but we have not pursued the matter further.

The plan of the paper is as follows. In section 2 we introduce the module formalism and we axiomatize the theories of valued fields with an isometry or a contractive derivation as modules over a (Ore) skew polynomial ring. In section 3 we introduce the formalisms of valued modules, first as two-sorted structures with a valuation map between the module sort and the ordered set sort, and then as modules with a chain of distinguished additive subgroups indexed by their value group which yield an abelian structure. In section 4 we prove quantifier elimination in the abelian structure formalism, and then in section 5 for the two-sorted one. In section 6 we apply quantifier elimination to prove that in the two-sorted language, these structures do not have have the independence property (and so their definable sets can be endowed with a VC-dimension, see e.g. [21]). Finally, in section 7, we observe that all this allows to prove a transfer principle between Witt vectors and power series.

NOTATION AND TERMINOLOGY.

We will use boldface notation for tuples, e.g. $\boldsymbol{x} = (x_1, \ldots, x_n)$. All our modules will be right modules. For an element r of a ring, $ann(r)$ will denote its annihilator in a given module. For a valued field (K, v), we denote its value group by vK, its valuation ring by $\mathcal{O}_K$, its residue field by $\bar{K}$. The natural residue map from $\mathcal{O}_K$ to $\bar{K}$ is denoted by $\bar{\ }$, and will be used for various reductions. For example if $q(X) \in \mathcal{O}_K[X]$, then $\bar{q}$ is the reduced element in $\bar{K}[X]$ obtained by reducing the coefficients. An isometry is an automorphism of K such that $v(\sigma(x)) = x$. A contractive map is a map $\partial : K \to K$ such that $v(\partial(x)) \geq v(x)$. If f is a function on a set X we will sometimes use the notation x^f for $f(x)$, $x \in X$.

Let F be a perfect field of characteristic $p > 0$, we denote by $W[F]$ the ring of Witt vectors over F. It can be seen as a ring structure on F^ω given by universal polynomials $s_n(X_0, Y_0, \ldots, X_n, Y_n), p_n(X_0, Y_0, \ldots, X_n, Y_n)$ with coefficients in the integers and without constant terms: $(x_0, x_1, \ldots)+(y_0, y_1, \ldots) = (x_0+y_0, s_1(x_0, y_0, x_1, y_1), s_2(x_0, ...), \ldots)$ and similarly for the product using the p_n's. It is a complete discrete valuation ring of characteristic 0 with uniformizing parameter p and residue field F. Its field of fractions is denoted by $W(F)$. From the point of view above, the corresponding valuation is $v(x_0, x_1, \ldots) = \min\{n : x_n \neq 0\}$, and it is apparent that the map $(x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots)$ is an automorphism of $W[F]$ which induces an isometry on $W(F)$. We call this map the Witt Frobenius. If $F = \mathbb{F}_p$, then $W[\mathbb{F}_p]$ is the ring $\mathbb{Z}_p$ of p-adic integers, $W(\mathbb{F}_p)$ is the field $\mathbb{Q}_p$ of p-adic numbers, and the Witt Frobenius coincides with the identity. We refer to [19] for basics on Witt vectors.

A field of characteristic p is p-closed if it does not have any finite algebraic extension of degree divisible by p. G. Whaples showed that this is equivalent to the fact that every polynomial of the form $\sum c_i x^{p^i} = c$ has a root (for a proof "by hand" see *Afterthought: Maximal fields with valuation* in [20]). In particular, a p-closed field is perfect.

A ring is said to be right Ore, if any two elements have a common right multiple.

We will assume known classical results on the model theory of modules, and use the usual "pp" for "positive primitive", e.g. pp-formula, etc.. For the model theory of modules we refer to [24].

2. Modules

In order to deal simultaneously with the cases of a valued field with an isometry or with a contractive derivation, it will be convenient for us to follow the formalism of Cohn's book [9]. Other formalisms are possible to cover the two cases, e.g. D-valued fields (cf. [27]) or valued fields with a twist (cf. [25]), but Cohn's seems more fitting to our context.

Definition 2.1. Let K be a field and σ an automorphism of K. A map $\partial : K \to K$ is called a σ-derivation, if $\partial(x + y) = \partial(x) + \partial(y)$ and $\partial(xy) = \partial(x)y^\sigma + x\partial(y)$, for all $x, y \in K$.

When σ is the identity, this is just a standard derivation. Note that since for us K is always commutative, when σ is not the identity the σ-derivations are all of the form $\partial(x) = (\sigma(x) - x)c$ for some nonzero $c \in K$ ([9], Theorem 2.1.3).

We now fix (K, v, σ, ∂), namely a valued field (K, v) with distinguished isometry σ and contractive σ-derivation ∂.

If ∂ is the zero derivation, then we just have a valued field with an isometry, and if σ is the identity map, then we just have a valued field with a contractive derivation.

The operators ∂ and σ have an induced action on $\bar{K}$, that will be denoted by $\bar{\sigma}$ and $\bar{\partial}$ respectively, namely $\bar{\sigma}(\bar{a}) = \overline{\sigma(a)}$ and $\bar{\partial}(\bar{a}) = \overline{\partial(a)}$, which make $\bar{\partial}$ into a $\bar{\sigma}$-derivation. More generally, given any polynomial $q[X] \in \mathcal{O}_K[X]$, we can consider the action of $q(\sigma)$ on K and the induced action of $\bar{q}(\bar{\sigma})$ on $\bar{K}$.

Definition 2.2. ([9]) We let $A = K[t; \sigma, \partial]$ be the skew polynomial ring in the variable t with the commutation rule for $k \in K$:

$$kt = tk^\sigma + \partial(k).$$

When $\partial = 0$, we will simply write $A = K[t; \sigma]$.

We let $A_0 = \mathcal{O}_K[t; \sigma, \partial]$ be the subring of A consisting of polynomials with coefficients from $\mathcal{O}_K$, and we let $\mathcal{I}$ be the set of polynomials from A_0 with at least one coefficient of valuation 0.

In A we will write the polynomials in the form $\sum t^i k_i$.

First, let us assume that $\partial = 0$. Then K can be considered as a module over A, with t acting as σ, in the following way :

$$k.(\sum_{i=0}^{n} t^i k_i) = \sum_{i=0}^{n} \sigma^i(k)k_i.$$

Let $\partial \neq 0$ be a non-trivial σ-derivation, then K can be considered as a module over A, with t acting as ∂, in the following way :

$$k.(\sum_{i=0}^{n} t^i k_i) = \sum_{i=0}^{n} \partial^i(k)k_i.$$

Note that when σ is not the identity, then as noted above ∂ is of the form $\partial(x) = (\sigma(x) - x).c$, and we are back to the first case. If we make the following change of variable, setting $t' := t.c^{-1} + 1$, we have that A is isomorphic to $K[t';\sigma]$ with the commutation rule $kt' = t'k^{\sigma}$. So, we basically have two cases: either t acts as σ (or a linear polynomial in σ) with $\partial = 0$, or t acts as a classical derivation with σ being the identity.

Let $(F((x^{-1})), \tau)$ be the field of Laurent series in x^{-1} over a field F, let τ be an automorphism of $F((x^{-1}))$. Let $K := F((x^{-1}))(T)$ with the T-adic valuation, let σ be the automorphism of K defined by $\sigma(\sum_i c_i T^i) := \sum_i \tau(c_i)T^i$. Then, with $\partial = 0$, t-motives in [15] are a special kind of A_0-modules.

Let $\bar{A} = \bar{K}[t;\bar{\sigma},\bar{\partial}]$. The reduction map $\bar{}$ makes $\bar{K}$ into a corresponding $\bar{A}$-module via the action of $\bar{\sigma}$ or $\bar{\partial}$.

The skew polynomial ring A is right Euclidean and left Euclidean (see [9], Chapter 2). In particular it is an integral domain and right Ore. The degree of an element $q(t) = \sum_{i=0}^{d} t^i k_i$ with $k_i \in K$ and $k_d \neq 0$, is equal to d.

The center of A is contained in $Fix(\sigma) \cap K_\partial$, where $K_\partial = \{x \in K : \partial(x) = 0\}$ and $Fix(\sigma) = \{x \in K : \sigma(x) = x\}$. Note that since a σ-derivation is of the form $c(\sigma - 1)$, whenever $\sigma \neq 1$, the subfield $Fix(\sigma) \cap K_\partial$ is either equal to K_∂ or to $Fix(\sigma)$.

Definition 2.3. Let $q(t) \in A$. We will say that $q(t)$ is *irreducible* if it cannot be expressed as a product of two elements of A of degree bigger than or equal to 1. We will say $q(t)$ is *separable* if $q(0) \neq 0$.

Since K is a valued field, we can extend the map v on the ring A as follows :

$$v(\sum_i t^i k_i) := \min_i \{v(k_i)\}$$

where $k_i \in K \setminus \{0\}$ and $v(0) = \infty$ (see [9], Chapter 9). In the case where $\partial = 0$, since $v(k^\sigma) = v(k)$ for any $k \in K$, this is a valuation (ibid., pp. 425-426), which can be extended to the fraction field of A (ibid., Proposition 9.1.1). For the convenience of the reader we will show that this is the case in general.

Note that $A_0 = \{q \in A : v(q) \geq 0\}$ and $\mathcal{I} = \{q \in A : v(q) = 0\}$.

Lemma 2.1. *The map v as defined above is a valuation on A.*

Proof: We need only check the valuation of a product. By induction on n, let us first show that

$$at^n = \sum_{i=0}^{n} t^i \, (\sum_{m_i \in C_i^n} a^{m_i(\sigma,\partial)})$$

where $m_i(\sigma, \partial)$ denotes a monomial of degree n in σ, ∂ with i occurrences of σ and C_i^n denotes the set of such monomials, whose cardinal is the corresponding binomial coefficient. We have

$$\begin{aligned} at^{n+1} &= (at^n)\,t = (\sum_{i=0}^{n} t^i\,(\sum_{m_i\in C_i^n} a^{m_i(\sigma,\partial)}))\,t \\ &= \sum_{i=0}^{n} t^{i+1}\,(\sum_{m_i\in C_i^n} a^{m_i(\sigma,\partial)\sigma}) + \sum_{i=0}^{n} t^i\,(\sum_{m_i\in C_i^n} a^{m_i(\sigma,\partial)\partial}) \\ &= t^{n+1}a^{\sigma^{n+1}} + \sum_{i=0}^{n-1} t^{i+1}\,(\sum_{m_i\in C_i^n} a^{m_i(\sigma,\partial)\sigma}) + \sum_{i=0}^{n-1} t^{i+1}\,(\sum_{m_i\in C_{i+1}^n} a^{m_i(\sigma,\partial)\partial}) + a^{\partial^n} \\ &= \sum_{i=0}^{n+1} t^i\,(\sum_{m_i\in C_i^{n+1}} a^{m_i(\sigma,\partial)}). \end{aligned}$$

Now consider

$$\begin{aligned} \sum_{i=0}^{n} t^i a_i \cdot \sum_{j=0}^{m} t^j b_j &= \sum_{i=0}^{n} t^i \sum_{j=0}^{m} a_i t^j b_j \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m}(\sum_{\ell=0}^{j} t^{\ell+i}\,(\sum_{m_\ell\in C_\ell^j} a_i^{m_\ell(\sigma,\partial)})\,b_j) \end{aligned}$$

Let $0 \le i_1 < \cdots < i_\mu \le n$ be the indices where $v(a_i)$, $i \in \{0,\cdots,n\}$, is minimum and let $0 \le j_1 < \cdots < j_\mu \le m$ be the indices where $v(b_j)$, $j \in \{0,\cdots,m\}$, is minimum. Let $k = i_\mu + j_\mu$. Then the coefficient of t^k is equal to $a_{i_\mu}{}^{\sigma^{i_\mu}} b_{j_\mu}{}^{\sigma^{j_\mu}} + \sum_{i,j} a_i^{m_i}.b_j^{m_j}$, where ∂ occurs at least once in m_i and m_j and $i+j>k$, so either $i > i_\mu$ and so $v(a_i^{m_i}) > v(a_{i_{k_1}})$ or $j > j_\mu$ and so $v(b_j^{m_j}) > v(b_{j_\mu})$. Hence the minimum valuation possible $v(a_{i_\mu}{}^{\sigma^{i_\mu}}) + v(b_{j_\mu}{}^{\sigma^{j_\mu}}) = v(a_{i_\mu}) + v(b_{j_\mu})$ is attained, as wanted. □

We will show that A_0 satisfies a generalized right Euclidean algorithm and so is right Ore. We will state that result in a more general setting.

Lemma 2.2. *Let D be a right Ore domain, α a monomorphism of D and ∂ a α-derivation. Then the skew polynomial ring $D[t;\alpha,\partial]$ satisfies a generalized right Euclidean algorithm. Namely, given any $q_1(t)$, $q_2(t)$ with $deg(q_1) \ge deg(q_2)$, there exist $c \in D\setminus\{0\}$ and $f,r \in D[t;\alpha,\partial]$ with $deg(r) < deg(q_2)$ such that $q_1c = q_2f + r$.*

Proof: (See [18], Theorem 2.14, p.128). By induction on $deg(q_1)+deg(q_2)$. Write $q_1 = \sum_{i=0}^{n} t^i a_i$ and $q_2 = \sum_{j=0}^{m} t^j b^j$, with $n \ge m$. Then $q_1 b^{\alpha^{n-m}} - q_2 t^{n-m} a_n$ is an element of A_0 of degree strictly less than n. So we apply the induction hypothesis to this polynomial. The degree 0 case is handled by the fact that D is right Ore. □

Note that $\mathcal{O}_K$ is a commutative integral domain and so it is certainly right Ore.

Definition 2.4. Let L_A be the language of A-modules and let T_A be the L_A-theory of right A-modules.

(1) Let T_d be the theory T_A together with the axioms $\forall m\ \exists n\ (n.q(t) = m)$, where $q(t)$ varies over the irreducible polynomials of A.

(2) Let $T_{d,\sigma}$ be the theory T_A together with:
   (2.1) $\forall m\, \exists n\ (m = n.t),\ \&\ \forall m\ (m.t = 0 \rightarrow m = 0)$,
   (2.2) $\forall m\, \exists n\ (n.q(t) = m)$, where $q(t)$ varies over the separable irreducible polynomials of A.

(3) Let T_{Ore} (respectively $T_{Ore,\sigma}$) be the theory T_d (respectively $T_{d,\sigma}$) with the axioms $\exists n \neq 0\ (n.q(t) = 0)$, where $q(t)$ varies over all irreducible (respectively separable irreducible) polynomials of A.

Let M be an A-module. We will denote by $Tor(M)$ the torsion part of M, namely $Tor(M) = \{m \in M : \exists q(t) \in A,\ m.q(t) = 0\}$. Note that since A is right Ore, $Tor(M)$ is a submodule of M. For each $a \in A$, $ann_M(a)$ is a $Fix(\sigma) \cap K_\partial$-vector space.

Since A is right and left Euclidean, the pp-formulas have a very simple form (see for instance [18], Theorem 3.8, p.181).

Recall that a theory of modules admits positive quantifier elimination if any positive primitive formula is equivalent to a finite conjunction of atomic formulas.

Proposition 2.3. *The theory T_d of A-modules admits positive q.e. Each completion of T_d admits quantifier elimination.*

Proof: This follows from classical results in the model theory of modules over right and left Euclidean rings and in particular from the fact that any existential formula is equivalent to annihilators conditions on the parameters (see [17], Proposition 2.7). □

We will now determine the completions of T_d. We will say that a right A-module is divisible if it is a model of T_d.

Lemma 2.4. *For any pair of elements $\{q_1(t), q_2(t)\}$ with $deg(q_1(t)) > deg(q_2(t))$ of A, we have that the following equivalences in any divisible A-module M:*

$$q_2(t) \text{ divides } q_1(t) \text{ if and only if } ann_M(q_2) \subseteq ann_M(q_1).$$

Moreover, if $q_1(t) = q_2(t).r(t)$ and if the cardinality of the quotient $ann(q_1(t))/ann(q_2(t))$ is finite, then $|ann(q_1(t))/ann(q_2(t))| = |ann(r(t))|$.

Proof: See [17], Lemma 2.9 and Proposition 2.10. □

Corollary 2.5. *If the subfield $Fix(\sigma) \cap K_\partial$ of K is infinite, then the completions of T_d are obtained by specifying for which irreducible polynomials $q(t)$ whether $ann(q(t)) \neq \{0\}$. In particular under this assumption T_{Ore} (respectively $T_{Ore,\sigma}$) admits quantifier elimination. If $Fix(\sigma) \cap K_\partial$ is finite, then the completions of T_d are obtained by specifying for each polynomial $q(t)$ if the cardinal $|ann(q(t))|$ is finite and giving its value.*

Proof: First, we observe that if an element m does not belong to a pp-definable subgroup S, then $\lambda.m$ does not belong to that subgroup for any $\lambda \in Fix(\sigma) \cap K_\partial$: let $\lambda \neq \mu\ \in Fix(\sigma) \cap K_\partial$ and assume that $\lambda.m$ and $\mu.m$ belong to the same coset, then $(\lambda - \mu).m \in S$, so $m \in S$ a contradiction.

So, if $Fix(\sigma) \cap K_\partial$ is infinite, then the index of a pair of pp-definable subgroups is either 1 or infinite.

Second, by the positive quantifier elimination result, the proper definable subgroups are the annihilators of elements of A.

So in the first case, in order to determine the completions, by the above lemma, it suffices to determine which irreducible elements $q(t)$ of A have a non-zero annihilator.

When $Fix(\sigma) \cap K_\partial$ is finite, the completions are determined by the cardinalities of non-trivial quotients of the annihilators of elements of A, which by the above Lemma can be reduced to the cardinalities of the annihilators. $\square$

Let N be a pure-injective indecomposable model of T_d. Let $J_N := \{a \in A : \exists n \in N \setminus \{0\}, n.a = 0\}$ and suppose $Tor(N) \neq \{0\}$. Denote the subset of separable irreducibles elements of A by $\mathcal{P}$.

Proposition 2.6. *Let $N_1,\ N_2$ be two pure-injective indecomposable models of T_d such that $Tor(N_1) \neq \{0\}$. Then, there exists $q_0(t) \in A$ such that $J_{N_1} \cap \mathcal{P} = \{q(t) \in A : deg(q(t)) = deg(q_0(t))\ \&\ \exists q_1 \exists q_2\ q(t).q_1 = q_2.q_0,\ \ \text{with } deg(q_1), deg(q_2) < deg(q.q_0)\}$. Also, for any $a_1,\ a_2 \in J_{N_1} \cap \mathcal{P}$, we have that $ann_{N_1}(a_1) \cong ann_{N_1}(a_2)$.*

Moreover, if $J_{N_1} \cap J_{N_2} \cap \mathcal{P} \neq \{0\}$, then $N_1 \cong N_2$.

Proof: See [10], Lemma 3.14 . $\square$

Lemma 2.7. *Let M be an A-module, where each element is divisible by t. Then, M is A-divisible iff it is $\mathcal{I}$-divisible.*

Proof: Let $r(t) := \sum_i t^i k_i \in A \setminus A_0$, let λ such that $v(\lambda) = \min_i\{v(k_i)\} \neq 0$. W.l.o.g., we may assume that $r(t)$ is separable. Set $r_1(t) := \lambda^{-1} r(t)$, we have $v(r_1(t)) = 0$ i.e. $r_1(t) \in \mathcal{I}$. Let $m \in M$. Since M is $\mathcal{I}$-divisible, there exists n such that $m = n.r_1(t) = n.\lambda^{-1}.r(t)$. $\square$

We will now determine conditions which ensures that (K, v, σ, ∂) is a model of T_d or $T_{d,\sigma}$. We will state everything directly in terms of modules. The reader is invited to make the direct translation in terms of linear σ-equations and linear ∂-equations.

Definition 2.5. Let $M = (K, v, \sigma, \partial)$ be as above and viewed as an A-module. We will say that M has the *linear Hensel property*, if for any $q(t) \in \mathcal{I}$ and $m \in \mathcal{O}_K$, if there is $y \in \mathcal{O}_K$ such that $\bar{y}.\bar{q}(t) = \bar{m}$, then there exists $x \in \mathcal{O}_K$ such that $x.q(t) = m$ and $\bar{x} = \bar{y}$.

In our basic case of the action of the isometry σ, this is the σ-Hensel scheme in [6] but restricted to linear σ-polynomials. In the case of the action of ∂, this is the D-henselian property of [27] but restricted to linear D-polynomials. A henselian property for polynomial equations involving contractive endomorphisms (even many at a time) has also been considered in [25].

Proposition 2.8. *Any $M = (K, v, \sigma, \partial)$ such that (K, v) is a complete discrete valued field and $\bar{M} = (\bar{K}, \bar{\sigma}, \bar{\partial})$ is a divisible $\bar{A}$-module, has the linear Hensel property.*

Proof:

Let $q(t) \in \mathcal{O}_K[t; \sigma, \partial]$ such that $\bar{q}(t) \neq 0$. Let $m \in \mathcal{O}_K$, and assume that there exists $u_0 \in \mathcal{O}_K$ such that $\bar{u}_0.\bar{q}(t) = \bar{m}$. We want to find $u \in \mathcal{O}_K$ such that $u.q(t) = m$ and $v(u - u_0) > 0$.

Set $Q_0 = u_0.q(t) - m$, we have that $v(Q_0) > 0$.

We look for an element $x \in \mathcal{O}_K$ such that

$$v((u_0 + xQ_0).q(t) - m) > v(Q_0).$$

We have $u_0.q(t) + xQ_0.q(t) - m = Q_0 + x.Q_0q(t)Q_0^{-1}Q_0 = (1 + x.Q_0q(t)Q_0^{-1})\, Q_0$. Note that $v(Q_0q(t)Q_0^{-1}) = v(q(t)) = 0$.

Since $\bar{K}$ is $\bar{A}$-divisible, there exists $x_1 \in \mathcal{O}_K$ such that

$$v(1 + x_1.Q_0q(t)Q_0^{-1}) > 0$$

So, by letting $h_1 = x_1Q_0$ and $u_1 = u_0 + h_1$, we get

$$u_1.q(t) - m = (1 + x_1.Q_0q_1(t)Q_0^{-1})\, Q_0$$

and the desired inequality $v(u_1.q(t) - m) > v(Q_0) = v(u_0.q(t) - m)$.

We now generate, by induction, sequences $u_n, h_n, Q_n, x_n \in \mathcal{O}_K$ such that $u_{n+1} = u_n + h_{n+1}, h_{n+1} = x_{n+1}Q_n, Q_n = u_n.q(t) - m$ and that the sequence $v(Q_n)$ is strictly increasing. Indeed, as above, there exists $x_{n+1} \in \mathcal{O}_K$ such that $v(1 + x_{n+1}.Q_nq(t)Q_n^{-1}) > 0$, and letting $h_{n+1} = x_{n+1}Q_n, u_{n+1} = u_n + h_{n+1}, Q_{n+1} = u_{n+1}.q(t) - m$, we get the desired inequality $v(Q_{n+1}) > v(Q_n)$. Now, the sequence $(u_n)_{n\in\omega}$ converges since $u_{n+1} - u_n = x_{n+1}Q_n$ with $x_{n+1} \in \mathcal{O}_K$ and $v(Q_n) \to +\infty$. Let $u = \lim_n u_n$. Then, $u.q(t) - m = u.q(t) - u_n.q(t) + (u_n.q(t) - m) = (u - u_n).q(t) + Q_n$. Since $v((u - u_n).q(t)) \geq v(u - u_n)$, we get that $u.q(t) - m = 0$. Moreover, $v(u_0 - u) > 0$. □

Corollary 2.9. *Assume that $M = (K, \sigma, \partial, v)$ has the linear Hensel property, each element is $t-$divisible and that $\bar{M}$ is $\bar{A}$-divisible. Then M is a model of T_d.*

Proof: By the linear Hensel property and $\bar{A}$-divisibility, M is $\mathcal{I}$-divisible. Then apply Lemma 2.7. □

Corollary 2.10. *Let F be a field of characteristic p which is p-closed.*

1. *Let $K = W(F)$ with σ the Witt Frobenius. Then $W(F)$ as an A-module with t acting as the Witt Frobenius is a model of $T_{d,\sigma}$.*
2. *Let $K = F((x))$, the field of Laurent series, with σ defined as $\sigma(\sum c_i x^i) = \sum c_i^p x^i$. Then $F((x))$ as an A-module with t acting as σ is a model of $T_{d,\sigma}$.*

Proof: Axiom (a) is direct since we have an automorphism. Note that $\bar{K} = F$ and $\bar{\sigma}(x) = x^p$. We then use the fact that p-closed implies that every equation $\sum c_i x^{p^i} = c$, with $c_i, c \in F$ has a solution in F. But this amounts to $\bar{K}$ being $\bar{A}$-divisible, and Axiom (b) follows from the linear Hensel property and lemma 2.7. □

We will return at the end of this paper to the relation between these two models. It is interesting to note that the general setting in e.g. [6] does not cover the case of equal characteristic p, like the field of Laurent series $F((x))$ above (but see [3] in that direction.)

We may apply these results to $F = \tilde{\mathbb{F}}_p$, the algebraic closure of the prime field $\mathbb{F}_p$, and to $F = k_p = \bigcup_{n\in\omega} \mathbb{F}_{p^{p^n}}$ which is the minimal p-closed field (N.B. Its theory is decidable ([2])) . So, both $W(\tilde{\mathbb{F}}_p)$, $W(k_p)$ are models of the theory $T_{d,\sigma}$ with the corresponding skew polynomial ring. Since $ann(q(t)) \neq \{0\}$ with $q(t)$ a separable irreducible polynomial, their theories are axiomatized by $T_{Ore,\sigma}$ and they admit quantifier elimination.

Note that there are non algebraically closed fields of characteristic p where each separable additive polynomial has a non trivial zero (see [6]). We don't know whether if we require in addition the field to be p-closed that it implies it is algebraically closed. We also don't know if it is possible to characterize the elements of $W(\tilde{\mathbb{F}}_p)$ which belong to the torsion submodule and to $ann(q(t))$ for a specific $q(t)$.

The question whether the theory $T_{Ore,\sigma}$ is decidable (at least in the classical sense) only makes sense if the ring A is countable and such that some part of its existential theory is decidable. Another way to proceed would be to consider decidability questions working with BSS-machines instead of Turing machines (see [8]). In particular, this implies that the elements of the field are given.

Corollary 2.11. *Let K be an elementary substructure of $W(F)$ where F is a field of characteristic p which is p-closed. Assume that K is recursively presented with decidable word problem. Let $B := K[t;\sigma]$. Then, the corresponding theory $T_{Ore,\sigma}$ is consistent and decidable.*

Proof: One checks that the ring B satisfies the required condition D in [24], Chapter 17, p. 334. □

Here are assumptions on K which ensure that $Fix(\sigma)$ is infinite. In case $K = F((x^{-1}))$, this is a special case of a theorem of Hellegouarch (see [14], Theorem 1).

Proposition 2.12. *Consider (K, v, σ). Assume that $\bar{\sigma}$ is surjective on $\bar{K}$ and that K has the linear Hensel property with t acting as σ. Then, $Fix(\sigma)$ is infinite whenever the valuation is non trivial.*

Proof: We can assume K has finite characteristic. Let $x \in K$ with $v(x) > 0$. This implies in particular that x is transcendental over the prime subfield of K. Since σ is an isometry, we have that $v(\sigma(x)) = v(x)$. In other words there exists u with $v(u) = 0$ such that $\sigma(x) = x.u$. First, assume that $\bar{u} = 1$. We look for an element $a \in K$ with $v(a) = 0$ such that $\sigma(x.a) = x.a$. So, $x.u.a^{\sigma} = x.a$, i.e. $a^{\sigma}.u = a$. We have to check that we can solve the equation $a.(tu - 1) = 0$ residually, but since we assumed that $\bar{u} = 1$, we can solve it residually by setting $\bar{a} = 1$. So by the linear Hensel property, we get that K has a solution of that equation. Since the solution is again necessarily transcendental over the prime field, this ensures that $Fix(\sigma)$ is infinite. Suppose now that $\bar{u} \neq 1$, then we look for an element u_1 such that $\sigma(x.u_1) = x.u.u_1^{\sigma}$ with $\overline{u.u_1^{\sigma}} = 1$. But we can solve the equation $\bar{u}_1^{\sigma} = (\bar{u})^{-1}$, since $\bar{\sigma}$ is surjective on $\bar{K}$. □

Now we give some examples involving a derivation ∂.

Let (F, ∂) be a differential field of characteristic 0. Assume its field of constants is algebraically closed. Then, it is known that there exists an extension $\tilde{F}$ of F with no new constants such that any finite subset of $\tilde{F}$ is included in a finite sequence of successive Picard-Vessiot extensions and such that $\tilde{F}$ has no proper Picard-Vessiot extensions (see [22], Theorem 3.34).

First consider $(\tilde{F}((x)), v)$ with the x-adic valuation and ∂ defined by $\partial(\sum_{i\geq m} x^i.a_i) = \sum_{i\geq m} x^i.\partial(a_i)$. Then, ∂ is contractive. Second, consider $(\tilde{F}((x^{-1})), v)$ with the x^{-1}-adic valuation and ∂ defined by $\partial(\sum_{i\geq m} a_i.x^{-i}) = (\sum_{i\geq m}(-i.a_i + \partial(a_{-i+1})).x^{-i+1}$. Then, again ∂ is contractive.

So, we obtain the following corollary.

Corollary 2.13. *Let K be either the field of Laurent series $(\tilde{F}((x)), v, \partial)$ or $(\tilde{F}((x^{-1})), v, \partial)$. Then K, viewed as an A-module with t acting as ∂, is a model of T_{Ore}.*

Proof: We will prove at the same time that they are models of T_d and T_{Ore}. By Proposition 2.8 and Corollary 2.9, to prove that they are models of T_d it suffices to check that they are t-divisible and that $\tilde{F}$ is $\bar{A}$-divisible.

First note that any inhomogeneous linear differential equation of order $n \geq 1$, of the form $L(y) = b$, $b \in \tilde{F}$, can be reduced to the homogeneous equation $b.\partial(1/b.L(y)) = 0$. Now $\tilde{F}$ has no proper Picard-Vessiot extension. So, we get all the solutions in $\tilde{F}$, so it is $\bar{A}$-divisible and moreover we get non-trivial solution to any homogeneous linear differential equation.

Then, it remains to check that for any element $c \in \tilde{F}((x))$ there is a a such that $\partial(a) = c$. In the first case, this follows directly from the fact that $\tilde{F}$ has no proper Picard-Vessiot extension. In the second case, let $c = \sum_{i\geq m} c_i.x^{-i}$, then we look for elements $a_i \in \tilde{F}$ such that $\sum_{i\geq m} c_i.x^{-i} = \sum_{i\geq m}(-i.a_i+\partial(a_{-i+1})).x^{-i+1}$. Equivalently, for $i \geq m$; $c_i = (i+1).a_{i+1} + \partial(a_i)$. So, if $i+1 \neq 0$, we set $a_{i+1} = \frac{c_i}{i+1}$, and if $i+1 = 0$, then we look for an element a_i such that $\partial a_i = c_i$. $\square$

A subgroup H of a group G is definably connected (in G) if it does not contain any proper relatively definable subgroup of finite index (a relatively definable subgroup of H is the intersection of H with a definable subgroup of G). Recall that in any module, a definable subgroup has a pp-definable subgroup of finite index ([16], p.140) as a consequence of B.H. Neumann's Lemma. So, if we assume that $Fix(\sigma) \cap K_\partial$ is infinite (for instance in the case where $K = W(\tilde{\mathbb{F}}_p)$ with the Witt Frobenius, $Fix(\sigma) = \mathbb{Q}_p$), then no definable subgroup has a proper definable subgroup of finite index. Since any pp-definable subgroup is $\emptyset$-definable, then any definable subgroup is connected and $\emptyset$-definable.

Corollary 2.14. *Assume that $Fix(\sigma) \cap K_\partial$ is infinite. Then the valuation ring $\mathcal{O}_K$ of K is not definable in the language L_A of modules.*

Proof: By the above, it would be pp-definable, and so invariant by multiplication by elements of $Fix(\sigma) \cap K_\partial$, a contradiction. $\square$

3. Valued modules and abelian structures.

We will now fix the structures of valued modules we will be dealing with.

We keep the same notation as in the previous section, with a fixed (K, v, σ, ∂), A the skew polynomial ring $K[t; \sigma, \partial]$, etc.

Definition 3.1. (cf. [11]) A *valued A-module* is a structure $(M, \Delta, \leq, +, w, \infty)$, where M is an A-module, $\infty \in \Delta$, $(\Delta, \leq)$ is a totally ordered set for which ∞ is a maximum, $+$ is an action of vK on $(\Delta, \leq)$ and w is a surjective map $w : M \to \Delta$ such that

(1) For all $\delta_1, \delta_2 \in \Delta, \gamma_1, \gamma_2 \in vK$, if $\delta_1 \leq \delta_2$ and $\gamma_1 \leq \gamma_2$ then $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$.
(2) For all $m_1, m_2 \in M$, $w(m_1 + m_2) \geq \min\{w(m_1), w(m_2)\}$, and $w(m_1) = \infty$ iff $m_1 = 0$.
(3) For all $m \in M, w(m.t) \geq w(m)$.
(4) For all $m \in M, \forall \lambda \in K, \lambda \neq 0, w(m.\lambda) = w(m) + v(\lambda)$.

Taking $M = (K, v, \sigma, \delta)$ and $w = v$, it is a valued A-module in either case of t acting as σ or ∂.

From the axioms above, we deduce as usual the following properties : $w(m) = w(-m)$, and if $w(m_1) < w(m_2)$, then $w(m_1 + m_2) = w(m_1)$.

Definition 3.2. We let L_w be the two-sorted language of valued A-modules obtained from L_A, with a sort M for the underlying module, and a sort Δ for the ordered set of valuations, a constant symbol ∞ of sort Δ, and unary function symbols $+\gamma$ for each $\gamma \in vK$. We define the following L_w-theories:

- let T_w be the theory of valued A-modules obtained by translating the required axioms in L_w.
- let $T_w^* = T_w \cup T_d$.

Let $M \models T_w$, then we define

$$M_\delta := \{m \in M : \ w(m) \geq \delta\}$$

This is not only a subgroup of M, but an A_0-submodule.

In order to go into the setting of abelian structures, we will now introduce another (less expressive) language. This is the language of [26], where Rohwer was considering the field of Laurent series over the prime field $\mathbb{F}_p$ with the usual Frobenius map $y \mapsto y^p$ (which is not an isometry). If (M, Δ, w) is a valued A-module, we have in mind the structure $(M, (M_\delta)_{\delta \in \Delta}))$.

Definition 3.3. Let $(\Delta, \leq)$ be a fixed totally ordered set with an action $+$ of vK on Δ such that for all $\delta_1, \delta_2 \in \Delta, \gamma_1, \gamma_2 \in vK$, if $\delta_1 \leq \delta_2$ and $\gamma_1 \leq \gamma_2$ then $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$. We let L_V be the language consisting of the language L_A of A-modules together with a set of unary predicates V_δ, indexed by the elements of Δ.

Definition 3.4. Let T_V be the L_V-theory obtained from T_A together with the following axioms $(1) - (7)$. Let T_V^* be the theory T_V together with axiom scheme (8).

(1) $\forall m \, \exists n \ (m = n.t)$.
(2) $\forall m \, \exists n \, (n.q(t) = m)$, where $q(t) \in \mathcal{P}$.
(3) $\forall m \, (V_{\delta_1}(m) \to V_{\delta_2}(m))$, whenever $\delta_1 \leq \delta_2$.
(4) $\forall m \ (V_\delta(m) \to V_\delta(m.t))$.
(5) $\forall m_1 \, \forall m_2 \ (V_\delta(m_1) \,\&\, V_\delta(m_2) \to V_\delta(m_1 + m_2))$.
(6) $\forall m \ (V_\delta(m) \to V_{\delta + v(\lambda)}(m.\lambda))$, where $\lambda \in K$.
(7) $\forall m \ (V_\delta(m) \to V_{\delta + v(q(t))}(m.q(t)))$, where $q(t) \in K[t, \sigma]$.
(8) $\forall m \in V_\delta \, \exists n \in V_\delta \ n.q(t) = m$, where $q(t) \in \mathcal{I}$.

If (M, Δ, w) is a valued A-module, let $\mathcal{M}$ be the L_V-structure $(M, +, 0, (.r)_{r\in A}, (M_\delta)_{\delta\in\Delta})$, where V_δ is interpreted as M_δ.

Example 1. *Let F be a p-closed field of characteristic p. If $M = K = W(F)$ with t acting as the Witt Frobenius, or if $M = K = F((x))$ with t acting as $\sum c_i x^i \mapsto \sum c_i^p x^i$, then $\mathcal{M} \models T_V^*$, as in Corollary 2.10.*

Example 2. *Let (F, ∂) be a differential field of characteristic 0 whose field of constants is algebraically closed. Let K be either the field of Laurent series $(\tilde{F}((x)), v, \partial)$ or $(\tilde{F}((x^{-1})), v, \partial)$ as in Corollary 2.13. Then K, viewed as an A-module with t acting as ∂, is a model of T_V^*.*

The structure $\mathcal{M}$ is an abelian structure and one gets as in the classical case of (pure) modules that any formula is equivalent to a boolean combination of pp-formulas and index sentences (namely, sentences telling the index of two pp-definable subgroups in one another; for all this see [24]). Moreover, this elimination is uniform in the class of such structures.

Note that a positive primitive formula $\phi(\mathbf{x})$ is now of the form:

$$\exists y_1 \exists y_2 \cdots \exists y_n \bigwedge_i (\mathbf{x}, \mathbf{y}).B = 0 \;\&\; V_{\gamma_i}(t_i(\mathbf{x}, \mathbf{y})),$$

where B is a matrix with coefficients in A, $\gamma_i \in \Gamma$ and $t_i(\mathbf{x}, \mathbf{y})$ is a term in the language L_A.

In the next section, we will prove a positive quantifier elimination result for these abelian structures.

4. Quantifier elimination for valued modules considered as abelian structures.

This section will be devoted to the proof that T_V^* admits positive quantifier elimination. This means that for any formula $\phi(\mathbf{x}, \mathbf{y})$ which is a conjunction of atomic formulas, the existential formula $\exists \mathbf{x}\, \phi(\mathbf{x}, \mathbf{y})$ is equivalent to a conjunction of atomic formulas. By our previous remark about general pp-elimination in abelian structures, it will imply that any formula is equivalent to a quantifier free formula and index sentences.

The subgroups defined by the unary predicates V_δ are A_0-modules. We will use the fact that A_0 satisfies a generalized right (henceforth g.r.) Euclidean algorithm (cf. Lemma 2.2).

Proposition 4.1. *T_V^* admits positive quantifier elimination.*

Proof: We can proceed by induction on the number of existential quantifiers, so it suffices to consider a formula existential in just one variable $\exists x \phi(x, \mathbf{y})$, where $\phi(x, \mathbf{y})$ is a conjunction of atomic formulas. Then, since A is right Euclidean, we can always assume that we have at most one equation involving x.

Let $\phi(x, \mathbf{y})$ be a positive quantifier-free formula of the form

$$x.r_0 = t_0(\mathbf{y}) \;\&\; \bigwedge_{i=1}^{n} V_{\delta_i}(x.r_i - t_i(\mathbf{y})) \;\&\; \theta(\mathbf{y})$$

where $r_i \in A$, $\theta(\mathbf{y})$ is a quantifier-free pp-formula, the $t_i(\mathbf{y})$ are L_A-terms, and $\delta_i \in \Delta$ with $\delta_1 \geq \delta_2 \geq \cdots \delta_n$. Consider $\exists x\, \phi(x, \mathbf{y})$. It suffices to show that any such formula is equivalent to a positive quantifier-free formula.

First, we note that for each i there exists a non-zero $\lambda_i \in K$ such that $r_i\lambda_i \in \mathcal{I}$. Then, in T_V^*, we can replace $V_{\delta_i}(x.r_i - t_i(\mathbf{y}))$ by $V_{\delta_i+v(\lambda_i)}(x.r_i.\lambda_i - t_i(\mathbf{y}).\lambda_i)$ and $x.r_0 = t_0(\mathbf{y})$ by $x.r_0.\lambda_0 = t_0(\mathbf{y}).\lambda_0$, so we can always assume that the $r_i \in \mathcal{I}$, for all i.

Note also that we can always assume that $deg(r_0) > deg(r_i)$, for all i. Indeed, suppose that $deg(r_0) \leq deg(r_i)$, for some i, say $i = 1$. By the g.r. Euclidean algorithm in A_0, there exists $\lambda \in \mathcal{O}_K$ such that $r_1.\lambda = r_0.r + r_1'$ with $deg(r_1') < deg(r_0)$ and $r,\ r_1' \in A_0$. So, we have that $x.r_1.\lambda = x.r_0.r + x.r_1 = t_0.r + x.r_1'$, and we can replace $V_{\delta_1}(x.r_1 + t_1)$ by $V_{\delta_1+v(\lambda)}(x.r_1' + t_0(\mathbf{y}).r - t_1(\mathbf{y}))$.

We will call *normalization* the process of going through the last two reductions and re-indexing if necessary to keep the condition $\delta_1 \geq \delta_2 \geq \cdots \delta_n$. We will use the notation $x.r \equiv_\delta u$ to mean that $V_\delta(x.r - u)$ holds.

First, we will assume that there is one equation present in $\phi(x, \mathbf{y})$. We will concentrate on the system formed by this equation and the "congruences".

Consider the system (1):

$$(1)\ :\ x.r_0 = t_0,\ x.r_1 \equiv_{\delta_1} t_1,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

with $deg(r_0) > deg(r_i)$, $r_0, r_i \in \mathcal{I}$, $t_0 = t(\mathbf{y}), t_i = t_i(\mathbf{y})$, $1 \leq i \leq n$.

Applying the g.r. Euclidean algorithm, we get some $\lambda \in \mathcal{O}_K$ and $s,\ s_1 \in A_0$ such that $r_0.\lambda = r_1.s + s_1$ with $deg(s_1) < deg(r_1)$.

We claim that system (1) is equivalent to the following system (2) :

$$(2)\ :\ x.r_1 = t_1,\ x.s_1 \equiv_{\delta_1+v(\lambda)} t_0.\lambda - t_1.s,\ x.r_2 \equiv_{\delta_2} t_2,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

(1) $\rightarrow$ (2)

Let x satisfy (1). Then $x.r_1 = t_1 + a$ for some $a \in V_{\delta_1}$. By the divisibility condition on V_{δ_1} there exists $u \in V_{\delta_1}$ such that $u.r_1{=}a$. Then, we obtain $(x - u).r_1 = t_1$ and $(x-u).s_1 = (x-u).r_0.\lambda - (x-u).r_1.s$. So, $(x-u).s_1 = t_0.\lambda - t_1.s + (-u).r_0.\lambda$. Since $(-u).r_0 \in V_{\delta_1}$, we get that $(-u).r_0.\lambda \in V_{\delta_1+v(\lambda)}$. Since $\delta_1 \geq \delta_2 \cdots$, we have $u \in V_{\delta_i}$ and $u.r_i \in V_{\delta_i}$, for $i \geq 2$, so that $x - u$ still satisfies the other congruence conditions.

(2) $\rightarrow$ (1)

Let x satisfy (2). We look for an element $u \in V_{\delta_1}$ such that $(x + u).r_0 = t_0$, or equivalently $(x+u).r_0.\lambda = t_0.\lambda$. Replacing $r_0.\lambda$ by $r_1.s+s_1$, we obtain: $(x+u).r_1.s+(x + u).s_1 = t_0.\lambda$. We have: $t_1.s + u.r_1.s + t_0.\lambda - t_1.s + a + u.s_1 = t_0.\lambda$, for some $a \in V_{\delta_1+v(\lambda)}$. So, u has to satisfy $u.r_0.\lambda + a = 0$, but by the divisibility property of $V_{\delta_1+v(\lambda)}$, we can find such an element. It remains to check that $x + u$ satisfies the other conditions. But, $x.r_1 = t_1$ and since $u \in V_{\delta_1}$ we have that $(x + u).r_1 \equiv_{\delta_1} t_1$, and similarly for the other congruence conditions since $u \in V_{\delta_i}$.

By this device, we replace a system where the couple (r_0, r_1) occurred by a system where (r_1, s_1) occurs with $deg(r_0) > deg(r_1) > deg(s_1)$. We might have to normalize the new system.

Second, we will consider the case where there are only congruence relations in the system.

Consider the following system (3):

$$(3)\ :\ x.r_1 \equiv_{\delta_1} t_1,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

with $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, $r_i \in \mathcal{I}$, $1 \leq i \leq n$.

i) We assume that $deg(r_1) \geq deg(r_2)$.

Applying the g.r. Euclidean algorithm, we get some $\lambda \in \mathcal{O}_K$ and and $s, s_2 \in A_0$ such that $r_1\lambda = r_2 s + s_2$ with $deg(s_2) < deg(r_2)$.

We claim that system (3) is equivalent to the following system (4_1):

$$(4_1)\ :\ x.r_2 = t_2,\ x.s_2 \equiv_{\delta_2+v(\lambda)} t_1.\lambda - t_2.s,\ x.r_3 \equiv_{\delta_3} t_3,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

$(3) \to (4_1)$

Let x satisfy (3). Then $x.r_2 = t_2 + a$ and $x.r_1 = t_1 + b$ for some $a \in V_{\delta_2}$ and $b \in V_{\delta_1} \subseteq V_{\delta_2}$. By the divisibility condition on V_{δ_2} there exists $u \in V_{\delta_2}$ such that $u.r_2{=}a$. Then, we obtain $(x-u).r_2 = t_2$ and $(x-u).s_2 = (x-u).r_1\lambda - (x-u).r_2 s = t_1.\lambda - t_2.s + b.\lambda + (-u).r_1.\lambda$. Since $(-u).r_1 \in V_{\delta_2}$, we get that $(-u).r_1.\lambda,\ b.\lambda \in V_{\delta_2+v(\lambda)}$. So $(x-u) \equiv_{\delta_2+v(\lambda)} t_1.\lambda - t_2.s$. The other congruence conditions are still satisfied by $x-u$ since $u \in V_{\delta_2} \subseteq V_{\delta_i}$, for $i \geq 3$.

$(4_1) \to (3)$

Let x satisfy (4_1). We look for an element $u \in V_{\delta_2}$ such that $(x+u).r_1 \equiv_{\delta_1} t_1$ or equivalently $(x+u).r_1.\lambda \equiv_{\delta_1+v(\lambda)} t_1.\lambda$. Replacing $r_1.\lambda$ by $r_2.s + s_2$, we obtain $(x+u).r_2.s + (x+u).s_2 \equiv_{\delta_1+v(\lambda)} t_1.\lambda$. So we have $t_2.s + u.r_2.s + t_1.\lambda - t_2.s + a + u.s_2 \equiv_{\delta_1+v(\lambda)} t_1.\lambda$, for some $a \in V_{\delta_2+v(\lambda)}$. In fact, we can ask that u satisfies $u.r_1.\lambda + a = 0$. But by the divisibility property of $V_{\delta_2+v(\lambda)}$ we can find such an element. It remains to check that $(x+u).r_i \equiv_{\delta_1} t_i$, for $i \geq 2$. But this follows from $x.r_i \equiv_{\delta_i} t_i$ and $u \in V_{\delta_2} \subseteq V_{\delta_j}$, $j \geq 3$.

By this device, we replace a system where the couple (r_1, r_2) occurred by a system where (r_2, s_2) occurs with $deg(r_1) \geq deg(r_2) > deg(s_2)$. Then, we normalize the system and we note that system (4_1) is of the same kind as system (1).

ii) Assume that $deg(r_1) < deg(r_2)$.

Applying the g.r. Euclidean algorithm, we get $\lambda \in \mathcal{O}_K$, and $s,\ s_2 \in A_0$ such that $r_2.\lambda = r_1.s + s_2$ with $deg(s_2) < deg(r_1)$.

We claim that (3) is equivalent to the following system (4_2):

$$(4_2)\ :\ x.r_1 = t_1,\ x.s_2 \equiv_{\delta_2+v(\lambda)} t_2.\lambda - t_1.s,\ x.r_3 \equiv_{\delta_3} t_3,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

$(3) \to (4_2)$

Let x satisfy (3). Then $x.r_1 = t_1 + a$ and $x.r_2 = t_2 + b$, for some $a \in V_{\delta_1}$ and $b \in V_{\delta_2}$. By the divisibility condition on V_{δ_1} there exists $u \in V_{\delta_1}$ such that $u.r_1{=}a$. Then, we obtain $(x-u).r_1 = t_1$ and $(x-u).s_2 = (x-u).r_2.\lambda - (x-u).r_1.s = t_2.\lambda + b.\lambda - t_1.s + (-u).r_2.\lambda$. Since $(-u).r_2 \in V_{\delta_1}$, we get $(-u).r_2.\lambda \in V_{\delta_1+v(\lambda)} \subseteq V_{\delta_2+v(\lambda)}$ and $b.\lambda \in V_{\delta_2+v(\lambda)}$. The other congruence conditions are still satisfied by $x-u$ since $u \in V_{\delta_1} \subseteq V_{\delta_i}$, for $i \geq 2$.

$(4_2) \to (3)$
Let x satisfy (4_2). Replacing $r_2.\lambda$ by $r_1.s + s_2$, we obtain $x.r_1.s + x.s_2 \equiv_{\delta_2+v(\lambda)} t_2.\lambda$. So we have $t_1.s + t_2.\lambda - t_1.s + a \equiv_{\delta_2+v(\lambda)} t_2.\lambda$, for some $a \in V_{\delta_2+v(\lambda)}$. So, $x.r_2 \equiv_{\delta_2} t_2$.

Note that in each case, we have decreased the degree of each of the coefficients, and we strictly decreased the sum of the degrees of the elements occurring in the congruence conditions. In the end, we will decrease the degree of the element occurring in the congruence relation corresponding to the smallest subgroup V_δ.

So, we may assume that we reduce ourselves to a system consisting of possibly a conjunction of congruence conditions on the parameters and:

$$(*) \ : \ x.r = u(\mathbf{y}),\ x.\mu \equiv_\delta t'(\mathbf{y}),\ x.r'_3 \equiv_{\delta'_3} t'_3(\mathbf{y}),\ \cdots,\ x.r'_\ell \equiv_{\delta'_\ell} t'_\ell(\mathbf{y})$$

where $\mu \in \mathcal{O}_K$, $r, r'_i \in \mathcal{I}$, $\delta \geq \delta'_3 \geq \cdots \geq \delta_\ell$, $3 \leq i \leq \ell \leq n$. Note that if $\mu = 0$, we strictly decreased the number of congruence conditions in our system. If $\mu \neq 0$, we replace the first congruence by $x \equiv_{\delta+v(\mu^{-1})} t'(\mathbf{y}).\mu^{-1}$. If $V_{\delta+v(\mu^{-1})}$ is no longer the smallest subgroup we continue the above procedure with the smallest subgroup. But after a finite number of steps, we will obtain a system of the form $(*)$ with the second formula of the form $x.\mu \equiv_\delta t'(\mathbf{y})$, where we may now assume w.l.o.g. that $\mu = 1$.

To finish, we observe that the above system $(*)$ with $\mu = 1$ is equivalent to the following congruence conditions on the parameters:

$$(**) \ : \ t'(\mathbf{y}).r \equiv_\delta u(\mathbf{y}),\ t'(\mathbf{y}).r'_3 \equiv_{\delta'_3} t'_3(\mathbf{y}),\ \cdots,\ t'(\mathbf{y}).r'_\ell \equiv_{\delta'_\ell} t'_\ell(\mathbf{y})$$

Indeed, assume that $(**)$ holds, in particular $t'(\mathbf{y}).r \equiv_\delta u(\mathbf{y})$. So we have that $t'(\mathbf{y}).r = u(\mathbf{y}) + b$ for some $b \in V_\delta$. By the divisibility property of V_δ there exists $c \in V_\delta$ such that $c.r = b$. Then $(t'(\mathbf{y}) - c).r = u(\mathbf{y})$. Since V_δ is the smallest subgroup, the element $t'(\mathbf{y}) - c$ still satisfies the other conditions of $(*)$ with $\mu = 1$. □

Corollary 4.2. *Let F be a field of characteristic p which is p-closed.*

1. *The L_V-structure $(W(F), (W(F)_\delta)_{\delta\in\mathbb{Z}})$ admits quantifier elimination, for the Witt Frobenius action.*
2. *The L_V-structure $(F((x)), (F((x))_\delta)_{\delta\in\mathbb{Z}})$ admits quantifier elimination, for the action by $\sum c_i x^i \mapsto \sum c_i^p x^i$.*

Corollary 4.3. *Let (F, ∂) be a differential field of characteristic 0 whose field of constants is algebraically closed. Let K be either the field of Laurent series $(\tilde{F}((x)), v, \partial)$ or $(\tilde{F}((x^{-1})), v, \partial)$ as in Corollary 2.13. Then the L_V-structure $(K, (K_\delta)_{\delta\in\mathbb{Z}})$ admits quantifier elimination, with t acting as ∂.*

5. Model-completion

Let M be a A-module. Let $X := \{t^n : n \in \mathbb{N}\}$, this is a right denominator set (see [12], Lemma 9.1). So, there exists a right ring of fractions AX^{-1} (ibid., Theorem 9.7). It is isomorphic to the set of equivalence classes in $A \times X$ of the following equivalence relation: $(a, t^{n_1}) \sim (a', t^{n_2})$ iff there exist $s \in A$ and $n \in \mathbb{N}$ such that $at^n = a's$ and $t^{n_1}t^n = t^{n_2}s$. We may extend the valuation on A to AX^{-1}.

The module M has a right module of fractions M_X with respect to X and it is isomorphic to $M \otimes_A AX^{-1}$, into which it embeds if M is X-torsion-free. Moreover, any element of M_X has the form $m \otimes t^{-n}$, for some $m \in M$, $n \in \mathbb{N} \setminus \{0\}$. (ibid., Theorem 9.13, Proposition 9.14).

Proposition 5.1. *T_V^* is the model-completion of the theory $T_{V,X}$ consisting of T_V together with the axioms $\forall m\ (m.t = 0 \rightarrow m = 0)$ and $\forall m\ V_\delta(m.t) \leftrightarrow V_\delta(m)$, for each $\delta \in \Delta$.*

Proof: Let now M be a model of $T_{V,X}$. We will embed M in a model of T_V^*.

First, we extend the predicates V_δ, $\delta \in \Delta$ on M_X by $V_\delta(m \otimes t^{-n})$, $m \in M$, $n \in \mathbb{N} \setminus \{0\}$, whenever $V_\delta(m)$. This is well-defined since $V_\delta(m.t)$ iff $V_\delta(m)$.

Now, consider M_X^ω the direct product of ω copies of M_X. Let $\mathcal{F}$ be the Fréchet filter on ω. We endow M_X^ω with a structure of A-module as follows. Let $(m_i)_{i\in\omega} \in M_X^\omega$, define $(m_i).t := ((m_{i+1}.t)_{i\in\omega})$ and extend it by linearity on A. Then define $V_\delta((m_i)_{i\in\omega})$ iff $V_\delta(m_i)$, for every $i \in \omega$.

Finally, consider the quotient of M_X^ω by $\mathcal{F}$ and the diagonal embedding of Δ into $\Delta^\omega/\mathcal{F}$. Define $(m_i)_{\mathcal{F}}.t := ((m_{i+1}.t)_{i\in\omega})_{\mathcal{F}}$ and $V_\delta((m_i)_{i\in\omega})_{\mathcal{F}}$ iff $\{i \in \omega : V_\delta(m_i)\} \in \mathcal{F}$. Let us show that this is a model of T_V^*. Let $q(t) = \sum_{i=0}^d t^i.a_i$ with $a_i \in K$. Given $m \in \mathcal{M}$, we wish to find $n \in \mathcal{M}$ such that $m = n.q(t)$. Suppose we have chosen $n_0, \cdots, n_{d-1}$, then $m_0 = n_d.t^d.a_d + \sum_{i=0}^{d-1} n_i.t^i.a_i$. So, define $n_d := (m_0.a_d^{-1} - \sum_{i=0}^{d-1} n_i.t^i.a_i.a_d^{-1}).t^{-d}$. Since M_X is X-divisible we can find such an element. □

6. Two-sorted valued structures.

In this section we revert to the two-sorted language L_w of valued A-modules. In L_w, x, y, z will denote variables of sort M and δ will denote variables of sort Δ. We keep the same notation as before, with a fixed (K, v, σ, ∂), $A = K[t; \sigma, \partial]$, etc.

In order to get more precise quantifier elimination results for valued A-modules $(M, \Delta, \leq, +, w, \infty)$, it is useful to look more closely at the structure $((\Delta, \leq), (+\gamma)_{\gamma\in vK})$.

Definition 6.1. Let $(\Gamma, +, 0, \leq)$ be a fixed totally ordered abelian group. Let L_Δ be the language of sort Δ with a binary predicate $\leq$ and unary function symbols $+\gamma$ for each $\gamma \in \Gamma$. We will write the action of $+\gamma$ from the right. Let T_Δ be the following L_Δ-theory.

(1) $\leq$ is a total order.
(2) $\forall \delta\ ((\delta + \gamma_1) + \gamma_2 = \delta + (\gamma_1 + \gamma_2))$, where $\gamma_1, \gamma_2 \in \Gamma$.
(3) $\forall \delta_1 \forall \delta_2\ (\delta_1 \leq \delta_2 \rightarrow \delta_1 + \gamma_1 \leq \delta_2 + \gamma_2)$, where $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \leq \gamma_2$.

For a valued A-module $(M, \Delta, +, w)$ and $\Gamma = vK$, we see that $((\Delta, \leq), (+\gamma)_{\gamma\in vK}) \models T_\Delta$.

We will consider two conditions under which the corresponding extensions of T_Δ to valued A-modules admit quantifier elimination.

Definition 6.2. We define the following L_Δ-theories.

(1) Let $T_{\Delta,dense}$ be the theory T_Δ together with the axiom that the total order $\leq$ is dense.

(2) Assume that Γ has a smallest strictly positive element 1. Let $T_{\Delta,discrete}$ be the theory T_Δ together with the axiom

$$\forall\delta_1\exists\delta_2\forall\delta_3\ (\delta_2 > \delta_1\ \&\ (\ \delta_3 > \delta_1 \to (\delta_2 \leq \delta_3\ \&\ \delta_2 = \delta_1 + 1)))$$

In the case of $T_{\Delta,discrete}$, we will always make the assumption that we have a constant 1 in the language of the group Γ.

Proposition 6.1. (cf.[11]) *Both theories $T_{\Delta,dense}$ and $T_{\Delta,discrete}$ admit quantifier elimination.*

Proof: It suffices that any formula with free variables $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ of the form

$$\exists\alpha\ \ (\bigwedge_\ell \delta_{i_\ell} + \gamma_{i_\ell} \,\square\, \alpha \,\square\, \delta_{j_\ell} + \gamma_{j_\ell})$$

where $\square \in \{\leq, <\}$, $1 \leq i_\ell, j_\ell \leq n$, and $\gamma_{k\ell} \in \Gamma$, be equivalent to a quantifier-free formula in $\boldsymbol{\delta}$. It is equivalent to

$$\exists\alpha\ (\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} \,\square\, \alpha \,\square\, \min\{\delta_{j_\ell} + \gamma_{t_\ell}\}\ \ \&\ \theta(\boldsymbol{\delta}))$$

for some quantifier-free formula $\theta(\boldsymbol{\delta})$. In $T_{\Delta,dense}$, it is equivalent to

$$\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} \,\square\, \min\{\delta_{j_\ell} + \gamma_{t_\ell}\}\ \&\ \theta(\boldsymbol{\delta})$$

In $T_{\Delta,discrete}$, if one of the $\square$ is $\leq$, then it is treated similarly as above replacing the $\square$ in the resulting formula by a strict inequality if there was one. If both $\square$ stands for $<$, then we use the special axiom of $T_{\Delta,discrete}$ and the resulting formula is:

$$\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} + 1 < \min\{\delta_{j_\ell} + \gamma_{t_\ell}\}\ \&\ \theta(\boldsymbol{\delta})$$

$\square$

Definition 6.3. We define the following L_w-theories of valued A-modules.

(1) Let $T_{w,d}$ consists of the following:

(1.1) T_w.

(1.2) Divisibility axioms (DG). For each q in $\mathcal{I} \cup \{t\}$:

$$\forall x\ \ (x \neq 0 \to \exists y\ (\ x = y.q\ \&\ w(x) = w(y)\ ))\,.$$

(1.3) No residual identities axioms (IR). For each $p_1, \cdots, p_n$ in $\mathcal{I}$:

$$\forall\delta\forall x_1\cdots\forall x_n\exists x\ \left((\bigwedge_{i=1}^n w(x_i) \geq \delta\ ) \to \bigwedge_{i=1}^n\ w(x.p_i + x_i) = \delta\right).$$

(2) Let $T_{w,d,dense} = T_{w,d} \cup T_{\Delta,dense}$.

(3) Let $T_{w,d,discrete} = T_{w,d} \cup T_{\Delta,discrete}$.

Example 3. *Let F be a p-closed field of characteristic p. Then, as in Corollary 2.10, $M = K = W(F)$ with the action of the Witt Frobenius, and $M = K = F((x))$ with the action of $\sum c_i x^i \mapsto \sum c_i^p x^i$, yield models of $T_{w,d,discrete}$.*

Example 4. *Let (F, ∂) be a differential field of characteristic 0. Assume its field of constants is algebraically closed. Let K be either the field of Laurent series $(\tilde{F}((x)), v, \partial)$ or $(\tilde{F}((x^{-1})), v, \partial)$ as in Corollary 2.13. Then K, with t acting as ∂, yields models of $T_{w,d,discrete}$.*

We will abuse notation and continue to use the V_δ, and introduce the new $V_{\delta+}$, with $V_{\delta+}(x) \leftrightarrow w(x) > \delta$. We will also identify all these with the sets they define in any model, and we recall they are always A_0-modules.

Note that (IR) implies that $V_\delta/V_{\delta+} \neq \bigcup_{i=1}^n \{x + V_{\delta+} : x \in V_\delta \;\&\; p_i(x) \equiv 0 \bmod V_{\delta+}\}$ and that $|V_\delta/V_{\delta+}|$ is infinite. When the model of $T_{w,d}$ is K itself, it implies that $\bar{K}$ does not satisfy any *linear* $\bar{\sigma}$-identities or $\bar{\partial}$-identities, according to the action of t.

Definition 6.4. Let $r \in A_0$ and $n \in \omega$.

(1) A *residual index formula* $\mathrm{Indr}_{n,r}(\delta)$, is a L_w-formula which is existential in the module sort with a free variable in the ordered set sort, of the form

$$\exists x_1 \cdots \exists x_n \; (\bigwedge_{1 \le i < j \le n} w(x_i - x_j) = \delta \;\; \& \;\; \bigwedge_{i=1}^{n} (w(x_i.r) > \delta \,\&\, w(x_i) = \delta)).$$

(2) A *residual index sentence* is an existential sentence of the form $\exists \delta \; \mathrm{Indr}_{n,r}(\delta)$.

We will often use the following fact. In models of $T_w \cup \{(DG)\}$, if $r \in \mathcal{I}$ and $Indr_{n,r}(\delta)$ holds, then there exist n elements in $ann(r) \cap V_\delta$ which belong to different cosets of $V_{\delta+}$ in V_δ. In particular, index sentences (in the theory of modules) are special instances of residual index sentences.

Proposition 6.2. *In the theory $T_{w,d}$, any existential L_w-formula in the module sort is equivalent to a formula which is quantifier free in the module sort, existential in the totally ordered set sort, plus some residual index formulas and sentences.*

Corollary 6.3. *Let $(M, w, \Delta_M) \subset (N, w, \Delta_N)$ be valued A-modules satisfying $T_{w,d}$. Assume that both satisfy the same residual index formulas with parameters in Δ_M and that $(\Delta_M, \Gamma, +)$ is existentially closed in $(\Delta_N, \Gamma, +)$. Then (M, w, Δ_M) is existentially closed in (N, w, Δ_N).* $\square$

Corollary 6.4. *In the theory $T_{w,d,dense}$ (respectively $T_{w,d,discrete}$), any L_w-formula $\phi(\mathbf{x}, \boldsymbol{\delta})$ is equivalent to the conjunction of a quantifier-free L_w-formula $\theta(\mathbf{x}, \boldsymbol{\delta})$ together with residual index formulas and residual index sentences. Therefore the completions are given by residual index sentences and quantifier-free L_Δ-sentences.*

Proof: Apply Propositions 6.1 and 6.2. $\square$

Note that if the action of $v(K)$ is transitive on Δ, then the residual index formulas can be translated into statements in a fixed quotient.

Proof of Proposition 6.2.

In the language L_w, the only interaction between the two sorts M and Δ occurs in valuation equalities and inequalities. Let $r \in A$ and u a M-term where x does not occur, we will replace each valuation inequality where a term of the form $w(x.r + u)$ occurs, by a formula $\exists \delta (w(x.r + u) = \delta \,\&\, \psi)$, where δ is a new variable and ψ is obtained from the inequality by putting in δ for $w(x.r + u)$. In this way, we can always assume that terms of the form $w(x.r + u)$ only occur in equation of the form $w(x.r + u) = \delta$ (cf. [28]).

Also, by replacing each inequality $x.r \neq u$ by $w(x.r-u) \neq \infty$, we can always assume there are no such $x.r \neq u$. (N.B. We could handle these $x.r \neq u$ in the elimination process, but it would generate new index formulas of the form $|ann(r) \cap V_\delta| \geq n$.)

Therefore, by the preceding remarks and using the same normalization process as in Proposition 4.1, it suffices to consider formulas $\phi(x, \mathbf{y}, \boldsymbol{\delta})$ of the form

$$x.r_0 = t_0(\mathbf{y}) \;\&\; \bigwedge_{i=1}^{n} w(x.r_i + t_i(\mathbf{y})) = \delta_i \;\&\; \theta(\mathbf{y}, \boldsymbol{\delta})$$

where $r_i \in \mathcal{I}$, $0 \leq i \leq n$, $deg(r_0) > deg(r_i)$ $t_i(\mathbf{y})$ are L_A-terms, $1 \leq i \leq n$, $\infty \neq \delta_1 \geq \delta_2 \cdots \geq \delta_n$, and $\theta(\mathbf{y}, \boldsymbol{\delta})$ is a quantifier-free formula in the parameters $\mathbf{y}$ and existential in $\boldsymbol{\delta}$. For ease of notation, from now on we will simply denote the terms $t_i(\mathbf{y})$ by t_i, $0 \leq i \leq n$.

We want to show that $\exists x\, \phi(x, \mathbf{y}, \boldsymbol{\delta})$ is equivalent to a quantifier-free formula in the M-sort, existential in the Δ-sort, plus some residual index formulas and sentences.

First, we will assume that we have an equation $x.r_0 = t_0$ occurring in $\phi(x, \mathbf{y})$.

As in the previous elimination theorem, we will concentrate on the system of conditions where x appears. Using the normalization process of Proposition 4.1 if necessary, we can reduced ourselves to consider systems of the following form:

(1):

$$\exists x \quad \begin{cases} x.r_0 = t_0, \\ w(x.r_1 + t_1) = \delta_1, \\ \cdots \\ w(x.r_n + t_n) = \delta_n, \end{cases}$$

with $deg(r_0) > deg(r_i)$, $r_0,\ r_i \in A_0$, $\bar{r}_0 \neq 0, \bar{r}_i \neq 0 \in \bar{K}[t]$, $1 \leq i \leq n$, $\infty \neq \delta_1 = \cdots = \delta_m > \delta_{m+1} \geq \cdots \geq \delta_n$ and the t_ℓ are L_A-terms.

In order to proceed inductively, we will have to consider more general systems where also valuation inequalities of the form $w(x.r + t) > \delta$ or $w(x.r + t) \geq \delta$ may occur. Consider a system of the form:

$$x.r_0 = t_0 \;\&\; \bigwedge_{i=1}^{m} w(x.r_i + t_i)\Box\delta_1 \;\&\; \bigwedge_{i=m+1}^{n} w(x.r_i + t_i)\Box\delta_i$$

where $\Box$ is either one of $=, >, \geq$. We define its complexity as the pair $(deg(r_0), \sum_{i \in I_=} deg(r_i))$ in the lexicographic product of $(\mathbb{N}, <) \times (\mathbb{N}, <)$, where $I_=$ is the set of those i such that $w(x.r_i + t_i) = \delta_1$ occurs in the system. If $I_= = \emptyset$, the complexity is set to $(deg(r_0), -\infty)$, and if $r_0 = 0, I_= = \emptyset$ to $(-\infty, -\infty)$, and we extend the lexicographical order in the natural way.

To keep track of this complexity measure, it is worth recording the main effect of the normalization process of Proposition 4.1 which we will use again. Namely, it does not alter the degree of r_0, so that the first component of complexity is left unchanged.

We also note that normalization has the effect of altering the δ_i in a minor way : δ_i is transformed into $\delta_i + v(\lambda_i)$ for some nonzero $\lambda_i \in K$. It will be convenient to abuse notation and keep using the notation δ_i to denote these new terms along the way.

There are basic cases for the induction. One is when $deg(r_i) = 0, i = 1, \ldots, m$. W.l.o.g. we can then assume $r_i = 1, i = 1, \ldots m$, and this case will be dealt later as system (3) (see further below). Another case is the following.

Lemma 6.5. *Consider the system*

$$\exists x \quad \begin{cases} x.r_0 = t_0, \\ w(x.r + t) > \delta_1, \\ \bigwedge_{i=1}^m w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^n w(x.r_i + t_i) = \delta_i \end{cases}$$

where $r \in A_0, v(r) = v(r_0) = 0, deg(r) < deg(r_0)$. *Let* $\mu \in \mathcal{O}_K, s, r' \in A_0$ *such that* $r_0\mu = rs + r'$, *and* $r' = 0$ *or* $deg(r') < deg(r)$. *Then the above system is equivalent to the following system of strictly lower complexity*

$$\exists x \quad \begin{cases} x.r = -t, \\ w(x.r' - t.s - t_0\mu) > \delta_1 + v(\mu), \\ \bigwedge_{i=1}^m w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^n w(x.r_i + t_i) = \delta_i. \end{cases}$$

Proof: $(\rightarrow)$ Let x be a solution of the given system. By Axiom (DG) there exists u such that $w(u) > \delta_1$ and $u.r = -x.r - t$. We get $w((x+u).r' - t.s - t_0\mu) = w(u.r_0\mu) > \delta_1 + v(\mu)$, and $x + u$ is a solution of the new system.

$(\leftarrow)$ Let y be a solution of the new system. We look for an element u such that $(y + u).r_0.\mu = (y + u).r.s + (y + u).r' = t_0.\mu$. Equivalently, $u.r_0\mu = t_0.\mu + t.s - y.r'$. By Axiom (DG), the exists such an element u with valuation strictly bigger than δ_1. Then $y + u$ is a solution to the given system. $\square$

We now proceed to the proof by induction on complexity.

Induction hypothesis: every system of the form

$$\exists x \left(x.r_0 = t_0 \ \& \bigwedge_{i=1}^{m} w(x.r_i + t_i) \square \delta_1 \ \& \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) \square \delta_i \right)$$

where $\square$ is either one of $=, >, \geq$, is equivalent to a disjunction of formulas of the form

$$\bigwedge_j w(t_0.r'_j + t'_j) \square \delta_j \ \& \ \Theta(\boldsymbol{y}, \boldsymbol{\delta})$$

where $r'_j \in A_0$, $t'_j(\boldsymbol{y})$ are L_A-terms, the δ_j are among $\delta_1, \ldots, \delta_n$, $\square$ is either one of $=, >, \geq$, and $\Theta(\boldsymbol{y}, \boldsymbol{\delta})$ is quantifier free in the M-sort, existential in the Δ-sort, plus some residual index formulas and sentences.

We start with system (1) and we assume $r_0 \neq 0$.

Basis of induction. When there are no valuation conditions ($I_= = \emptyset$) the system is handled directly by the axioms. When $deg(r_0) = 0$, i.e. $r_0 \in K$, then $x = t_0.r_0^{-1}$ and we can eliminate x by substitution. When $deg(r_i) = 0, i = 1, \ldots, m$, then we can assume $r_i = 1$ and it reduces to system (3) below which will be treated later.

Induction step(s). We apply the normalization process if necessary. So consider system (1). We assume $I_= \neq \emptyset, deg(r_0) \geq 1$ and $deg(r_i) \geq 1$ for some $1 \leq i \leq m$.

Suppose first that some $r_i, 1 \leq i \leq m$, with $deg(r_i) \geq 1$ divides r_0, say $i = \ell$. Let $r_0\lambda = r_\ell s_\ell$, where $\lambda \in \mathcal{O}_K, s_\ell \in A_0$. Since $v(r_0) = v(r_\ell) = 0$, we get $v(\lambda) = v(s_\ell)$ and we can assume that $\lambda = 1$, i.e. $r_0 = r_\ell s_\ell$. We introduce momentarily a new module variable which will enable us to do induction. Let z be a new module variable. Then system (1) is clearly equivalent to the following system:

$(2)_z$:

$$\exists z \exists x \begin{cases} x.r_\ell = z, \\ \bigwedge_{1 \leq i \neq \ell \leq m} w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i, \\ z.s_\ell = t_0, \\ w(z + t_\ell) = \delta_1. \end{cases}$$

Now, by induction, the formula

$$\exists x \left(x.r_\ell = z \ \& \bigwedge_{1 \leq i \neq \ell \leq m} w(x.r_i + t_i) = \delta_1 \ \& \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i \right)$$

is equivalent to a disjunction of formulas of the form

$$\bigwedge_j w(z.r'_j + t'_j) \square \delta_j \ \& \ \Theta(\boldsymbol{y}, \boldsymbol{\delta})$$

where $r'_j \in A_0$, $t'_j(\boldsymbol{y})$ are L_A-terms, the δ_j are among $\delta_1, \ldots, \delta_n$, $\square$ is either one of $=, >, \geq$, and $\Theta(\boldsymbol{y}, \boldsymbol{\delta})$ is quantifier free in the M-sort, existential in the Δ-sort, plus some residual index formulas and sentences. We can now apply induction to

$$\exists z \left(z.s_\ell = t_0 \ \& \ w(z + t_\ell) = \delta_1 \ \& \bigwedge_j w(z.r'_j + t'_j) \square \delta_j \ \& \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i \right)$$

and we have completed that induction step.

We are left with the case where no nonconstant r_i divides r_0. Pick any such, say $i = \ell$, and do Euclidean division. Let $r_0\lambda = r_\ell s_\ell + r$, where $\lambda \in \mathcal{O}_K, s_\ell, r \in A_0, r \neq 0, deg(r) < deg(r_\ell) < deg(r_0)$. We claim that system (1) is then equivalent to the disjunction of the following two systems (2a) or (2b) :

$(2a)$:

$$\exists x \begin{cases} x.r_0 = t_0, \\ w(x.r - t_\ell.s_\ell - t_0\lambda) = \delta_1, \\ w(x.r_\ell + t_\ell) \geq \delta_1, \\ \bigwedge_{1 \leq i \neq \ell \leq m} w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i \end{cases}$$

$(2b)$:

$$\exists x \begin{cases} x.r_0 = t_0, \\ w(x.r - t_\ell.s_\ell - t_0\lambda) > \delta_1, \\ \bigwedge_{i=1}^{m} w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i. \end{cases}$$

Indeed, suppose x a solution of (1). We have $w(x.r_\ell + t_\ell) = \delta_1$, so that $w(x.r_\ell.s_\ell + t_\ell.s_\ell) \geq \delta_1$. If $w(x.r_\ell.s_\ell+t_\ell.s_\ell) = \delta_1$, we get $x.r_\ell.s_\ell = t_0\lambda - x.r$ and $w(x.r - t_\ell.s_\ell - t_0\lambda) = \delta_1$, and x is a solution of (2a). Similarly, if $w(x.r_\ell.s_\ell+t_\ell.s_\ell) > \delta_1$, we get $w(x.r_\ell - t_\ell.s_\ell - t_0\lambda) > \delta_1$, and x is a solution of (2b). Conversely, suppose x is a solution of (2a). We have $w(x.r_\ell.s_\ell+t_\ell.s_\ell) \geq w(x.r_\ell+t_\ell)$ and $w(x.r_\ell.s_\ell+t_\ell.s_\ell) = w(x.r_\ell - t_\ell.s_\ell - t_0\lambda) = \delta_1$, so that we cannot have $w(x.r_\ell + t_\ell) > \delta_1$. Hence $w(x.r_\ell + t_\ell) = \delta_1$ and x is a solution of (1). Note that a solution of (2b) is immediately a solution of (1).

Consider system (2a) and let $\mu \in \mathcal{O}_K$ such that $v(\mu) = v(r_\ell)$. Now if in system (2a) we replace $w(x.r_\ell - t_\ell.s_\ell - t_0\lambda) = \delta_1$ by $w(x.r_\ell\mu^{-1} - t_\ell.s_\ell\mu^{-1} - t_0\lambda\mu^{-1}) = \delta_1 - v(\mu)$, we obtain a system of strictly lower complexity. Indeed, if $v(r) = 0$ we replace a valuation equation involving δ_1 with another one where the coefficient of x has strictly smaller degree, and if $v(r) > 0$ we replace it by a valuation equation with a value strictly less than δ_1. So we have completed the induction step in case of system (2a).

Consider system (2b). Recall that $v(r_0) = 0$ and $r_0\lambda = r_\ell s_\ell + r$. So by scaling with a suitable element of K we can assume that either $v(r_\ell s_\ell) = 0$ or $v(r) = 0$. If $v(r) = 0$, then by Lemma 6.5 we can obtain an equivalent system of strictly lower complexity. If $v(r_\ell s_\ell) = 0$, then system (2b) is equivalent to the following system (2c):

$(2c)$:

$$\exists x \quad \begin{cases} x.r_\ell s_\ell = -t_\ell.s_\ell, \\ w(x.r - t_\ell.s_\ell - t_0\lambda) > \delta_1, \\ \bigwedge_{i=1}^{m} w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i \end{cases}$$

Indeed, if x is a solution of (2b), then by Axiom (DG) there exists u such that $u.r_\ell s_\ell = -x.r_\ell s_\ell - t_\ell.s_\ell$ and $w(u) > \delta_1$, and then $x + u$ is a solution of (2c). A similar computation works in the other direction. But now in system (2c), r_ℓ divides the coefficient x in the first equation and we are back to a case already treated at the beginning, the extra strict valuation inequality being harmless in that process. The induction is now completed.

Thus it remains to handle systems of the following form

(3):

$$\exists x \quad \begin{cases} x.r_0 = t_0, \\ \bigwedge_{i=1}^{m} \ w(x + u'_i) = \delta_1, \\ \bigwedge_{i=1}^{m} \ w(x.r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(x.r_i + t_i) = \delta_i. \end{cases}$$

where $u'_i = u'_i(\boldsymbol{y})$ is a M-term.

This system is equivalent to a disjunction of the following systems over the subsets B of $\{1, \ldots, m\}$ as follows:

$(3)_B$:

$$\exists x \begin{cases} x.r_0 = t_0, \\ \bigwedge_{i=1}^{m} \ w(x+u'_i) = \delta_1, \\ \bigwedge_{i=1}^{m} \ w(x.r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(x.r_i + t_i) = \delta_i, \\ \bigwedge_{i \in B} \ w(t_0 + u'_i.r_0) > \delta_1, \\ \bigwedge_{i \notin B} \ w(t_0 + u'_i.r_0) = \delta_1. \end{cases}$$

We first do the case $B = \emptyset$. So consider system

$(3)_\emptyset$:

$$\exists x \begin{cases} x.r_0 = t_0, \\ \bigwedge_{i=1}^{m} \ w(x+u'_i) = \delta_1, \\ \bigwedge_{i=1}^{m} \ w(x.r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(x.r_i + t_i) = \delta_i, \\ \bigwedge_{i=1}^{m} \ w(t_0 + u'_i.r_0) = \delta_1. \end{cases}$$

We claim that $(3)_\emptyset$ is equivalent to

$(4)_\emptyset$:

$$\begin{cases} \bigwedge_{i \neq j \leq m} \ w(u'_i - u'_j) \geq \delta_1, \\ \bigwedge_{i=1}^{m} \ w(u'_1.r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(u'_1.r_i - t_i) = \delta_i, \\ \bigwedge_{i=1}^{m} \ w(t_0 + u'_i.r_0) = \delta_1. \end{cases}$$

Indeed, suppose x is a solution of $(3)_\emptyset$. Then $w(u'_i - u'_j) = w(x + u'_i - x - u'_j) \geq \delta_1$, $w(u'_1.r_i - t_i) = w(x.r_i + u'_1.r_i - x.r_i - t_i) \geq \delta_1$ if $i \leq m$, and $w(u'_1.r_i - t_i) = w(x.r_i + u'_1.r_i - x.r_i - t_i) = \delta_i$ if $i > m$. Conversely, suppose $(4)_\emptyset$ is satisfied. By Axiom (DG) let u'' such that $u''.r_0 = t_0 + u'_1.r_0$ and $w(u'') = w(t_0 + u'_1.r_0) = \delta_1$. Then $u'' - u'_1$ is a solution of $(3)_\emptyset$: we have $(u'' - u'_1).r_0 = t_0$, and $w((u'' - u'_1) + u'_i) = w(u'' - (u'_1 - u'_i)) \geq \delta_1$, but the strict inequality would imply $w((u'' - u'_1).r_0 + u'_i.r_0) = w(t_0 + u'_i.r_0) > \delta_1$, which is not the case, and so $w((u'' - u'_1) + u'_i) = \delta_1$; on the other hand $w((u'' - u'_1).r_i + t_i) = w(u''.r_i - u'_1.r_i + t_i) \geq \delta_1$ if $i \leq m$, and $w((u'' - u'_1).r_i + t_i) = w(u''.r_i - u'_1.r_i + t_i) = \delta_i$ if $i > m$.

We must now consider the case $(3)_B$ where $B \neq \emptyset$. Note that in that case $(3)_B$ has a solution only if $ann(r_0) \neq \{0\}$. Indeed, let x be a solution and $i \in B$. By Axiom (DG) let u be such that $u.r_0 = t_0 + u'_i.r_0$ and $w(u) = w(t_0 + u'_i.r_0) > \delta_1$. Then $x - (u - u'_i) \in ann(r_0)$ and $w(x - (u - u'_i)) = w(x + u'_i - u) = \delta_1 \neq \infty$, so $x - (u - u'_i) \neq 0$. We claim that $(3)_B$ is equivalent to a disjunction over the possible nonempty subsets J of $\{1, \ldots, m\}$ of the following systems

$(4)_{B,J}$:

$$\exists x \quad \begin{cases} x.r_0 = t_0, \\ w(x+u'_1) = \delta_1, \\ \bigwedge_{i\in J} \ w(x+u'_i) = \delta_1, \\ \bigwedge_{i\neq j\in J} \ w(u'_i - u'_j) = \delta_1, \\ \bigwedge_{i\notin J} \vee_{j\in J} \ w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i=1}^{m} \ w(u'_1.r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(u'_1.r_i - t_i) = \delta_i, \\ \bigwedge_{i\in B} \ w(t_0 + u'_i.r_0) > \delta_1, \\ \bigwedge_{i\notin B} \ w(t_0 + u'_i.r_0) = \delta_1 \end{cases}$$

Indeed, let x be a solution of $(3)_B$. We get a corresponding subset J, $w(u'_1.r_i - t_i) = w(x.r_i + u'_1.r_i - x.r_i - t_i) \geq \delta_1$ if $i \leq m$, and $w(u'_1.r_i - t_i) = w(x.r_i + u'_1.r_i - x.r_i - t_i) = \delta_i$ if $i > m$. Thus x is a solution of $(4)_B$. Conversely, let x be a solution of $(4)_{B,J}$. Let $i \leq m$, we have $w(x+u'_i) = \delta_1$ if $i \in J$, and if $i \notin J$ we have $j \in J$ such that $w(u'_i - u'_j) > \delta_1$ and then $w(x+u'_i) = w(x+u'_j+u'_i-u'_j) = \delta_1$. On the other hand, $w(x.r_i + t_i) = w(x.r_i + u'_1.r_i - u'_1.r_i + t_i) \geq \delta_1$ if $i \leq m$, and $w(x.r_i + t_i) = w(x.r_i + u'_1.r_i - u'_1.r_i + t_i) = \delta_i$ if $i > m$. Whence x is a solution of $(4)_{B,J}$.

We now claim that $(4)_{B,J}$ is equivalent to a disjunction over the nonempty subsets J' of J of the following systems

$(5)_{B,J,J'}$:

$$\begin{cases} Indr_{|J'|,r_0}(\delta_1), \\ \bigwedge_{i\in B} \bigvee_{j\in J'} w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i\neq j\in J} \ w(u'_i - u'_j) = \delta_1, \\ \bigwedge_{i\notin J} \vee_{j\in J} \ w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i=1}^{m} \ w(u'_1.r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} \ w(u'_1.r_i - t_i) = \delta_i, \\ \bigwedge_{i\in J'} \ w(t_0 + u'_i.r_0) > \delta_1, \\ \bigwedge_{i\notin B} \ w(t_0 + u'_i.r_0) = \delta_1 \end{cases}$$

Indeed, suppose x is a solution of $(4)_{B,J}$. Let $J' \subseteq J$ be such that $\wedge_{i\in B} \vee_{j\in J'} \ w(u'_i - u'_j) > \delta_1$ and $\wedge_{i\in J'} \vee_{j\in B} \ w(u'_i - u'_j) > \delta_1$.Then certainly $\wedge_{i\in J'} \ w(t_0 + u'_i.r_0) > \delta_1$. Fix $j \in J'$. Then $w((u'_j + x).r_0) > \delta_1, w((u'_j - u'_i).r_0) > \delta_1, j \neq i \in J'$, so that $Indr_{|J'|,r_0}(\delta_1)$ holds, and hence $(5)_{B,J,J'}$ is satisfied. Conversely, suppose $(5)_{B,J,J'}$ is satisfied. Fix $j \in J'$. Then again $w((u'_j - u'_i).r_0) > \delta_1, j \neq i \in J'$. The condition $Indr_{|J'|,r_0}(\delta_1)$ ensures that there is y such that $w(y) = \delta_1$ and $w(y.r_0) > \delta_1$ and $w(y - (u'_j - u'_i)) = \delta_1, j \neq i \in J'$. By Axiom (DG), let z be such that $z.r_0 = t_0 + (u'_j - y).r_0$ and $w(z) = w(t_0 + (u'_j - y).r_0) > \delta_1$. Then $(z - u'_j + y).r_0 = t_0$, $w((z - u'_j + y) + u'_j) = \delta_1$, $w((z - u'_j + y) + u'_i) = \delta_1$ si $j \neq i \in J'$. It follows that $w((z - u'_j + y) + u'_i) = \delta_1$ if $i \in B$. On the other hand, if $i \notin B$, we already have $w((z - u'_j + y) + u'_i) = w(z + y + (u'_i - u'_j)) \geq \delta_1$, but we cannot have $w((z - u'_j + y) + u'_i) > \delta_1$ because it would lead to $w((z - u'_j + y).r_0 + u'_i.r_0) = w(t_0 + u'_i.r_0) > \delta_1$ which is not the case. Thus $z - u'_j + y$ is a solution of $(4)_{B,J}$.

It remains to handle the case when we have no equation in our system. We will show how to reduce to the first case, using axiom (IR). We have to consider systems of the following form :

(1'):

$$\exists x \quad \begin{cases} \bigwedge_{i=1}^{m} w(x.r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i, \end{cases}$$

with the same notation as before, in particular $\delta_1 = \delta_2 = \cdots = \delta_m > \delta_{m+1} \geq \cdots \geq \delta_n$.

We claim that (1') is equivalent to the following system (1"), which brings us back to system (1), so we are done.

(1"):

$$\exists x \quad \begin{cases} x.r_1 + t_1 = 0, \\ \bigwedge_{i=2}^{m} w(x.r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^{n} w(x.r_i + t_i) = \delta_i. \end{cases}$$

Indeed, suppose x is a solution of (1'). By axiom (DG), there exists y such that $y.r_1 = x.r_1 + t_1, w(y) = \delta_1$, and $w(y.r_1) = \delta_1$. We get that $\bigwedge_{i=2}^{m} w((x-y).r_i + t_i) \geq \delta_1$ and $\bigwedge_{i=m+1}^{n} w((x+y).r_i + t_i) = \delta_i$. So $x - y$ is a solution of (1"). Conversely, suppose that x is a solution of(1"). Then, thanks to axiom scheme (IR), we may add to x an element u with $w(u) = \delta_1,\ w(u.r_1) = \delta_1$ and $\bigwedge_{i=2}^{m} w(u.r_i + (x.r_i + t_i)) = \delta_1$. □

7. NIP

As a reference about the independence property see [23], Chapter 12, section 12.d. We will abbreviate the property of "not having the independence property" by NIP. Recall that the following theories do have NIP: any stable theory, the theory of a chain (ibid., section 12.f), the theory of abelian totally ordered groups ([13]). It is known that for most known valued fields for which the Ax-Kochen-Ershov principle holds, its theory has NIP iff the theory of its residue field does (see [4]). This applies to $W(F)$, but leaves $F((x))$ open if F is of characteristic $p > 0$.

Note that by a similar proof that the theory of a chain has NIP, we can deduce that the theories $T_{\Delta,dense}$ and $T_{\Delta,discrete}$ have NIP too. Indeed, the quantifier elimination result for these theories implies that the types are determined by the quantifier-free parts and so the type of an element over a model is determined by its cut in that model. Therefore a type has at most two co-heirs and so we get the NIP property by Theorem 12.28 of [23].

Proposition 7.1. *Let (M, w, Δ) be either a model of $T_{w,d,dense}$, or $T_{w,d,discrete}$. Then the theory of M has NIP.*

Proof: We will proceed by contradiction. Assume that we have a formula $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$ witnessing the independence property in the single variable α (e.g. see [23], Theorem 12.18), where α is of either sort M or Δ and the variables $\mathbf{y}$ are of sort M and the $\boldsymbol{\delta}$ of sort Δ. Denote by T either $T_{w,d,dense}$ or $T_{w,d,discrete}$.

By compactness, we have tuples $(a_i, \mathbf{b}_m, \mathbf{c}_m)_{i\in\omega, m\in 2^\omega}$ in some model of T such that the following holds :

$$\bigwedge_{i\in m} \phi(a_i, \mathbf{b}_m, \mathbf{c}_m), \bigwedge_{i\notin m} \neg\phi(a_i, \mathbf{b}_m, \mathbf{c}_m), \quad i \in \omega, m \in 2^\omega.$$

First, suppose that α is of sort Δ.

Since T admits quantifier elimination (Corollary 6.4), we may assume that $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$ is a finite disjunction of quantifier free formulas ϕ_ℓ, of the form:

$$\phi_l(\alpha, \mathbf{y}, \boldsymbol{\delta}) := \bigwedge_{i=1}^{n(\ell)} w(t_i(\mathbf{y})) = \delta_i \;\&\; \theta_\ell(\mathbf{y}) \;\&\; \delta_{\ell 1} + \gamma_{\ell 1} \,\square\, \alpha \,\square\, \delta_{\ell 2} + \gamma_{\ell 2} \;\&\; \psi_\ell(\boldsymbol{\delta})$$

where $\square \in \{<, \leq\}$, $\theta_\ell(\mathbf{y})$ is a quantifier-free pp-formula, $t_i(\mathbf{y})$ are L_A-terms of sort M, and $\psi_\ell(\boldsymbol{\delta})$ is a quantifier free formula (recall that $T_{\Delta,dense}$ and $T_{\Delta,discrete}$ both admit quantifier elimination).

We introduce new variables β_h of sort Δ that we substitute to each $w(t_i(\mathbf{y}))$ and $\delta_{\ell j} + \gamma_{\ell j}$.

Let $\phi'_\ell(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta})$ be the formula we obtain from $\phi_\ell(x, \mathbf{y}, \boldsymbol{\delta})$ by making the above substitution and leaving out all the subformulas not involving either α, $\boldsymbol{\delta}$ or $\boldsymbol{\beta}$. Set $\phi'(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta}) := \bigvee_\ell \phi'_\ell(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta})$.

Let $\mathbf{d}_m$ be the value of $\boldsymbol{\beta}$ obtained by making the above substitutions with the values $\mathbf{b}_m$ of $\mathbf{y}$ and $\mathbf{c}_m$ of $\boldsymbol{\delta}$.

Therefore the tuples $(a_i, \mathbf{d}_m, \mathbf{c}_m)_{i\in\omega, m\in 2^\omega}$ satisfy the following

$$\bigwedge_{i\in m} \phi'(a_i, \mathbf{d}_m, \mathbf{c}_m), \bigwedge_{i\notin m} \neg\phi'(a_i, \mathbf{d}_m, \mathbf{c}_m), \quad i \in \omega, m \in 2^\omega$$

contradicting the property that no chain has the independence property.

Now we suppose that α is of sort M. Then we may assume that $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$ is a disjunction $\bigvee_{\ell=1}^k \phi_\ell(\alpha, \mathbf{y}, \boldsymbol{\delta})$ of quantifier-free formulas of the form:

$$\phi_l(\alpha, \mathbf{y}, \boldsymbol{\delta}) :=$$

$$\bigwedge_{i=1}^{n(\ell)} w(\alpha.r_i + t_i(\mathbf{y})) = \delta_i \;\&\; \alpha.r_0 = t_0(\mathbf{y}) \,\&\, \theta_\ell(\mathbf{y}) \;\&\; \psi_\ell(\boldsymbol{\delta})$$

with the same notation as before and $r_0, r_i \in A$.

We vary $j \in n$ and by the pigeonhole principle, there exist pairwise distinct elements $j_4, j_3, j_2 \in n$ and some atomic formula χ occurring in ϕ such that, setting $S_1 = n - \{j_3, j_4\}$, $S_3 = n - \{j_2, j_4\}$, $S_2 = n - \{j_2, j_3\}$, we have that $\bigwedge_{i=2,3} \neg\chi(a_{j_2}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})$, $\bigwedge_{i=1,2} \neg\chi(a_{j_3}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})$ and $\neg\chi(a_{j_4}, \mathbf{b}_{S_1}, \mathbf{c}_{S_1})$ are satisfied.

Let $j_1 \in S_1 \cap S_2 \cap S_3$.

Either χ is of the form $\alpha.r_0 = t_0(\mathbf{y})$ and then we get $(a_{j_1} - a_{j_2}).r_0 = 0$ and $a_{j_1}.r_0 + t_0(\mathbf{b}_{S_2}) = 0$ but $a_{j_2}.r_0 + t_0(\mathbf{b}_{S_2}) \neq 0$, which is a contradiction.

Or, χ is of the form $w(\alpha.r_i + t_i(\mathbf{y})) = \delta_i$. We will denote by c_{S_j} the i^{th} component of $\mathbf{c}_{S_j}$. Then we have $w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_1})) = c_{S_1}$ and $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_1})) = c_{S_1}$. So, $w((a_{j_1} - a_{j_2}).r_i) \geq c_{S_1}$.

Similarly, we get $w(a_{j_1}.r_i+t_i(\mathbf{b}_{S_3})) = c_{S_3}$ and $w(a_{j_3}.r_i+t_i(\mathbf{b}_{S_3})) = c_{S_3}$. So, $w((a_{j_1} - a_{j_3}).r_i) \geq c_{S_3}$.

On the other hand since $j_3 \notin S_1$, $w(a_{j_3}.r_i + t_i(\mathbf{b}_{S_1})) \neq c_{S_1}$ and $w(a_{j_3}.r_i + t_i(\mathbf{b}_{S_1})) \geq \min\{w((a_{j_1} - a_{j_3}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_1}))\}$, and since $j_2 \notin S_3$, $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_3})) \neq c_{S_3}$ and $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_3})) \geq \min\{w((a_{j_2} - a_{j_1}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_3}))\}$.

So we get $c_{S_1} \geq c_{S_3}$. Suppose now that $c_{S_1} > c_{S_3}$, then $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_3})) = w((a_{j_2} - a_{j_1}).r_i + a_{j_1}.r_i + t_i(\mathbf{b}_{S_3})) = \min\{w((a_{j_2} - a_{j_1}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_3}))\} = c_{S_3}$, a contradiction.

So we have $c_{S_1} = c_{S_3}$ which denote henceforth c, and by the above, we necessarily get that $w(a_{j_3}.r_i + t_i(\mathbf{b}_{S_1})) > c$ and $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_3})) > c$.

Using these two strict inequalities, we get that $w((a_{j_1} - a_{j_3}).r_i) = c$ and $w((a_{j_2} - a_{j_3}).r_i) = c$.

Since $j_1 \in S_2$, $w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_2})) = c_{S_2}$.

Suppose $c_{S_2} < c$, then $w(a_{j_3}.r_i + t_i(\mathbf{b}_{S_2})) = \min\{w((-a_{j_1} + a_{j_3}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_2}))\} = c_{S_2}$, which contradicts the fact that $j_3 \notin S_2$. Therefore, $c_{S_2} \geq c$. Suppose $c_{S_2} > c$. First note that $w((-a_{j_1} + a_{j_4}).r_i) \geq c_{S_2}$, since $w((-a_{j_1} + a_{j_4}).r_i) \geq \min\{w(a_{j_1}.r_i+t_i(\mathbf{b}_{S_2})), w(a_{j_4}.r_i+t_i(\mathbf{b}_{S_2}))\}\}$. Then, $w(a_{j_4}.r_i+t_i(\mathbf{b_{S_1}})) = \min\{w((-a_{j_1}+a_{j_4}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b_{S_1}}))\} = c$, a contradiction since $j_4 \notin S_1$.

So, we get that $c_{S_2} = c$.

Now, we have $w(a_{j_3}.r_i + t_i(\mathbf{b}_{S_2})) \geq \min\{w((a_{j_3} - a_{j_1}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_2}))\} = c$ and $w(a_{j_2}.r_i + t_i(\mathbf{b}_{S_2})) \geq \min\{w((a_{j_2} - a_{j_1}).r_i), w(a_{j_1}.r_i + t_i(\mathbf{b}_{S_2})\} = c$. Hence $w(a_{j_3}.r_i+t_i(\mathbf{b}_{S_2})) > c$ and $w(a_{j_2}.r_i+t_i(\mathbf{b}_{S_2})) > c$. But therefore, $w((a_{j_2}-a_{j_3}).r_i) > c$, a contradiction. □

It has been observed that formulas without the independence property are closed under boolean combinations (see e.g. [1]). So we could have directly considered only atomic formulas.

8. Ultraproducts.

Let $\mathcal{U}$ be a non-principal ultrafilter over the set of prime numbers p. Let F_p be a p-closed field of characteristic p of cardinality at most $\aleph_1$. By the Ax-Kochen-Ershov theorem the ultraproducts $\prod_{\mathcal{U}} W(F_p)$ and $\prod_{\mathcal{U}} F_p((x))$ are elementarily equivalent as valued fields, and $\prod_{\mathcal{U}} W(\mathbb{F}_p)$ and $\prod_{\mathcal{U}} \mathbb{F}_p((x))$ just as well. They are also $\aleph_1$-saturated. If we assume the continuum hypothesis[1], they are of cardinality $\aleph_1$ and so isomorphic. Similarly $\prod_{\mathcal{U}} W(\mathbb{F}_p)$ and $\prod_{\mathcal{U}} \mathbb{F}_p((x)))$ are isomorphic as valued fields, say via φ. Let σ_{pwf} be the Witt Frobenius on $W(F_p)$ and let σ_{pc} be the automorphism sending $\sum a_i x^i$ to $\sum a_i^p x^i$ in $F_p((x))$. Consider the valued fields with isometry $(W(F_p), \sigma_{pwf})$, and $(F_p((x)), \sigma_{pc})$. Denote respectively by σ_{wf} and σ_c the induced automorphisms on the ultraproducts of these fields.

Let $A = \prod_{\mathcal{U}} W(\mathbb{F}_p)[t]$, the polynomial ring over $\prod_{\mathcal{U}} \mathbb{Q}_p$. Consider $\prod_{\mathcal{U}} W(F_p)$ as an A-module with t acting as σ_{wf} on $\prod_{\mathcal{U}} W(F_p)$. Consider also $\prod_{\mathcal{U}} F_p((x))$ as an A-module via φ and t acting as σ_c, namely $m.\sum t^i c_i = \sum \sigma_c^i(m)\varphi(c_i)$.

[1]S. Shelah has constructed a model of ZFC where these ultraproducts are not isomorphic.

We will consider their theories first as modules, then as valued modules with the usual valuation map. As modules they are elementarily equivalent if for all $q(t) \in A$, $ann(q)$ is non-zero in $\prod_{\mathcal{U}} W(F_p)$ iff it is non-zero in $\prod_{\mathcal{U}} F_p((x))$ (see Corollary 2.5). Since both structures satisfy the linear Hensel property and since the residue fields are isomorphic to $\prod_{\mathcal{U}} F_p$ and since both $\bar{\sigma}_{wf}$ and $\bar{\sigma}_c$ act as the standard Frobenius on the residue fields, we get the result.

Concerning their theories as valued modules, first we have to check that they are models of the schemes (DG) and (IR), and by remarks on completions in Corollary 6.4, since in this case the image by the map w of the module is equal to the value group of the ring, it suffices to examine the cardinalities of the annihilators in the quotient V_0/V_0^+ or in the subgroup V_0; namely either in $\prod_{\mathcal{U}} F_p$ or in the subgroup $\prod_{\mathcal{U}} W[F_p]$ (respectively $\prod_{\mathcal{U}} F_p[[x]]$). Since $\prod_{\mathcal{U}} F_p$ is infinite, it does not satisfy any identities, so (IR) holds, and the axiom (DG) still holds using the linear Hensel property and the fact that F_p is p-closed. So we have elementary equivalence as valued modules as well. This follows also from [6], but it might be appropriate to notice that it already follows from the *linear theory*.

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