Faculté des Sciences

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Compact objects in modified theories of gravity

PhD Thesis submitted by

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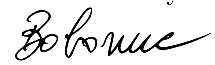
Dedicated to the beloved memory of



Smile, though your heart is aching
Smile, even though it's breaking
When there are clouds in the sky
you'll get by
If you smile through your fear and sorrow
Smile and maybe tomorrow
You'll see the sun come shining through
for you

Light up your face with gladness
Hide every trace of sadness
Although a tear may be ever so near
That's the time you must keep on trying
Smile what's the use of crying
You'll find that life is still worthwhile
If you'll just
Smile

Smile,
Lyrics by John Turner and Geoffrey Parsons,
On a music by Charles Chaplin.

Contents

Acknowledgements

The following text represents the completion of six years of a PhD thesis filled with emotion. Before switching to a formal “we” to design myself, I would like to dedicate the present pages to thank all the peoples who contributed to transmuting this dream into a reality. This text is written in English (as you may have noticed. . .) but, in order to give to the peoples involved the opportunity to appreciate the importance of their contribution, I will switch to French to thank those of you who may not be completely familiar with Shakespeare’s language. In those cases, an approximate translation in English will be given in a footnote so that everyone is included.

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[1] I thank my best friend and life partner, Marie-Aurore, for always supporting me through this thesis. . . and life in general.

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[4] Merci à tous les professeurs qui m'ont tant appris. Merci aussi aux étudiants pour m'avoir poussé à me poser des questions que je ne soupçonais pas toujours.

bienveillance générale si rare et si précieuse au sein du milieu universitaire.[5]

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Finally, all my thanks to any of you who have contributed directly or indirectly to the completion of this work and that I may have (to my greatest shame) forgotten to cite here.

[5]Many thanks to Anne Staquet for her valuable advices and for her constant kindness.

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Notations and Conventions

In this little note, we provide a presentation of the main notations and writing conventions of this text. In principle, most – if not all – of the symbols are (re)defined the first time that they appear in the body of the text of a given chapter but this note should facilitate the reading of independent parts of the text by allowing the reader to find a reference for our notations without having to extensively "scroll" in the different chapters.

1 Fundamental Constants and Natural Units

In this section, we list, for definiteness, the notations for the fundamental constants used in this text and recall their values in the international system of units (SI units). We also provide a quick recap of the so-called natural units that will be used through our research papers.

1.1 Fundamental Constants in SI Units

These values were checked from the NIST (National Institute of Standards and Technology) database and the BIPM (Bureau International des Poids et Mesures) definitions on 30 April 2022. Digits inside parentheses here denotes the uncertainty on a value in concise form[1].

Through this text :

1. Newton's constant of gravitation will be denoted $\mathcal{G}$. Its value in SI units is
$$\mathcal{G} = 6.674\ 30(15) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}.$$

2. The speed of light in vacuum will be denoted c. Its value in SI units is
$$c = 299\ 792\ 458\ \mathrm{m}\ \mathrm{s}^{-1}.$$

[1] For example, 1.993 1000(05) is a shortcut for $1.993\ 1000 \pm 0.000\ 0005$.

3. Planck's constant will be denoted h. Its value in SI units is

$$h = 6.626\ 070\ 15 \times 10^{-34}\ \mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-1}.$$

We should also rely on the reduced Planck constant $\hbar \coloneqq h/(2\pi)$.

4. The vacuum permittivity will be denoted ε_0. Its value in SI units is

$$\varepsilon_0 = 8.854\ 187\ 8128(13) \times 10^{-12}\ \mathrm{m}^{-3}\ \mathrm{kg}^{-1}\ \mathrm{s}^4\ \mathrm{A}^2.$$

In case it may be of use, we should define that the vacuum permeability will be denoted by μ_0 and recall that one has the relation $\varepsilon_0\mu_0 = 1/c^2$.

In the above expressions, some constants have an error while others do not. Let us recall that this comes from the fact that the modern definition of the SI units fixes the value of some of the fundamental constants to an exact value as a way to define the SI units themselves. It is the case of c and $\hbar$ here above. It is also the case of the electric charge of the electron e and the caesium hyperfine frequency $\Delta\nu_{\mathrm{Cs}}$. The value of $\Delta\nu_{\mathrm{Cs}}$ is used to define the Hertz [Hz] (and hence the second [s] from $1\ \mathrm{s} = 1\ \mathrm{Hz}^{-1}$). The value of c is then used to define the meter [m]. The kilogram [kg] is then obtained by fixing the value of h. Finally, one fixes the ampere [A] (and hence the Coulomb [C] via $1\ \mathrm{C} = 1\ \mathrm{s\ A}$) from e. Note that this list is not exhaustive[2] but is sufficient for the units of quantities that we will be interested in here.

The other constants present an error since they are measured experimentally (like $\mathcal{G}$), or derived from quantities that are measured experimentally (like μ_0)[3], and not used as part of the units' definitions.

1.2 Natural Units

In the realm of theoretical physics, to further simplify formulas and inspections on physical units, one usually introduces the so-called natural units. This corresponds to a system of units based on the natural constants $(c, \hbar, \mathcal{G}, \mu_0)$ *i.e.* a system of units for which the numerical values of the constants assume a simpler form compared to SI units.[4]

One thus usually defines natural units by setting

1. For the speed of light in vacuum

$$c = 1.$$

[2]One should, for example, discuss the value of the Boltzmann constant and how it allows to define the Kelvin unit.

[3]The vacuum permeability μ_0 might be defined from the fine structure constant α via the relation $\mu_0 = 4\pi\alpha\hbar/(e^2c)$. In other words, a measure of the fine structure constant also gives, indirectly, a measure of μ_0 – or the other way around. The same goes for the vacuum permitivity ε_0 from the relation $\varepsilon_0\mu_0 = 1/c^2$.

[4]Of course, due to their interconnection, one could also use h instead of $\hbar$ or ε_0 instead of μ_0, or the electric charge of the electron e instead of ε_0 or μ_0 to fix the unit system. The important point is the idea of constructing the unit system in terms of a coherent and complete set of fundamental physical constants.

2. For the reduced Planck constant

$$\hbar = 1.$$

In other words, one constructs the units such that $h = 2\pi$.

3. For Newton's constant of gravitation, at least two conventions coexist. First, there is the convention that mimics the previous ones

$$\mathcal{G} = 1,$$

in this case one also refers to the system as "geometric units". But one also frequently encounters the convention

$$\mathcal{G} = \frac{1}{8\pi}.$$

The interest of this convention is that, in this case, the constant appearing in Einstein's equations of general relativity $8\pi\mathcal{G}/c^4 = 1$. This is the choice we will make in this text when referring to natural units.

4. Finally, for the vacuum permittivity and/or permeability, there are also at least two competing conventions. The first one consists in setting

$$\varepsilon_0 = \frac{1}{4\pi},$$

so that the constant appearing in Maxwell's law of electrostatic is given by $1/(4\pi\varepsilon_0) = 1$. This convention thus also implies that one sets $\mu_0 = 4\pi$. Another choice consists in setting instead

$$\mu_0 = 1,$$

since this is the constant appearing in Maxwell's equations to relate the derivatives of the Faraday tensor to the 4-vector current density. On account of the previous choices, this also consists in setting $\varepsilon_0 = 1$. This is the convention that we should follow here when referring to natural units.

This choice of units also allows simplifying dimensional analysis. Indeed, from the condition that $c = 1$, one can always eliminate time dimensions in terms of length. With this correspondence in place, the condition $\hbar = 1$ or the chosen condition on $\mathcal{G}$ further allows eliminating mass units in terms of length. Finally, the condition on ε_0 or μ_0 allows to further eliminate the ampere (or Coulomb) units in terms of length. In natural units, every quantity can thus be expressed in terms of a (fractional) power of length units – or equivalently as a (fractional) power of mass units.

To compare the value of a quantity expressed in natural units to its value expressed in SI units, one should just reintroduce the natural constants in the expression to get the correct dimensions. For example, according to our definition of natural units ($c = 1, \mathcal{G} = 1/(8\pi)$), omitting its uncertainty, one might express the mass of the sun, denoted $M_\odot$, as

$$M_\odot[\text{NU}] = 37\ 113.3\ \text{m} = 37.1133\ \text{km}.$$

This quantity is thus expressed in length dimensions. Reintroducing the natural constants in the expression, one can easily check that it is actually the combination $\mathcal{G}M_\odot/c^2$ who has the dimensions of a length. One then have

$$\frac{\mathcal{G}M_\odot}{c^2}[\text{m}] = \frac{\mathcal{G}[\text{SI}]\ M_\odot[\text{SI}]}{c[\text{SI}]^2} = \frac{\mathcal{G}[\text{NU}]\ M_\odot[\text{NU}]}{c[\text{NU}]^2} = \frac{M_\odot[\text{NU}]}{8\pi}.$$

From this expression and the SI unit values of $\mathcal{G}$ and c, one thus recovers the well-known value

$$M_\odot[\text{SI}] = 1.9885 \times 10^{30}\ \text{kg}.$$

Following this line, we would also get that, in geometric units ($c = 1, \mathcal{G} = 1$)

$$M_\odot[\text{GU}] = 1476.69\ \text{m} = 1.476\ 69\ \text{km}.$$

We should conclude this brief summary of natural units by an important warning : different authors might have a different terminology. For example, some authors might refer to what we called geometric units as their natural units. The term “natural units” might be used to refer, collectively or individually, to any system of units based on a set of reference quantities that allows to cover all possible units of interest. One should then be careful with the use of this term.

2 Einstein Summation Convention

Through this thesis, we will make frequent use of the Einstein summation convention : we will omit the writing of the summation symbol when a summation appears in an expression in which the summation index is repeated once in an “upper” and once in a “lower” position.

For example, we should write

$$\sum_{i=1}^{n} x^i y_i = x^1 y_1 + x^2 y_2 + \cdots + x^n y_n =: x^\mu y_\mu.$$

The values in which the summation index ranges should be clear from the context *i.e.* from the definition of the indexed quantities x^μ and y_μ. Obviously, the summation index being “mute”, $x^\mu y_\mu = x^\alpha y_\alpha = x^\tau y_\tau = \cdots$.

This convention of notation will be ubiquitous in this thesis **except in chapter 3**. In this chapter, since we will be discussing the mathematical and practical working of the main algorithm used through this work, we decided to reintroduce the summation symbols. We found this more convenient since the number of terms in the summations will sometimes depend on the choice of a partition of a given interval of the real numbers and since we will have to compare different partitions.

3 Symmetric and Antisymmetric Parts of a Tensor

Another important convention that we will use concern the symmetry properties of a tensor. More precisely, to fix the notation, let us consider a tensor whose all indices are covariants. Let $T_{i_1 i_2 \cdots i_n}$ be the components of this tensor in an appropriate basis.

The symmetric part of the tensor of components $T_{i_1 i_2 \cdots i_n}$ is defined as the tensor of components

$$T_{(i_1 i_2 \cdots i_n)} \coloneqq \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_n)},$$

where S_n denotes the group of permutations of n elements.[5]

The antisymmetric part of the tensor of components $T_{i_1 i_2 \cdots i_n}$ is defined as the tensor of components

$$T_{[i_1 i_2 \cdots i_n]} \coloneqq \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)\ T_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_n)},$$

where $\epsilon(\sigma)$ denotes the signature of the permutation σ. So $\epsilon(\sigma) = +1$ if the permutation is even and $\epsilon(\sigma) = -1$ if the permutation is odd.[6]

A tensor will thus be completely symmetric if and only if it is equal to its symmetric part and completely antisymmetric if and only if it is equal to its anti-symmetric part. We could also note that any tensor twice covariant can be decomposed as the sum of its symmetric and antisymmetric part since

$$T_{ij} = T_{(ij)} + T_{[ij]}.$$

We should also note that, obviously, one can apply the above procedure to impose symmetrisation or anti-symmetrisation for only some of the indices. For example, one has that

$$T_{i[jk]} = \frac{1}{2}\left(T_{ijk} - T_{ikj}\right).$$

Finally, if we have to consider the symmetrisation or anti-symmetrisation over indices that are not adjacent, we will indicate the indices that are excluded from the operation by placing them between vertical bars. For example,

$$T_{(i|jk|lm)n} = \frac{1}{6}\left(T_{ijklmn} + T_{ijkmln} + T_{ljkimn} + T_{ljkmin} + T_{mjkiln} + T_{mjklin}\right).$$

[5] For example,

$$T_{(ijk)} = \frac{1}{3!}\left(T_{ijk} + T_{ikj} + T_{jik} + T_{jki} + T_{kij} + T_{kji}\right).$$

[6] For example,

$$T_{[ijk]} = \frac{1}{3!}\left(T_{ijk} - T_{ikj} - T_{jik} + T_{jki} + T_{kij} - T_{kji}\right).$$

4 Table of Notations

Hereunder, we offer a table of the conventions of notation used in the different chapters of this thesis.

Symbol	Concept
	Generic
$\mathbb{N}$	Natural numbers
$\mathbb{R}$	The set of real numbers
$[a, b] \subset \mathbb{R}$	Interval in on the real numbers
$\mathbb{C}$	The set of complex numbers
c	Speed of light in vacuum
$\mathcal{G}$	Newton's constant of gravitation
h, $\hbar$	Planck's constant and reduced Planck constant
ε_0	Vacuum permittivity
μ_0	Vacuum permeability
$\mathcal{O}$ (ex: $\mathcal{O}\left(\epsilon^2\right)$)	Big O notation (ex: Big O of ϵ square)
$\otimes$	Tensor product
$\mathscr{L}$	Lagrangian density
S	Action functional
	Chapter 1
(ct, x, y, z)	Inertial frame on Minkowski spacetime
τ	Proper time of a particle
$\mathrm{d}s^2$	Spacetime interval or line element on a curve
$\mathbb{M}_4$	Minkowski spacetime (as an affine space)
$\mathcal{V}^4$	Vector space associated to Minkowski spacetime
$\boldsymbol{\eta}$, η_{ab}	Minkowski metric and its components
$(\mathbb{M}_4, \boldsymbol{\eta})$	Minkowski spacetime
$\vec{u}(\tau)$	4-velocity of a curve
$\vec{P} = m\vec{u}$	4-momentum of a pointwise particle of mass m
E	Energy of a particle
$\vec{E}$, $\vec{B}$	Electric and magnetic fields
$\mathbf{F}$, F_{ab}	Faraday tensor and its components
J^μ	Components of the 4-vector density current
$\mathbf{A}$, A_a	Electromagnetic vector potential and its components
T_{ab} or Θ_{ab}	Components of the energy-momentum tensor
m_{I} or m, m_{G} or m	Inertial and gravitational mass of a particle
$\mathbf{N}^4$	Newtonian spacetime (only in remark 1.2)
$\mathbb{E}^3$	3D Euclidian space (only in remark 1.2)
$\mathcal{M}$	Manifold
(U, Φ)	Chart on a manifold
$\{\, (U_i, \Phi_i) \,\}$	Atlas on a manifold

Φ or $\{ x^\mu \}$	Local coordinate system on a manifold
$\mathfrak{X}^\mu : \mathbb{R}^n \to \mathbb{R}$	Projection onto the μ^{th} component on $\mathbb{R}^n$
$f : \mathcal{M} \to \mathbb{R}$	Real-valued function on a manifold
$\mathscr{C} : \mathbb{R} \to \mathcal{M}$	Curve on a manifold
$\boldsymbol{\partial}_\mu$	μ^{th} partial derivative on $\mathbb{R}^n$
$T_p\mathcal{M}$	Tangent space at a point $p \in \mathcal{M}$ of a manifold
$T_p^*\mathcal{M}$	Cotangent space at a point $p \in \mathcal{M}$ of a manifold
$\nabla_{\vec{v}} f\|_p$	Directional derivative of a function f in the direction of a vector $\vec{v} \in T_p\mathcal{M}$
$\Gamma\left(T\mathcal{M}\right)$	Smooth vector fields on a manifold
$[\vec{v}, \vec{w}]$	Lie bracket of two vector fields (here $\vec{v}$ and $\vec{w}$)
$\Gamma\left(T^*\mathcal{M}\right)$	Smooth covector fields on a manifold
$\mathrm{d}f$	Differential of a function f
$\{ \partial_\mu \}$	Coordinate basis *i.e.* natural basis associated to a coordinate system $\{ x^\mu \}$
$\vec{v}_\lambda$	Vector tangent to a curve $\mathscr{C}$ at point $\mathscr{C}(\lambda)$
$\mathscr{C}_p^{\vec{V}}$	Integral curve of a vector field $\vec{V}$ through a point $p \in \mathcal{M}$
$\{ \mathrm{d}x^\mu \}$	differentials of functions $\{ x^\mu \}$, dual basis of a coordinate basis $\{ \partial_\mu \}$
$\vec{v}_{\|\|\mathscr{C}}\left(p\right)$	Parallel transport of a vector $\vec{v}$ along a curve $\mathscr{C}$ up to a point p
$\mathcal{T}$	Generic tensor or tensor field
$\nabla_{\vec{V}} \vec{v}\|_p$	Covariant derivative of a vector field $\vec{v}$ along a vector $\vec{V}$ at point p
$\{ \vec{e}_{(a)} \}$	generic basis of vector fields
$\{ \underline{\theta}^{(a)} \}$	Dual basis of a basis of vector fields $\{ \vec{e}_{(a)} \}$
${e_a}^\mu$	Components of a basis $\{ \vec{e}_{(a)} \}$ in a coordinate basis $\{ \partial_\mu \}$
${e^a}_\mu$	Components of a dual basis $\{ \underline{\theta}^{(a)} \}$ in a dual coordinate basis $\{ \mathrm{d}x^\mu \}$
ω^b_{ac}	Connection coefficients in a generic basis $\{ \vec{e}_{(a)} \}$
$\omega^b_{a\mu}$, ${\omega^b}_a = \omega^b_{a\mu}\mathrm{d}x^\mu$	Spin connection coefficients and connections 1-forms
$\Gamma^\rho_{\mu\nu}$	Christoffel symbols *i.e.* connection coefficients in a coordinate basis $\{ \partial_\mu \}$
∇	Covariant derivative operator or covariant differential operator
$\mathcal{L}$	Lie derivative (only in remark 1.7)
d	Exterior differential
$\wedge$	Wedge product
$\mathbf{R}$, ${R^a}_{bcd}$	Curvature tensor and its components
R_{ab}	Components of the Ricci tensor

$\mathbf{T}$, $T^c{}_{ab}$	Torsion tensor and its components
$\boldsymbol{g}$, g_{ab}, g^{ab}	Metric tensor and its components and inverse components
$\flat$, $\sharp$	Musical isomorphisms (only in remark 1.10)
$\mathrm{O}(r,s)$	Group of generalised orthogonal transformations
$\mathbf{Q}$, Q_{abc}	Non-metricity tensor and its components
$\overset{\circ}{\omega}{}^b_{ac}$, $\overset{\circ}{\omega}{}^b_{a\mu}$, $\overset{\circ}{\Gamma}{}^\rho_{\mu\nu}$	Connection coefficients of the Levi-Civita connection (when compared to another connection)
$\mathbf{D}$, $D^a{}_{bc}$	Disformation tensor and its components
$\mathbf{K}$, $K^a{}_{bc}$	Contorsion tensor and its components
$L(\mathscr{C})$	Length of a curve $\mathscr{C}$
s	Arc-length of a curve
R	Ricci scalar
$\mathbf{G}$, G_{ab}	Einstein tensor and its components
T	Torsion scalar
T_{vec}	Vector torsion
B	Teleparallel boundary term
Chapter 2	
ψ or π or ϕ	Scalar field
Δ	Laplacian operator
$\Box$	Dalembertian operator
Chapter 3	
$\boldsymbol{x} = \{ x_1, x_2, \cdots, x_M \}$	Mesh of $M+1$ points
$S^{\boldsymbol{r}}_N(\boldsymbol{t})$	Splines of degree N, based on a mesh $\boldsymbol{t}$ and with continuity requirements $\boldsymbol{r}$
ρ_k	Gauss-Legendre collocation points on $[-1,1]$
$f^{(n)}$	n^{th} derivative of a function f
$\underline{u} = (u_1, \cdots, u_d)$	Unkwnown functions for COLSYS
$\underline{Z}(\underline{u})$	Vector of unknowns for COLSYS
$\hat{s}$	Error function

Introduction

What is gravity? One question, three words, a (possibly infinite) number of underlying interrogations. This question, so simple to phrase, is indeed at the core of a whole branch of physicists' endeavour since (and, arguably, still for) quite a long time.

To start this discussion with really down-to-earth considerations, we should acknowledge that the abyssal depth of this question is, at first, already tied to the very meaning of its central word: "gravity". A first touch on this question would indeed be to rephrase it as "What does the word 'gravity' means?". This should reasonably be the first (sub-)question to answer to enter this vast subject and it is, in a sense, already a dead end.

Indeed, as for most – if not any – physical concepts, we should acknowledge that a precise and definitive definition of what the term "gravity" means is a hopeless task prior to any sort of modelling. It is so because the way we will conceive gravity inherently depends on a subtle mix between mathematical tools and the physical interpretation we aim to invest these objects with. This does not mean, of course, that we cannot develop theories about gravity that are precise in both the mathematical and physical sense. Nevertheless this already suggests from the very beginning that this question relies on a precarious balance and is not guaranteed to have a simple and/or single answer.

That being said, sticking to down-to-earth and intuitive considerations, something remains certain: things fall … and we would like to understand how.

1 A Small History of Gravity

In fact, even this last statement could, in bygone days, have been subject to discussions.

1.1 Gravity Before Newton

Aristotle

In the, outdated, Aristotelian view of the world, the concept of gravity didn't really make sense. According to Aristotle, phenomena should be structured by means of a (mostly *ad hoc*[1]) framework, whose Earth occupies the centre, divided into two impermeable parts:

1. The sublunar world, which corresponds to the part of the world[2] that consists of the earth and its "atmosphere"[3] up to, and including, the moon.

2. The supralunar world, which corresponds to the rest of the world and namely refers to the realm of celestial bodies.

In the sublunar world, all observable objects were supposed to be composed of different amounts of four elements: earth, water, air and fire. To these four elements, Aristotle assigns a natural state (the state describing what these elements, and bodies built from them, will do if nothing provides them to do so): to be at rest in their "natural position". The notion of natural position depends on the nature of the elements one aims to consider. The two heavy elements, earth and water, would tend to be as close as possible from the centre of the Earth, with earth having the most prominent tendency to do so (so that earth sink in water). On the contrary, the two light elements, air and fire, would tend to be as far away as possible from the centre of the Earth, while remaining in the sublunar world, with fire having the most prominent tendency to do so (so that fire should ultimately end above air).

Of course, even if the natural state of bodies were to be at rest, they could be set in motion. Here again, Aristotle proposed a distinction based on two classes: "natural" and "unnatural" (or "violent") motions. Natural motions consist in any vertical motion in which a body tends to reach (or at least to get closer from) its natural position. For example, if one holds a stone in his hand and drop it, the motion will be a natural motion as the stone will fall on the ground and thus reach his natural position. On the opposite, unnatural motions encompass all motions that move a body away from its natural position. This then includes vertical motions that goes opposite to natural ones – such as the one consisting of raising a stone from the ground – or all possible horizontal motions – such as sliding a stone on the ground.

In Aristotle's view, the distinction between natural and unnatural motions is based on the idea that natural motions do not require any explanation (they are

[1] By this, we mean that aristotelian view is based on assumptions that, for the most part and definitely for what may concern gravity, cannot be experimentally disputed. It is in fact, to some extent, more a philosophical dogma than a scientific theory.

[2] Here, especially while we will be discussing old conceptions, we will be using generic words like "world" or "universe" to informally refer to the physical reality. We may also, sometimes, use words that refer to concepts more modern than the theory we expose. This should allow us to make it more accessible and put it in perspective.

[3] See previous remark.

"built-in" properties of matter) while, on the contrary, any unnatural motion should be explained by an external cause as they should never happen "by themselves". There is also the idea that no action at a distance should be possible so that any unnatural motion should ultimately be explained through a direct contact of something with the object. Finally, one should note that any motion, whether natural or not, was assumed to have a beginning and an end so that perpetual motion did not make sense.

This description was thus supposed to allow describing physical motions on Earth. Without entering the details, as our main purpose is not to do philosophy nor history of Science, this description also relied on the idea that objects on Earth ($\sim$ in the world of men) were imperfect and subject to changes and alterations. This was stated in contrast with the celestial world ($\sim$ the world of gods) which was assumed to be perfect and immutable. This idea has his direct reflection in the way Aristotle structures the supralunar world.

In the supralunar world, celestial bodies were supposed to be made of a fifth element, called æther[4]. Since celestial phenomena seemed to occur in an unchanging and regular way, æther was endowed with different properties than the sublunar world's elements. First, æther's natural state was supposed to be a uniform circular motion (centred around the Earth). In addition, on account for its "perfection", the notion of unnatural motion, and alterations in general, did not apply to it. This allowed to explain the apparent steady circular motion of the sun and distant stars around the earth.

Aristotelian view of the world, despite its pleasant logical self-consistency, suffered from several problems and difficulties to easily explain some phenomena – such as the fact that an arrow remains in motion even after it left the arrow bow that shot it out – but a detailed analysis of these flaws is not aimed here. For the sake of our pedestrian preamble, we mostly need to emphasise one key aspect of Aristotle's construction: to state it in a catchy way, there is no such a concept as gravity in this picture.[5] Indeed, either the freefall of objects at the surface of Earth or the motion of celestial bodies were postulated as inherent properties of matter (*i.e.* the five elements) that did not require an explanation.

In a totally *a posteriori* analysis, it is amusing to remark that, in Aristotle's philosophy, what we would today name "gravity" was incorporated as an internal property of matter (*i.e.* of the object that is moving) while, in modern views, instead, gravity is ultimately encoded in properties of space(time). This is, of course, far from being the main difference between the modern and Aristotelian approaches! In our modern-day constructions, in contrast with the Aristotelian view, it is in particular always possible, at least in principle, to consider that things could go in a different way that what is proposed by a given model.

[4]In Aristotle's classification, æther actually appears as the first element. First in that it was supposed to be superior to the other four.

[5]We should slightly nuance this assertion below. In any case, we can say that, should a property be defined a "Aristotelian gravity", it cannot be understood as an interaction (since it is related to a purely internal property of bodies). Also, in this case, one should even talk of "Aristotelian gravities" in the sense that its description for bodies on Earth is assumed to be completely unrelated to its description for celestial bodies.

From Copernicus to Descartes

Despite the aforementioned problems, Aristotelian view of the world remained prevailing, at least in western Europe, for more than a millennium. Critics of Aristotle's ideas arose gradually across the Middle Ages but it was not until the advent of heliocentrism that they really started to wobble on their pedestal.

The development of increasingly more reliable tools to perform astronomical observations challenged Aristotle's view of a perfect and never changing celestial world. In addition, the simplicity offered by a heliocentric description of the observations of planetary motions slowly drove Earth from the centre of the universe. This then led to Copernicus's heliocentrism.

As (more or less) always in science, these new discoveries raised new questions. With the idea that the Sun was at the centre of the solar system, Earth had to be seen as a planet among others and (to preserve the ability to explain the cycle of days and nights) it had to turn on itself and achieve a full rotation each day. According to the Aristotelian view of motion, such a rotary motion should have induced observable effects that were not detected in actual experiments. As an example, combining the idea that Earth was turning on itself with the Aristotelian one stating that the unnatural motion of an object should stop when contact with the object stops, one arrived at the conclusion that a stone thrown in the air vertically should land far away from its departure point since the "horizontal" motion of the object should immediately stop after leaving the surface of the Earth who would, on the contrary, continue its displacement.

As widely known, the resolution of this kind of apparent paradox was made possible with the advent of the principle of inertia. A first, slightly incorrect[6], version was proposed by Galileo Galilei whose claim can be synthesised as "*Once thrown, a body will keep moving at a constant speed on a horizontal plane free of friction*". It is clear that Galileo was really close to the, say, modern version of the principle and that his idea of a "neutral" motion – referring to Aristotle's idea, that was neither natural (towards Earth's centre) nor unnatural (away from Earth's centre) – as a kind of motion on its own that do not necessitate external influence to maintain itself was a key step in the development of the principle. That being said, the first correct statement of the principle of inertia is usually attributed to René Descartes. Descartes changed the "horizontal plane" of Galileo for the more geometric and abstract notion of a "straight line". He also aimed to apply it without distinction to terrestrial and celestial bodies. These two key contributions from Descartes led to the first statement of *the* principle of inertia; namely that "*A body should keep moving along a straight line at a constant speed in the absence of any exterior influence*".

The advent of the principle of inertia is really what gave birth to the modern

[6] The incorrectness lies in the use of the term "horizontal plane" which refers to a surface equidistant form Earth's center. At large distances, this version of the principle of inertia thus allows for (almost) circular motions around the Earth without need of an external influence.

One should also mention that Galileo's principle was only supposed to apply to massive bodies at Earth's surface and that, perhaps rightly given the epoch, it thus makes no statment about the motion of celestial bodies, thus maintaining the aristotelian wall between terrestrial (sublunar) and celestial (supralunar) bodies.

conceptions on gravity. Indeed, from this principle, it finally comes that both the motion of freefall at Earth's surface (which will happen on a straight line if the body has no initial speed but, as already proved by Galileo in some of his work with inclined planes, not at a constant speed) and the motion of celestial bodies around the sun (which, even with the approximation that it is circular and uniform, is not on a straight line) should be explained by means of an external influence – in modern terms, an interaction.

1.2 Newtonian Gravity

The question of the nature of these interactions[7] remained unsolved for a few more years until the publication of Newton's work. As we know, rooted in the principle of inertia, Newton built his (Newtonian) mechanics. He also identified the interaction (the force) explaining freefall at Earth's surface and the motion of celestial bodies around the sun to be the same through his law of universal gravitation.

It is interesting to note in passing that, in the context of Newtonian mechanics, adopting a modern point of view, what Newton's first law (*a.k.a.* the principle of inertia) do is postulating the existence of inertial frames as the frames in which a body "should move along a straight line at constant speed in the absence of any external influence".[8] These frames then play a central role in the formulation of Newton's second law (*a.k.a.* – in an inertial frame – $\vec{F} = m\vec{a}$). Indeed, inertial frames are seen, among all possible reference frames, as the ones allowing to *really* measure external interactions (*a.k.a.* forces). Finally, Newton's third law (*a.k.a.* the law of action and reaction) postulates that, if a body 1 interacts with a body 2 so that 2 experiences a force $\vec{F}_{2\leftarrow 1}$ due to the presence of 1, in return 1 experiences a force $\vec{F}_{1\leftarrow 2}$ due to the presence of 2 such that $\vec{F}_{2\leftarrow 1} = -\vec{F}_{1\leftarrow 2}$. With the so-called principle of superposition, it reflects the infinite speed of propagation of interactions in Newtonian mechanics. The principle of superposition relies on the fact that a force is mathematically a vectorial quantity and, basically, says that if several external interactions, modelled individually by $\vec{F}_1, \vec{F}_2, \cdots, \vec{F}_n$, are applied to a body, the resulting net interaction on the body is given by $\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$. This, and Newton's other laws, also makes the force a measurable quantity (with a dynamometer, for example).[9]

In the context of Newtonian mechanics, thanks to the law of universal grav-

[7] Note that, according to the principle of inertia, it could have been that freefall at Earth's surface and the motion of celestial bodies around the sun relied on different interactions.

[8] We should note that, in Newton's version of the principle of inertia, a distinction is made between objects at rest and objects having a linear uniform motion. Since being at rest can be seen as a linear uniform motion at zero speed, we will avoid making the text heavier and omit such a distinction here.

[9] Note, however, that what one will measure is not "directly" the force but other properties (like the acceleration of an object or the lengthening of a spring). One will then use these data to reconstruct informations on the force by means of Newton's second law. One will then not be able to perform measurements of a force independently of the relation $\vec{F} = m\vec{a}$.

itation, the term gravity finally reaches a precise, quantitative, meaning and we can thus use it to ask precise questions about gravity.

A good example, related to the first formulation of Newtonian mechanics, concerns the means by which gravity propagates. Indeed, for Newton as for Aristotle[10], the idea of an interaction at a distance without any other form of explanation was highly unsatisfactory. At the same time, it was totally clear that gravity does not happen thanks to a direct contact between the objects attracting each other. This question led to several, unsuccessful but very interesting, attempts to explain the origin of the gravitational force based on the existence of a form of material filling in all the space called the gravitational æther. The properties of this material medium were supposed to explain those of the gravitational force. Using, again, an *a posteriori* analysis, it is interesting to see how, between Aristotle and Newton, gravity has moved from an internal property of bodies (and hence not even an interaction) to an external force applying to the bodies (thus really an interaction) but whose existence is attributed to the internal properties of a third party. That being said, let us stress that these models of gravitational æther revealed a challenge of interpretation – whose resolution was only provided a few centuries later with the development of the notion of field as a physical entity whose existence does not rely on the presence of a medium – but never reached a broad acceptance.

Other fundamental questions rely on the status and properties of gravity as a universal force. As we should see, these are key questions to ask about gravity but we should stop here this little introductory history since those questions and subsequent developments (especially general relativity and the physical and mathematical ideas leading to its formulation) are discussed as a central theme of chapter 1.

2 Gravity in Modern Days

Doing a little jump in time, to continue our introduction, we would like to review how gravity takes place in the modern picture of physics. This should help to get a better understanding of modern unsolved questions about gravity.

2.1 The Four Fundamental Interactions

Following, in a sense, the legacy of Newtonian mechanics, gravity is still considered in modern days as an interaction. According to our current understanding, it is one of the four fundamental interactions of Nature. To state it simply, the term “fundamental” here refers to the fact that, to this date, every physical phenomenon that we can model can be understood as a – possibly very intricate – consequence of some of these four interactions.

[10]The same goes for contemporary physicists, including ourself, which aims to describe the laws of physics by means of equations that relates quantities all expressed at the same point.

The four fundamental interactions

These four fundamental interactions are gravity, electromagnetism, the weak and strong nuclear interactions.

1. Gravitational interaction: It constitutes the main topic of this thesis. It is the fundamental interaction responsible for the falling of objects near Earth's surface, the motion of planets within the solar system, the formation and dynamics of galaxies, ... Theoretically, the range of this interaction is infinite, meaning that it should be present to describe the attraction of massive objects for every possible distance between these objects. Being the first discovered fundamental interaction, just for the seek of comparison with the other fundamental interactions, will take the intensity of the gravitational interaction as reference here. This means that its relative intensity will be taken to be 1.

2. Electromagnetic interaction: It is the fundamental interaction responsible for the fact that electric charges can attract or repel each other depending on their sign, for the existence and properties of electromagnetic waves (visible light, gamma rays, microwaves, ...) which fundamentally governs optical phenomena, ... Theoretically, the range of this interaction is also infinite. Actually, the electromagnetic interaction is the one for which the behaviour at different scales is better understood to this date. When compared to gravity, the relative strength of the electromagnetic interaction is about 10^{36}.

The other two fundamental interactions do not show up directly in day-to-day experiments but these are of great importance to understand the behaviour of matter at atomic and subatomic levels.

3. Weak nuclear interaction: It is the fundamental interaction responsible for some types of disintegrations such as the β-decay[11]. The range of this interaction is limited to a distance of about 10^{-18}m that confines its range to subatomic distances. When compared to gravity, the relative strength of this interaction is about 10^{25}.

4. Strong nuclear interaction: It is the fundamental interaction responsible for the cohesion of the atomic nucleus, and for the cohesion of the nucleons themselves. Its existence was first postulated from the necessity to understand the stability of atomic nuclei despite the electric repulsion of their components (a property that cannot be attributed to gravity due to its relative strength with respect to the electromagnetic interaction). It was later realised that the strong interaction was also responsible for the

[11] The β-decay is the process by which a neutron n^0 can decay to form a proton p^+, an electron e^- and an electron anti-neutrino $\bar{\nu}_e$. This process is usually symbolized as

$$n^0 \to p^+ + e^- + \bar{\nu}_e.$$

confinement of quarks (to simplify, the constituents of protons and neutrons – and also of other particles that, with protons and neutrons, forms the family of hadrons). In theory, the range of this interaction could be infinite but, in practice, it seems to be limited to distances of about 10^{-15}m that confines it into the nucleus. When compared to gravity, the relative strength of this interaction is about 10^{38} (which makes it stronger than the electromagnetic interaction, allowing it to explain the stability of atomic nuclei).

Why gravity is different

This super brief presentation of the four fundamental interactions calls for a few comments.

The first remark concerns the notion of the relative strength between the interactions. Of course, one cannot define the strength of an interaction in absolute terms, apart from any sort of context. The intensity of the interaction between two particles will depend on the distance by which they are separated and also on the coupling constants characterising how these particles react to the interaction under study. To compare the strength of two interactions one thus needs to consider particles that will interact with each other by means of both interactions and one should ideally refer to a clear test situation where the distance and the coupling constants can be estimated.[12] When we talk about the relative strength of two interactions here, we thus implicitly refer to mere estimations that can be made under specific circumstances. In any case, this small lack of precision does not ruin the idea of the comparison to show how one interaction can overcome another in situations where both apply; especially considering that these differences are always of several orders of magnitude. In particular, in the context of this discussion centred on gravity, these estimations emphasise two distinguishing properties of gravity when compared to the other interactions.

First, we see that, when compared with the other interactions (in situations where they apply), gravity is always significantly weaker. This observation could lead to wonder how it is possible that gravity is the first fundamental interaction that we ever discovered and that seems to rule the biggest structures in the Universe.

This then leads to the second observation. Gravity is a universal interaction (in the sense that every massive particle is subject to it) with infinite range of

[12] Considering the most intuitive example, for two protons distant by about 10^{-15}m, the ratio between the strength of the gravitational $|F_{\mathrm{G}}|$ and electromagnetic $|F_{\mathrm{EM}}|$ forces is defined as

$$\frac{|F_{\mathrm{G}}|}{|F_{\mathrm{EM}}|} = \frac{\mathcal{G}\ (m_p)^2}{k\ e^2} \approx 10^{-36},$$

where $\mathcal{G}$ is Newton's constant of gravitation, m_p the proton mass, $k = 1/(4\pi\varepsilon_0)$ is the constant involved in the law of the electrostatic force and e is the electric charge of the proton. Note that here the ratio is actually independent of the distance due to the $1/r^2$ behaviour of both the gravitational and electrostatic forces. When comparisons should be made with the weak and strong interactions, the situation gets trickier but the principle remains similar.

action and that is always attractive (indeed, the coupling constant to gravity is basically the mass of the particle, which is always positive).

This makes a significant difference with the weak and strong interactions which only apply at a finite and quite small range (for either theoretical or experimental reasons). On the contrary, the electromagnetic interaction also has an infinite range of applications (both theoretically and in practice as far as we know). Yet, the sources of this interaction (namely electric charges) come in two "species" (positive and negative charges) that allow explaining that the interaction of two electrically charged particles can be either attractive or repulsive. Since these two types of charged particles appears to be present in the same amount in the Universe, at large scales, the net effects of the electromagnetic interaction also tend to disappear, leaving gravity as the dominant interaction to rule large-scale structures or, at least, electrically neutral macroscopic ones.

The second remark that should be made concerns the theoretical framework used to describe these interactions. The electromagnetic, weak and strong interactions are – with the classification of the elementary particles subject to these interactions – the keystones of the standard model of particle physics. This theory allows describing these three interactions as gauge theories. Following the work of Sheldon Glashow, Abdus Salam and Steven Weinberg, the electromagnetic and weak interactions can even be unified as a single "electroweak" interaction in this framework. The standard model of particle physics offers an extremely successful theory at both the theoretical and experimental level that allows ordering the behaviour of elementary particles in any known situations where gravity can be neglected.

On the contrary, the gravitational interaction is not included in the standard model of particle physics. On account for the above analysis, it should make sense that, to describe elementary particles that are subject to interactions that will so massively overcome gravity, the influence of the gravitational interaction can be consistently neglected. This is also emphasised by the excellent experimental adequacy of the standard model with experiments performed with particle accelerators.

Grand Unified Theory

Reasons to be dissatisfied with this fact may come from elsewhere. There is an important "meta-principle" in theoretical physics that aims to search for frameworks that allows for a simple and unified formulation of the laws of Physics. We called it a meta-principle since it is not a real physical principle (as the principle of inertia for example). It does not impose any constraint on the laws of physics themselves. It is more of a philosophical mindset. This idea that there must be a single and elegant (*i.e.* maybe conceptually complicated but well adapted to the task) framework allowing us to study the laws of Physics does not rely on a purely rational reasoning. Indeed, there is no guarantee at all that such a framework exists – unless, maybe, the fundamental idea that the Universe exists and is unique and that, consequently, there must be a way to depict it with respect to this "unity". That being said, even though there is no firm guarantee, this principle does not only rest on mere æsthetics considerations.

In the History of Physics, several examples have demonstrated the enhanced explicative power of unified frameworks. In this respect, one should cite the example of Maxwell's electromagnetism which, by unifying the laws of electricity, magnetism and optics, allowed to predict the existence of electromagnetic waves that were unknown in the former (incomplete) formulation. This then allows the above meta-principle to rest on a reasonably optimistic basis.

The absence of gravity from the standard model of particle physics then revealed the lack of a unified framework to express the laws of nature. This observation raised (from the '60s on) a lot of activity trying to formulate gravity explicitly as a gauge theory, similarly to the other fundamental interactions. Despite its intrinsic interest, we should not enter a detailed discussion of this point here. Instead, we will continue our discussion by focussing on the standard formulation of the gravitational interaction offered by Albert Einstein.

The above presentation replaced gravity in the physical panorama. A few more words on Einstein's theory of gravity should reveal that, independently from its relation with the theory describing the other interactions, there are some mysteries left unsolved that motivates further explorations on the subject.

2.2 General Relativity

Offering a decent while informal presentation of general relativity in a few paragraphs requires to sacrifice a lot of interesting (if not important) aspects of the topic. Indeed, general relativity and its consequences did not only fundamentally revolutionised our understanding of gravity. They also restructured most of our ideas about the Universe. In the following, to focus on our task to introduce our research subject, we should mostly concentrate on some of the main consequences and challenges revealed by the theory across the XX$^{\text{th}}$ century when confronted to experimental data. This presentation should allow us to introduce the main puzzles left unsolved to this date, each of which being a possible motivation for the work of this thesis. A few more theoretical and technical questions should be addressed later on, through the rest of this thesis.

General relativity in the nutshell of a nutshell

The most successful theory to describe the gravitational interaction to this date is general relativity. The first formulation of the theory is famously due to Albert Einstein who completed a first presentation in 1915. Some key motivations to Einstein's construction are detailed in chapter 1 but we can already mention that the construction of general relativity was shaped by the necessity to formulate the gravitational interaction in a relativistic context (such as Maxwell's equations for electromagnetism which can be elegantly formulated on Minkoswki spacetime) and by the very specific properties of gravity revealed by the Newtonian theory of gravity.

As explained above, general relativity is not (at least explicitly) formulated as a gauge theory. In general relativity, gravity is not understood as a force any more (even though it remains a physical interaction) but as an effect related to the curved nature of spacetime. In other words, the gravitational interaction merges with the structure of spacetime, which is then not rigid, *i.e.* fixed, any more. The curvature of spacetime is induced by the source of gravity, the energy-momentum content of matter.

General relativity at the solar system scale

Apart from its theoretical elegance, the main strength of general relativity is its ability to match with experiments. The experimental successes of general relativity at the solar system scale started very soon after its formulation with the explanation of the advance of Mercury's perihelion and the correct prediction of the deflection of light by massive objects.

In this respect, a famous experiment was conducted during a total solar eclipse on May 29, 1919, by Frank Dyson and Arthur Eddington who aimed to confirm general relativity's predictions. The results of this experiment helped to popularise Einstein's theory worldwide. Even though the reliability of this set of measurements have been questioned, the experiment has been repeated in subsequent times and confirmed adequacy to the predictions of general relativity.

General relativity and the universe: cosmology and its first successes

The consequences of general relativity when used to describe the gravitational interaction at much broader scales also played an important role in its success and gave birth to cosmology as a physical endeavour. Indeed, when studied at very large scales, general relativity predicted for the first time the dynamical character of the Universe itself.

Here again, an important historical anecdote should be mentioned. When realising that its theory predicted a universe that was either in expansion or in contraction, Einstein modified its equations to add to it a cosmological constant that was fine-tuned to allow for a(n unstable) static solution. When the expanding character of the universe was experimentally highlighted, Einstein famously came back to its original introduction of the cosmological constant by calling this move "the biggest mistake of his career".

The characteristics of solutions of Einstein's equations were studied (independently) by Karl Friedmann, who provided a detailed analysis of the different types of solutions that one can find assuming a homogeneous and isotropic universe, and Georges Lemaitre who first interpreted the apparent motion of recession of distant galaxies, as observed experimentally by Edwin Hubble, as coming from the expansion of space(time) itself. This successful interplay between general relativity and the numerous new experimental data available at that time thus really founded Einstein's theory as the basis for modern cosmology.

These findings raised the idea that the Universe itself have indeed a dynamics – it is expanding – and that the key quantity to understand the behaviour of this expansion was the matter density of the Universe. This then placed the question to characterise this matter density at the centre of cosmological considerations.

In addition, the fact that the Universe is expanding also suggested that, in the distant past, it should have been much smaller and hence hotter and denser. A detailed study of this "primal state" of the Universe permitted, among other things, to explain the relative abundance of chemical elements as a consequence of nucleosynthesis processes that were possible in these extremal conditions. It also led to the idea that, as an imprint of these first moments of the universe, one should expect to observe a residual radiation that would have been produced at that time and still be present to these days. This is the well-known cosmic microwave background (or CMB) whose existence was first postulated in 1948 by Ralph Alpher and Robert Herman and which was first observed around 1965 by Arno Penzias and Robert Wilson.

General relativity and cosmological puzzles: dark matter and dark energy

All of this constitute a non-exhaustive list of successes whose either discovery or interpretation is directly related to Einstein's general relativity. As we already saw, if general relativity is very well tested at the solar system scale, some of its most impactful discoveries are related to the field of cosmology. A detailed history of cosmology across the XX$^{\text{th}}$ century is not aimed here, but some of the main consequences of this fruitful activity are; more precisely, unsolved questions that arose as consequences of these studies. Indeed, if general relativity raised important ideas about the nature and evolution of the Universe, further investigations in these directions revealed intriguing phenomena.

Further investigations of the motion of distant galaxies, once compared to the predictions made by general relativity by taking into account only the visible matter, showed some important discrepancies and raised the question of the existence of a (*a priori* new) form of "invisible" matter that could explain the differences.

The first observation of the problem of missing mass can be traced back to the work of Fritz Zwicky in 1933. By means of the measure of the speed of galaxies within the Coma galaxy cluster, Zwicky concluded that the gravitational interaction associated to the visible matter in the cluster was not sufficient to reconcile the measured speeds of the galaxies with the fact that the cluster was not scattering.

Completing these observations, similar conclusions were obtained by Jeremiah Ostriker, James Peebles and Amos Yahil in 1974, and further confirmed from a broader set of data by the team of Vera Rubin and by Albert Bosma around 1978 when they studied the rotation curve of spiral galaxies. When studying the profile of the orbital speed of visible stars in the galaxy as a function of their radial distance to the centre of the galaxy, experimental data showed that the speed was more or less constant (or even slightly increasing) when one increases the distance to the galaxy centre. This observation was in total disagreement with the expected profile in which the speed was supposed to decrease roughly as the inverse square root of the distance to the galaxy centre. Since one expects the major part of the matter in the galaxy to be located near the centre[13], this theoretical expectation can also be rephrased as the fact that one expected the velocity to decrease with the matter density.

In an attempt to solve this puzzle, if we trust our model of the gravitational interaction in the system, one thus arrives at the conclusion that the discrepancies between the expected and observed dynamics comes from the presence of a form of massive material that could not be detected from the light it emits. If such an extra matter content, that was not taken into account in the theoretical matter density profile, was present in the system, one could possibly reconcile the theoretical and experimental results.

The automatic questions following from this conclusion concerns the nature of this extra matter content and, related to that, the reasons for our inability to detect it from the light it emits.

If it was first proposed to be due to ordinary matter (*i.e.* matter content predicted in the standard model of particle physics) undergoing specific conditions, in 1982, James Peebles introduced the so-called cold dark matter (or CDM) model. In this model, the missing matter is modelled by means of a new type of particle which has a high mass, does not interact electromagnetically and whose dynamics within the galaxies is such that its speed is very small compared to the speed of light. This hypothetical kind of matter was dubbed "weakly interacting massive particles" (WIMPs). Subsequently, in 1984, Peebles extended his CDM model to include a cosmological constant, leading to the ΛCDM model.

The update of the status of the cosmological constant as a necessary piece to understand the dynamics of the Universe happened in 1998, following the work of the teams of Saul Perlmutter, on one hand, and of Adam Riess and Brian Schmidt, on the other hand. By performing a systematic study on type Ia supernovæ, both teams were able to reveal an accelerated feature of the expansion of the Universe that would argue in favour of a Universe with a cosmological constant $\Lambda > 0$. This discovery then replaced the cosmological constant at a central place in cosmology and, once again, revealed the necessity to answer questions regarding the origin and precise meaning of this new form of energy. This question is usually referred to as the study of dark energy.

2.3 An Always More Mysterious Universe

As we saw, in less than a century, the study of general relativity in the context of the dynamics of the entire Universe reshaped most of our conceptions of it. This study led to astonishing discoveries but also motivated the introduction of some quite mysterious pieces to describe the content of the Universe and raised the question for their nature.

This question became even more fundamental when taking into account the great success of the ΛCDM model. Indeed, thanks to the WMAP and Planck

[13] At least, this is what is suggested from the observation of the light emitted from visible matter.

space missions, a detailed study of the spectrum of the CMB (characterising its small but crucial inhomogeneities) have been made possible. This spectrum was compared to predictions from the ΛCDM model and showed an excellent agreement when the parameters of the model were fixed so that the relative amount of the different pieces of the matter energy content of the model – namely normal matter, dark matter and dark energy (modelled by the cosmological constant) – were taken such that normal matter account for $\sim$ 4.9%, dark matter for $\sim$ 26.6% and dark energy for $\sim$ 68.5% of Universe's content.

This thus argues in favour of the description offered by the ΛCDM model but, consequently, also gives a central place to questions related to dark matter and dark energy. Indeed, according to this, one arrives at the conclusion that the very successful description of physical processes offered by the standard model of particle physics informs us on less than 5% of the content of the observable universe.

One thus faces a portrait of the Universe which is bigger and more vivid than ever expected but also full of deep mysteries...

2.4 Alternative Theories of Gravity and Compact Objects

Through the above presentation, we gave a brief history of the main questions that arose all along the XX$^{\text{th}}$ century when using general relativity in the context of cosmology. We did so in order to set the context in which the mysteries of dark matter and dark energy take place. However, as the very title of this thesis suggests, our focus here will not directly be on cosmology but on compact objects. While reading the above material, one could maybe wonder how the problems of dark matter and dark energy arise in this context.

A second look at the ΛCDM model

This question gives us the opportunity to take a step back regarding the ΛCDM model. As we tried to briefly emphasise here, dark matter and dark energy are pieces that one introduces to remove the important discrepancies that arise when comparing predictions made using general relativity and the matter content known from the standard model of particle physics with experimental data. Of course, once again, on account of the excellent agreement of the best fit from the ΛCDM model with the experimental data from recent spacecraft missions, these are more than mere speculations. It is nevertheless very important to point out the following: apart from the matter content of the Universe (and a few other parameters), the ΛCDM model is based on the description of gravity offered by general relativity. This means that the results of this "best fit from ΛCDM" ultimately rely on the trust that we have in our gravity theory as much if not more as they rely on the assumed matter-energy content.

Stated in this way, one sees that the discrepancies with experimental data obtained by combining general relativity with the standard model of particle physics may as well come from general relativity itself. This then reveals the possibility to study alternative theories of gravity. In this respect, even if one should still analyse with care their cosmological implications, compact objects are extremely useful laboratories to test such alternative theories.

Compact objects

The term "compact object" usually refers to the endpoint of stellar evolution that results in a configuration with very high density compared to ordinary atomic matter. The term compact thus refers to the fact that these objects will concentrate a high mass within a radius significantly smaller than that of a usual star. Typical examples include white dwarfs, neutron stars or black holes. In the context of this thesis, the term "compact object" will be used as a collective way to refer to neutron stars, boson stars and black holes more specifically.

In the study of any alternative theory of gravity, these thus provide privileged laboratories for both theoretical and experimental investigations. Indeed, the fact that compact objects allow us to look at the behaviour of a theory when the most extreme conditions are imposed on matter also allows revealing possible limits of these alternative formulations.

In this respect, black holes, in particular, are especially powerful tools. In a sense, they are the most compact objects conceivable. They are expected to form as the result of the gravitational collapse of the most massive stars.[14] At the theoretical level, the study of black holes is at the same time quite simple and very rich. The most famous property of these objects is that they correspond to a configuration in which a region of spacetime – potentially containing a singularity (see below) – is causally disconnected from the rest of the Universe by an event horizon. In the context of general relativity, several quite generic properties of black holes are known.

It has been proved that the most general family of stationary (*i.e.* where no dynamical, time-dependent, process is happening) and asymptotically flat (*i.e.* such that the curvature of spacetime decreases as one get away from it) black hole solutions in general relativity is characterised by only three parameters – namely the mass, electric charge and angular momentum of the black hole. This property is responsible for the arguable simplicity of the description of (stationary + asymptotically flat) black holes in the context of general relativity. A few more details regarding this result are given in section 1.5 of chapter 1.

That being said, also in the context of general relativity, black holes contains some weird subtleties. Indeed, under some quite generic hypothesis, it has been proved by Roger Penrose and Stephen Hawking in the early 1970s that the final state of a gravitational collapse should lead to the formation of a spacetime containing a singularity (intuitively speaking, a point where the geometry of spacetime breaks).

[14]Let us note that the type of black holes described here, those forming as the result of a gravitational collapse, should more specifically be called *stellar* black holes to distinguish them from *primordial* black holes (black holes that are assumed to form soon after the big bang) and *micro* black holes (black holes of small mass for which quantum mechanical effects are supposed to play an important role and that could potentially form in particles accelerators). In the following, since we will not consider the dynamical processes leading to their formation, we will omit such a distinction and only talk about black holes in general.

Also in the context of general relativity, black holes are at the centre of the so-called black hole information paradox which reveals the difficulty to reconcile the geometry of black holes with quantum mechanical effects.

The study of how these properties of black holes known to general relativity can be modified or violated can be cited as a worth-investigating question in the context of modified theories of gravity from a theoretical perspective. In addition to that, the study of black holes in a given theory usually allows us to put constraints on the construction of non-trivial solutions for our modified theories of gravity. We should come back to this last property in chapter 2; see the no-hair theorems.

From an experimental perspective, there is also a lot of interesting work to do. Indeed, compared to cosmological observations, experimental facilities regarding black hole astronomy took a bit more time to arrive at maturity, but they eventually did. In this respect, we should mention the first experimental construction of the image of a black hole (via gravitational lensing) performed by the Event Horizon telescope in 2019. This very recent development of black hole astronomy will allow putting further constraints on alternative theories of gravity or, to put it the other way around, raises the need for theoretical predictions to be confronted with the experimental data.

Independent motivations to investigate alternative theories of gravity

Apart from its cosmological implications, the two most noteworthy predictions of general relativity are the existence of black holes and gravitational waves. We have already commented on the interest of alternative theories of gravity in the context of black hole physics. Even though this aspect has not been investigated in this thesis, we should also say a word on gravitational waves astronomy. Similarly to the development of black hole astronomy, due to the very challenging precision required, gravitational wave astronomy took a while to develop but, about a century after their first theoretical formulation by Einstein, gravitational waves have been directly observed for the first time from the LIGO and VIRGO collaborations in 2016.[15] Up to now, all results seem to be in agreement with general relativity. So, just as for black holes, this new era in the investigation of gravity through gravitational waves should open (and close) several ways of study for alternative theories of gravity.

Up to now, we tried to demonstrate the interest of the study of alternative theories of gravity at a classical (*i.e.* non-quantum) level. As a final motivation to the study of alternative theories of gravity, we should, of course, mention the elusiveness, through the last century, of a consistent quantum description of gravity. In this respect, alternative theories of gravity in general can either provide different starting points for an attempt of quantisation or, if they prove useful at the classical level, describe some low energy effective limits of the quantum theory of gravity to be found, leading to possible constraints in its construction.

[15] Note that indirect evidences were already accessible in 1974.

3 Structure of This Thesis

As we tried to emphasise during this introduction, the aim of this thesis is to discuss modified theories of gravity. More precisely, our focus has been the study of compact objects in alternative theories of gravity including a scalar field by means of both numerical and analytical methods.

To achieve this goal, one important step consists in emphasising the specificities and limits of the standard theories of gravity (nominally GR). To this purpose, we devoted chapter 1 to a review of some of the main physical and mathematical features of general relativity. This chapter also contains a first discussion on alternative theories of gravity by means of theories that can be proved equivalent to the GR description while being constructed from different geometrical structures. On account for the needs of this thesis, we choose to focus our presentation of the geometrical tools directly on spacetime (*i.e.* the – base – differential manifold). We thus avoided the introduction of the more recent and powerful (but arguably unnecessary in the context of this text) tools offered when studying physical theories through the lens of principal bundles. We thus reached a chapter in which most if not all of the material can be considered pretty standard. We nevertheless tried to make its presentation somehow valuable. We thus wrote this chapter with in mind the idea to be self-content and the, perhaps more conceited, objective to give our readers an opportunity to think twice on things they already know and to see them, here and there, from a hopefully different point of view. This personal presentation has also been designed with the hope to be helpful for less experienced readers who already had a first contact with general relativity. On account of the above remarks, slightly experienced readers could decide to skip sections 1.1, 1.2 and perhaps 1.4 and very experienced readers could decide to focus only on section 1.6 which present the teleparallel equivalents of general relativity.

With this geometrical framework set up, we can then turn to the question of the introduction of scalar fields to encode the modification with respect to general relativity. This is the topic of chapter 2, where we recall the basic properties of scalar fields on flat spacetime, the basic features of scalar fields on curved spacetime (*i.e.* in the context of GR) and the most important kinds of non-minimal couplings that were investigated in this work. Section 2.5 also contains a summary of the original results obtained during this thesis.

Finally, since most of our original results were obtained from numerical investigations, we devoted chapter 3 to a review of the main characteristics of the numerical method used during this thesis. We did our best to reach a presentation that could help beginners to start with this very versatile and powerful tool with more confidence regarding both the theoretical and practical operation of the algorithm.

Conclusions and perspectives for subsequent work are then presented at the end of this text, in a different chapter.

Last, but – needless to say – not least, the papers presenting the original results of this thesis are placed in the annexes, see appendices B, C, D, E and F. We favoured this type of presentation since all these results led to publications in peer-reviewed journals and since the chapters in the body of the text should provide a sufficient introduction to allow for their understanding. In order to facilitate the reading of the papers, we adopted an independent page numbering and an internal bibliography for each paper.

Chapter 1

Around General Relativity

In this chapter, we present a synopsis of the main features of the better-established theory of the gravitational interaction nowadays : General Relativity (GR). Although we assume our readers to already have some knowledge in GR, this review will allow the present text to be more self-contained while giving us the opportunity to fix useful notations for later use. It is also a good occasion to emphasise the main assumptions of the theory. The latter is very useful in the context of this thesis since we will call some of these assumptions into question as a basis of our presentation of alternative theories of gravity. This will be done, first, at the end of this chapter where we introduce teleparallel theories of gravity. This inspection of the hypothesis will also be at the core of the next chapter in the context of scalar-tensor theories of gravity.

We open this chapter, in section 1.1, with a quick reminder of the key structures of the theory of Special Relativity (SR), trying to emphasise how this theory paved the road to the development of GR. This part of the discussion is completed in section 1.2 with a brief discussion on the equivalence principle. In section 1.3, we reintroduce elements of the mathematical structure inherent to the formulation of GR and modern theories of gravity in general. This will allow us to emphasise some of the main assumptions of GR on a firmer mathematical basis. Following this review on the formalism, that might be seen as the kinematics of GR, we devote section 1.4 to the dynamical content of the theory, reintroducing Einstein's equation and the associated Lagrangian. With this at hand, we will dispose of the necessary tools to discuss the status of compact objects in GR and connect to the main topic of this thesis. In section 1.5, we propose a discussion on black holes in GR. Finally, in section 1.6 we present the main features of alternative theories of gravity based on different geometrical setups. This part of the discussion will allow us to further emphasise the specificities of the usual formulation of GR and to present the necessary material for some of the original work of this thesis.

The material of this chapter owes a lot to [Carroll, 1997] and [Wald, 1984], where less experienced readers should find very useful material and in depth explanations for the physical content, and to [Kobayashi and Nomizu, 1996], where more advanced readers can find wonderfully rigorous mathematical formulations for the more mathematical content.

1.1 Worth-sharing Lessons from Special Relativity

The theory of special relativity (SR) is a well-known theory that is already taught to undergraduate students in physics. In this respect, we should assume our readers to be well used to it. That being said, it is also well known that the advent of special relativity was a turning point in the history of physics in many respects so that SR is the theory in which much of the motivations in the construction of general relativity are anchored. On account for this fact, we decided to devote the present section to it.

This section thus serves two goals. First, it should give us an opportunity to fix some notations and to reintroduce, for completeness, some important ideas from SR. This first goal is motivated by the second one : in this chapter, we should present the relativistic theories of gravity on which this thesis is based. If much of our work have been done using the usual formulation of general relativity (which will be discussed in a rather consensual way in section 1.4), some of our recent work involves alternative theories of gravity (that are reviewed in section 1.6) that rely on a different formulation but are argued to lead to an equivalent description of the gravitational interaction. To be able to compare these formulations and their physical interpretation, we thus found it important to show how they are both, in different (but coherent) ways, rooted in SR. This then motivated to rediscuss a bit of SR itself to avoid ambiguities in the discussion.

1.1.1 Lorentz Transformations

A bit of history :
Maxwell, Michelson, Morley and the constancy of the speed of light

The XIX$^{\text{th}}$ century has seen the advent of many important discoveries regarding the (classical) nature of light. Young's experiment (1801) first reached a conclusion on the (classical) nature of light. Light behaves like a wave. A few decades later (1864), Maxwell arrived at its unified theory of electricity and magnetism : Maxwell's electromagnetism. After some work out, this unification suggested the existence of electromagnetic waves (waves produced by oscillations of the electric and magnetic fields). Also, an inspection of the equations pointed towards a speed of propagation for these waves curiously similar to the accepted value of the speed of light. This was enough for Maxwell (and others) to wonder if light itself could be such an electromagnetic wave. The confirmation of this fact was made possible by the first production of Hertzian waves (1886), by Hertz, who showed that these waves exhibit all the same features as light. Light was then a wave, but more precisely an electromagnetic one.

These important discoveries raised evenly important questions. All the waves known at the time (mechanical waves) needed the presence of a medium to exist. In other words, these were vibrations of this medium (like the sound or the wave propagating in a rope when you shake its ends). Remove the medium, you incidentally remove the possibility for the wave to propagate; or to even make sense. It was then highly natural, by analogy, to raise the question of the nature of the medium in which electromagnetic waves propagate. In addition to that, there was a related question attached to Maxwell's theory itself. Maxwell's equations – describing how electric and magnetic materials would produce electric and magnetic fields and, far away from such sources, describing the behaviour of electromagnetic waves – where not invariant under the Galilean transformations of Newtonian mechanics. In other words, their form could not be expected to hold in all inertial frames, according to Newtonian mechanics. Regarding electromagnetic waves, this question was tied to that of knowing in which (inertial) frame the speed of light could assume the value $c = 1/\sqrt{\varepsilon_0 \mu_0}$ predicted by Maxwell's equations. Indeed, speed is never expected to have an absolute meaning in Newtonian mechanics.

The assumed common answer to these questions was related to the existence of some medium called the luminiferous æther (or just the æther, for short[1.1]). This medium was supposed to be composed of some kind of matter, filling in the entire universe and with some pretty bizarre properties, allowing the electromagnetic waves to propagate. In other words, according to this idea, electromagnetic waves should be vibrations of this æther. Since æther was thought of as a material medium, just as one calls "speed of sound" the speed of the wave arising from vibrations of the ambient air as measured in the referential where the air is globally at rest[1.2], the referential in which the speed of light would be c (also then, the referential in which Maxwell's equations are valid) had to be æther's proper frame.

According to the knowledge of that time, this was a bold but highly reasonable hypothesis who then needed to be confronted with experiments. This then famously led to the interferometer experiments conducted by Michelson and Morley. We will spare here the time and energy of an umpteenth discussion of Michelson-Morley's experiments as we should have enough material for the rest of our journey with their conclusions. The experiments were supposed to detect the æther by measuring its speed relatively to earth thanks to the assumed variations of the speed of light between earth's and æther's proper frames. If these were designed to do so, they did not. These were unable to detect any sizable effect revealing this relative speed.

The empirical robustness of the result being confirmed by a huge number of reiterations, physicists then add to understand this new puzzle. In this respect, Hendrick Lorentz and Henri Poincaré did some important work (that one would today consider as preliminary work for the advent of SR). In particular, Lorentz developed transformation laws for space and time that could explain the results

[1.1] Not to be confused with aristotelian æther anyway.

[1.2] One could say that this correspond to a referential where the center of mass of this gas is at rest.

of the experiments. These transformations suggested some weird phenomenons such as length contractions and time dilatations and were mainly viewed as mathematical "tricks". Poincaré further studied the mathematical structure of these transformations. He introduced the terminology "Lorentz transformations" to denote them. He also revealed the group structure of Lorentz transformations and introduced the name "Lorentz group". Finally, he went one step further in the physical interpretation and introduced the "principle of relativity". This principle postulated that all inertial frames should be regarded as physically equivalents, not only regarding the experiments of classical mechanics but also regarding electromagnetism and light[1.3].

These advances brings us very close to the original formulation of special relativity but, as we know, it is Einstein who went the extra conceptual mile and removed æther from the picture. Indeed, to Lorentz, the space and time coordinates involved in Lorentz transformations were the space and time coordinates measuring phenomenons in æther's proper frame on one side and mathematical artefacts with no physical reality on the other side. Poincaré went further by acknowledging that the quantities viewed as purely mathematical by Lorentz should really correspond to the space and time coordinates experienced by an observer but he kept the idea that this was an observer in motion with respect to the æther. We might be twisting a little bit the story by stating this as we are about to but what Einstein realised is that, if one really takes Lorentz transformations seriously while agreeing with the principle of relativity, then there was no means by which one could ever detect the æther whatsoever. The idea of detecting its motion (or the motion of objects relatively to it) became completely hopeless. But in this case, the idea of æther being a material medium in itself became useless. What kind of material object would it be if we cannot even apply the concept of motion to it ?

To summarise this presentation of Einstein's reasoning in a modern and apocryphal way, he threw out the æther and kept Lorentz transformations and the principle of relativity and there it was : special relativity. What this (again, apocryphal) statement hides is how conceptually advanced this move was at that time. It also fails to acknowledge that Einstein did not just decide to mathematically postulate Lorentz transformations and to make use of the principle of relativity on top of that. His brilliant contribution was also to realise that, by applying the principle of relativity with a, say realist, interpretation of Lorentz transformations, one would obtain that the speed of light should have the same value in each and every inertial frame and that this value would be c ... This, and that it was actually possible to work the other way around.

Lorentz transformations from inertial frames

In the first presentation that he gave of what we would today consider as the "birth certificate" of special relativity, Einstein then started from the hypothesis that

[1.3] and potentialy every other physical phenomenon that was left to discover

1. all inertial frames are completely equivalent to describe physical phenomenons (*i.e.* the laws of physics should assume the same form in all of these frames),

2. the magnitude of the speed of light in vacuum is c in every one of these inertial frames.

Imagine that we fix an inertial frame $\mathcal{R}$ *i.e.* that we have a Cartesian grid with associated Cartesian coordinates (x, y, z) to locate objects in space and that we have synchronised clocks that allows measuring the time t in this referential (at any point in space if ever needed). Now, further imagine that we aim to study the trajectory of a light ray in this referential. More precisely, if we try to locate the position of the "most advanced point" of this light ray compared to some point of reference corresponding to the place he was occupying in his trajectory at a given time, according to hypothesis 2, we should find that

$$\left|\left|\frac{\Delta\vec{x}}{\Delta t}\right|\right| = c, \tag{1.1}$$

where $\Delta\vec{x}$ is the position of the most advanced point of our light ray at a given time relatively to the reference point taken on its trajectory, Δt is the time interval between the time at which we want to study the system and the time at which the light ray was at the reference point and $||\cdot||$ is the usual (3D) Euclidian norm. This relation can be rewritten as

$$c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0, \tag{1.2}$$

or simply, if we fix our frame such that the light ray is at $(0, 0, 0)$ at time $t = 0$,

$$c^2t^2 - x^2 - y^2 - z^2 = 0. \tag{1.3}$$

Equation (1.3) then relates the space and time coordinates of the tip of our light ray as measured in our inertial frame.

Of course, according to postulate 1, this procedure should equally hold in any other inertial frame $\mathcal{R}'$ with its Cartesian coordinates (x', y', z') and its measure of time t'. The laws relating the situation in $\mathcal{R}$ and $\mathcal{R}'$ should then be such that

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2, \tag{1.4}$$

where the c on the left-hand side, of course, does not need a $'$ due to our hypothesis. In other words, to express the relation between two inertial frames, one should then look for transformations that preserve the quadratic expression in the left-hand side of (1.3).

To cut a long story short here, these transformations are precisely Lorentz transformations. For example, if we consider two inertial frames $\mathcal{R}$ and $\mathcal{R}'$ such that $\mathcal{R}'$ is moving at constant speed $\vec{V}$ along the x-axis of $\mathcal{R}$ and that, for simplicity, the frames are constructed such that the spatial frames coincide at time $t = 0 = t'$, we should find that the relation between the coordinates

(x, y, z) and t identifying a physical event in $\mathcal{R}$ and the coordinates (x', y', z') and t' identifying the same physical event in $\mathcal{R}'$ is given by

$$\begin{cases} ct' = \gamma\,(ct - \beta x) \\ x' = \gamma\,(x - \beta ct) \\ y' = y \\ z' = z \end{cases}, \tag{1.5}$$

where $\beta := V/c$, with $\vec{V} = V\ \vec{1}_x$ and $\gamma := 1/\sqrt{1-\beta^2}$. These are the Lorentz transformations associated to a setting in motion at a constant speed without rotation of the referential[1.4]. This type of Lorentz transformation is called a boost. This is the type of transformations that Lorentz constructed to explain the result of Michelson-Morley's experiments.

Of course, following the work of Lorentz, due to this structure, we will then recover in special relativity phenomenons such as contraction of length, dilatation of time and the relativity of simultaneity. Also, here, our interpretation forces us to consider these effects as really describing how the points of view of observers placed in different inertial frames would differ from one another. As we know, this leads to a breakdown of our daily life intuition in many situations where the effects of special relativity should be taken into account. We do not aim to enter the discussion of all these details here. Our main interest is in the structure of space and time revealed by special relativity.

1.1.2 Spacetime Interval

In this respect, the above reasoning already illustrates the most important ideas of special relativity.

Spacetime (an informal definition)

First of all, on account of (1.5), we see that, contrary to the situation in Newtonian physics, the border between space and time is not impermeable anymore. It is porous ! That is to say : space and time cannot be thought of as completely separated concepts anymore. This is, with no doubt, the most striking result from special relativity. The notions of space and time can, of course, still make sense to describe physical phenomenons from the point of view of any given observer but, to reconcile their respective points of view, the old conceptions of space and especially of time should melt and merge into an entirely new construct. This construct – whose name also arises from a melting-and-merging process – is called spacetime. This concept (to which we should give an even more formal meaning hereafter and in section 1.3) allows describing physical events in a given inertial frame in terms of their spacetime coordinates, written (ct, x, y, z).[1.5] We already naturally made use of this term but let us stress

[1.4] Here, we have simply expressed it in frames for which the motion happens along the x-axis.

[1.5] Note how time is included in the first coordinate and that this coordinate is written as ct so that it has the same physical units than the spatial coordinates.

that points in spacetime (representing things happening "somewhere in space at some time") are usually called events.

Spacetime interval

The above reasoning also shows that a quantity of fundamental interest to discuss this spacetime structure is the so-called spacetime interval defined as the quantity written (in an inertial frame) as

$$c^2 (\Delta\tau)^2 \coloneqq c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 . \tag{1.6}$$

Expressed in this way, the spacetime interval is written in terms of the finite differences between the space and time coordinates of two events. For practical purposes, it is also very interesting to look at the infinitesimal version of this relation *i.e.* to look at the spacetime interval between an event of spacetime coordinates (ct, x, y, z) and another event infinitesimally close, whose spacetime coordinates are $(c(t + \mathrm{d}t), x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ [1.6]

$$c^2 \mathrm{d}\tau^2 \coloneqq c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{1.7}$$

Just as for the finite version, this infinitesimal spacetime interval is thus the quantity whose form in an inertial frame should be invariant under Lorentz transformations.

Before going further, it is worth noting that writing (1.7) as the square of a quantity makes sense as long as we use it to study physical motions. By this, we mean the following : Another conclusion from special relativity that we haven't gone through is related to the law of transformation of the speed of a given object when looked at in different inertial frames. In a nutshell, the study of this relation shows that, if an object is perceived as moving more slowly than the speed of light in one inertial frame, it will be perceived as moving more slowly than the speed of light in all inertial frames.[1.7] In principle, the same should apply to particles moving faster than the speed of light, but no such particle or physical process has ever been observed in practice. The condition that a particle moves more slowly than the speed of light will correspond to the condition

$$\left\| \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \right\| \leq c, \tag{1.8}$$

which is thus equivalent to

$$c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \geq 0. \tag{1.9}$$

In other words, if (ct, x, y, z) and $(c(t + \mathrm{d}t), x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ are any two infinitesimally close events along the trajectory of an object, the fact that the object propagates more slowly than the speed of light (that is (1.8)) is equivalent

[1.6] We should give a more formal version of these ideas in section 1.3.

[1.7] Unless the speed of propagation of the object equals the speed of light, its speed will be different in different inertial frames; but it will never reach nor exceed c in intensity.

to state that the spacetime interval relating these two events is positive (that is (1.9)). It then makes sense to write the spacetime interval as the square of a quantity for physical purposes. Note that, in the context of special relativity, when describing the motion of an object in spacetime, instead of using the term trajectory (which is more rooted in the idea of splitting space and time in the description), one usually uses the term worldline.

In fact, one can push the interpretation of the spacetime interval even further. Indeed, if one considers a "random" physical observer that has a world line in spacetime which can be arbitrary except that, to be physical, it always satisfies (1.9), the time interval experienced by this observer between two points on its world line is given by

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{1}{c^2}\left|\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t)\right|\right|^2}\,\mathrm{d}t = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau, \tag{1.10}$$

where t_1 and t_2 respectively correspond to the time locating one of the two corresponding points of the observer's world line when observed in a given inertial frame where he is seen as following the curve of equation $\vec{x} = \vec{x}(t)$. The time interval experienced by this "random" observer, *i.e.* the one measured in its proper frame (the frame in which the observer is at rest at the origin), is called the proper time of the observer. From (1.10) we then see that, when referring to the motion of an observer, the quantity $\mathrm{d}\tau$ in (1.7) can be interpreted as the amount of (proper) time that has passed for that observer between two infinitesimally close events of its world line.

Relation (1.10), among other things, imply that the proper time measured by a physical observer between two events of its world line is always lower or equal to the corresponding amount of time in an inertial frame. Indeed, since the integrand in (1.10) is always smaller than 1, we get that

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{1}{c^2}\left|\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t)\right|\right|^2}\,\mathrm{d}t \leq \int_{t_1}^{t_2} \mathrm{d}t = \Delta t.$$

In passing, we can mention that this result allows resolving the twin paradox.

Note also that, to shorten the notation, the spacetime interval (1.7) is also denoted $\mathrm{d}s^2$ instead of $c^2\mathrm{d}\tau^2$. With this notation, one thus have that the increment of proper time for an observer is given by $\mathrm{d}s/c$.

Causality

This idea that the spacetime interval can be interpreted as a physical quantity measured by an observer in its proper frame really reinforces the idea that it corresponds to a physical notion intrinsically associated with the structure of the spacetime described in special relativity.

More than that, the fact that, on a given world line, the study of the sign of this spacetime interval allows us to determine if this world line describes a physical motion – or, more generally, any physical process that would be described by the events of this specific worldline – *i.e.* one that satisfies (1.8), should make us realise that the spacetime interval encodes what we should call the causal structure of spacetime in special relativity.

Indeed, according to special relativity, given two events, if we want to express the physical idea that one of them has "caused" the other (that it has led to it through a physical process), we should encode it in that, on the world line describing this process, the spacetime interval always satisfies (1.9). To state it differently, given two events, if we want to express the idea that one of them can be related to the other by means of a physical process, we should have that the spacetime interval between these two events satisfies (1.9).

1.1.3 Special Relativity and the Minkowski Spacetime

We thus saw that special relativity raises the necessity to formulate our physical theories on a construct that breaks the old distinction between space and time and we also saw that, in order to vividly incorporate the properties of electromagnetism (and more precisely of light) within the physical structure of this construct, we could rely on the properties of the spacetime interval $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$ as defined in (1.7).[1.8]

The recap proposed so far gave a central place to inertial frames in the formulation of the different concepts. It is then important to note that this is the old-fashioned way to present special relativity. We used it to do justice to the historical origins of the concepts but, of course, adopting a modern view of it, one would require to get rid of the central place given to inertial frames (see section 1.2). Hopefully, this – definitely not new – way of introducing the theory will do justice to the tangible content of the more abstract axiomatic formulation of special relativity that we should briefly review below and on which we should rely for the rest of the discussions of this chapter.

Spacetime (a more formal definition)

The first thing to specify in an axiomatic presentation of special relativity is the mathematical structure of spacetime as a set of points (still called events). One can then move to the question of encoding its causal structure.

From the above presentation, we see that, just as in Newtonian mechanics, we would like spacetime to have the mathematical structure of an affine space.[1.9]

Indeed, we would like to be able, given any two events P and Q, to make sense of the vector $\overrightarrow{PQ}$ that represents the unique (spacetime-)displacement that, when applied starting from P, brings you to Q. This also corresponds to the requirement that we would like to be able to construct reference frames by

[1.8] Of course, Maxwell equations of electromagnetism are not buit in the structure of spacetime. It is the invariance under Lorentz transformations, whose necessity was revealed by the properties of electromagnetism and the Michelson-Morley experiment, which is built in spacetime.

[1.9] An affine space is given by a set $\mathcal{P}$ and a vector space $\mathcal{V}$ together with a free and transitive action on $\mathcal{P}$ of the abelian group associated to $\mathcal{V}$. Here, for physical reasons, we should of course consider $\mathcal{V}$ to be a real vector space.

freely choosing an origin O and, once this origin fixed, by locating any event P in terms of a vector $\overrightarrow{OP}$ that would connect it to the origin. The definition of the frame would then be completed by means of a basis $\{\vec{b}_t, \vec{b}_x, \vec{b}_y, \vec{b}_z\}$ of the vector space associated to spacetime by fixing O as the origin in a way that should allow for space and time measurements. This way to construct frames corresponds to a – yet to be completed since we do not have any way to distinguish space and time so far – version of how one expects to construct inertial frames in special relativity.

This affine structure is also the kind of mathematical structure that we need to be able to make sense of the finite spacetime interval (1.6) as the "spacetime length" of the vector obtained by taking the difference between the two events P and Q that defines this spacetime interval. This notion of "spacetime length" is what we should specify hereinafter to encode the causal structure of the spacetime of special relativity. This is the structure that will really make it a spacetime since it will allow us to make the distinction between what a given observer would like to call "space" and "time".

Before doing so, let us just recall that, by construction, points in this spacetime will then be univoquelly associated to vectors once given an origin. Since, as we should expect, this vector space should be of dimension 4 (1 for time + 3 for space), the corresponding vectors are usually called four-vectors (most of the time even written 4-vectors). Due to this one-to-one correspondence between events seen as points and 4-vectors, one might do the slight abuse of terminology of using these terms interchangeably in special relativity. Yet, in the following, when the distinction will be useful, we shall use the notation $\mathbb{M}_4$ to denote the affine space of events and $\mathcal{V}^4$ for the associated vector space of 4-vectors.

Minkowksi metric

We thus saw the mathematical structure that we want for the set of events within special relativity. To really turn this affine space into a spacetime, we still need to encode its causal structure. In other words, we need to be able to associate a (finite) spacetime interval to a 4-vector.

This is done by means of a metric defined on the vector space of 4-vectors. More precisely, we should thus endow $\mathcal{V}^4$ with a map

$$\boldsymbol{\eta} : \begin{cases} \mathcal{V}^4 \times \mathcal{V}^4 \to \mathbb{R} \\ (\vec{u}, \vec{w}) \mapsto \boldsymbol{\eta}(\vec{u}, \vec{w}) \end{cases} \tag{1.11}$$

that is

1. bilinear :
$$\forall \vec{u}_1, \vec{u}_2, \vec{w} \in \mathcal{V}^4, \forall \alpha, \beta \in \mathbb{R},$$
$$\boldsymbol{\eta}(\alpha \vec{u}_1 + \beta \vec{u}_2, \vec{w}) = \alpha\, \boldsymbol{\eta}(\vec{u}_1, \vec{w}) + \beta\, \boldsymbol{\eta}(\vec{u}_2, \vec{w})$$
and the same for the second argument,

2. symmetric :
$$\forall \vec{u}, \vec{w} \in \mathcal{V}^4,\ \boldsymbol{\eta}\left(\vec{u}, \vec{w}\right) = \boldsymbol{\eta}\left(\vec{w}, \vec{u}\right),$$

3. non-degenerate :
$$\left(\exists \vec{u} \in \mathcal{V}^4 : \forall \vec{w} \in \mathcal{V}^4,\ \boldsymbol{\eta}\left(\vec{u}, \vec{w}\right) = 0\right) \Rightarrow \vec{u} = \vec{0}.$$

From the properties that we required on this map $\boldsymbol{\eta}$, linear algebra teaches us two things. First, due to property 1, once given a basis $\{\, \vec{b}_\mu \,\} := \{\, \vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3 \,\}$ of $\mathcal{V}^4$, the action of $\boldsymbol{\eta}$ on any pair of 4-vectors is entirely specified by the coefficients $\eta_{\mu\nu} := \boldsymbol{\eta}\left(\vec{b}_\mu, \vec{b}_\nu\right)$ specifying its action on the basis vectors. Second, there must be basis $\{\, \vec{b}_\mu \,\}$ of $\mathcal{V}^4$ in which the matrix $\eta := (\eta_{\mu\nu})$, characterising the action of $\boldsymbol{\eta}$ in that basis, is diagonal and present only ± 1 factors on its diagonal. The number of -1 and the number of $+1$ appearing in such a decomposition is always the same once given $\boldsymbol{\eta}$; the only thing that can change is at which place they appear on the diagonal. This then defines the signature of such a map $\boldsymbol{\eta}$, noted (r, s), where r denotes the number of -1 and s the number of $+1$ appearing in the decomposition. One can thus also always order the basis vectors so that all the -1 and then all the $+1$ appear on the diagonal of η.[1.10]

For the seek of our physical construction, we will then further demand that $\boldsymbol{\eta}$ is

4. Lorentzian : The signature of $\boldsymbol{\eta}$ is $(1, 3)$.

We will then obtain the existence of basis, that we should write as
$$\{\, \vec{1}_\mu \,\} := \{\, \vec{1}_{ct}, \vec{1}_x, \vec{1}_y, \vec{1}_z \,\}, \tag{1.12}$$
for which
$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.13}$$
The map $\boldsymbol{\eta}$ satisfying these 4 properties is known as the Minkowski metric.

Before going further, we should emphasise that the important is to have a relative sign between one of the numbers in the diagonal of (1.13) and the other three. As long as this property is respected, we will have the exact same structure on our spacetime. The number with the different sign will be associated with the time direction (see below). There is thus a sign convention to take. The sign convention adopted in this thesis is the so-called mostly plus convention since we have one -1 and three $+1$ in (1.13). We could just as well have adopted the mostly minus convention and required the signature of the metric to be $(3, 1)$. In fact, one can always navigate from the mostly plus to the mostly minus convention by applying the rule $\boldsymbol{\eta} \to -\boldsymbol{\eta}$ in every relation involving the

[1.10] Slightly misusing the terminology, these ± 1 factors will be called the eigen values of $\boldsymbol{\eta}$.

metric. That being said, here and in the following, we will stick to the mostly plus convention.

Together with a choice of origin O, a basis (1.12) for which (1.13) holds will define a frame $\left(O;\{\,\vec{1}_\mu\,\}\right)$ on $\mathbb{M}_4$. For an observer who would have this frame has proper frame, $\vec{1}_{ct}$ will be interpreted as pointing in the direction of time while $\vec{1}_x$, $\vec{1}_y$ and $\vec{1}_z$ will be interpreted as providing three linearly independent vectors pointing in the direction of space. In terms of spatial concepts, $\vec{1}_x$, $\vec{1}_y$ and $\vec{1}_z$ will be considered as orthonormal in terms of the Euclidian metric, denoted $\cdot$, induced by $\boldsymbol{\eta}$ (see remark 1.1) . Given an event P, we will be able to univoquely assign it to the 4-vector $\overrightarrow{OP} =: \vec{x} = x^\mu \vec{1}_\mu$ whose components are given by $(x^\mu) =: (ct, x, y, z)$.

These frames will correspond to the inertial frames of special relativity.

Minkowski spacetime

With the mathematical structure of $\mathbb{M}_4$ and $\boldsymbol{\eta}$ and the above interpretations, we will recover the entire structure of special relativity.

Indeed, given a frame $\left(O;\{\,\vec{1}_\mu\,\}\right)$ for which (1.13) holds, we will recover the spacetime interval (1.6) for two events P and Q by considering the 4-vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \Delta x^\mu\,\vec{1}_\mu$, with $(\Delta x^\mu) = (c\Delta t, \Delta x, \Delta y, \Delta z)$, and by computing

$$(\Delta s)^2 \coloneqq -\boldsymbol{\eta}\left(\overrightarrow{PQ}, \overrightarrow{PQ}\right) = c^2\left(\Delta t\right)^2 - \left(\Delta x\right)^2 - \left(\Delta y\right)^2 - \left(\Delta z\right)^2. \tag{1.14}$$

If we apply this to events P and Q which are infinitesimally close *i.e.* such that, in the basis (1.12), the components of the 4-vector $\overrightarrow{OP}$ associated to P are (ct, x, y, z) while the components of the 4-vector $\overrightarrow{OQ}$ associated to Q are given by $(c(t + \mathrm{d}t), x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$, we will recover from the vector $\overrightarrow{PQ} =: \overrightarrow{\mathrm{d}P}$ the infinitesimal spacetime interval

$$\mathrm{d}s^2 \coloneqq -\boldsymbol{\eta}\left(\overrightarrow{\mathrm{d}P}, \overrightarrow{\mathrm{d}P}\right) = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{1.15}$$

We thus have that the Minkowski metric $\boldsymbol{\eta}$ perfectly encodes the causal structure of special relativity.

We will also be able to express the idea that a world line (a curve on $\mathbb{M}_4$) describes the propagation of a light ray. Choosing an inertial frame whose origin is such that the light ray passes through $(0, 0, 0, 0)$, any event P of the world line will be associated to a 4-vector $\overrightarrow{OP} = \vec{x}$ whose components satisfy

$$\boldsymbol{\eta}\left(\vec{x}, \vec{x}\right) = x^\mu \eta_{\mu\nu} x^\nu = 0. \tag{1.16}$$

Using $(x^\mu) =: (ct, x, y, z)$, (1.16) is just a rewriting of (1.3).

Finally, according to our definition of inertial frames as frames constructed from an event O and a basis $\{\,\vec{1}_\mu\,\}$ of $\mathcal{V}^4$ such that (1.13) holds, we see that the transformations relating inertial frames with one another corresponds to a translation of the origin (which will be encoded in the 4-vector $\overrightarrow{OO'}$ joining the "old" and "new" origin) and a change of basis from $\{\vec{1}_\mu\}$ to $\{\vec{1}'_\mu\}$, two bases such that (1.13) holds. From this last point, we thus see that the matrix $\Lambda = \left(\Lambda_\mu{}^\nu\right)$ characterising that change of basis $\vec{1}'_\mu = \Lambda_\mu{}^\nu \vec{1}_\nu$ should satisfy

$$\Lambda_\mu{}^\alpha \; \eta_{\alpha\beta} \; \Lambda_\nu{}^\beta = \eta_{\mu\nu}, \tag{1.17}$$

where the coefficients $\eta_{\alpha\beta}$ and $\eta_{\mu\nu}$ are given by (1.13). Writing the inverse of this matrix as $\Lambda^{-1} = \left(\left(\Lambda^{-1}\right)_\mu{}^\nu\right) =: \left(\Lambda^\nu{}_\mu\right)$, we get that (1.17) can be rewritten as

$$\Lambda^\alpha{}_\mu \; \eta_{\alpha\beta} \; \Lambda^\beta{}_\nu = \eta_{\mu\nu}. \tag{1.18}$$

The relations (1.17) and (1.18) thus defines Lorentz transformations from the point of view of the mathematical structure given by $\mathbb{M}_4$ and $\boldsymbol{\eta}$. Their group structure then becomes obvious. One can easily verify that the transformation given in (1.5) is an example of Lorentz transformation.

We then see that this construct allows recovering all the important content of special relativity with firm mathematical roots. This mathematical structure is called the Minkowski spacetime and is usually denoted $(\mathbb{M}_4, \boldsymbol{\eta})$.

Remark 1.1. *Given an inertial frame* $\left(O; \{\vec{1}_\mu\}\right)$ *(a frame for which* (1.13) *holds),* $\boldsymbol{\eta}$ *will induce a Euclidian metric, denoted* $\cdot$*, on the set of* 4*-vectors* $\vec{w} \in \mathcal{V}^4$ *for which*

$$\boldsymbol{\eta}\left(\vec{w}, \vec{1}_{ct}\right) = 0.$$

According to the definition of the frame, the set of 4*-vectors satisfying this condition is actually* $\operatorname{Span}\{\vec{1}_x, \vec{1}_y, \vec{1}_z\} \subset \mathcal{V}^4$.

Given two 4*-vectors* $\vec{w}_1, \vec{w}_2 \in \mathcal{V}^4$ *satisfying the condition, the Euclidian metric will be defined as*

$$\vec{w}_1 \cdot \vec{w}_2 := \boldsymbol{\eta}\left(\vec{w}_1, \vec{w}_2\right).$$

From this definition, if we write $\{\vec{1}_i\} := \{\vec{1}_x, \vec{1}_y, \vec{1}_z\}$*, we will have that,*

$$\vec{1}_i \cdot \vec{1}_j = \delta_{ij}.$$

According to our interpretation of the frame $\left(O; \{\vec{1}_\mu\}\right)$ *as encoding an inertial observer,* 4*-vectors on* $\operatorname{Span}\{\vec{1}_x, \vec{1}_y, \vec{1}_z\}$ *can be interpreted as vectors in space. Indeed, if we fix a* $t_0 \in \mathbb{R}$*, the set of events* $P \in \mathbb{M}_4$ *defined by the property that their time coordinate relatively to* $\left(O; \{\vec{1}_\mu\}\right)$ *is* $t = t_0$ *should define "space at time* t_0*" for the observer. Vectors in space (at time* t_0*) should then be vectors joining two such points. One will then precisely obtain that this vector space always corresponds to* $\operatorname{Span}\{\vec{1}_x, \vec{1}_y, \vec{1}_z\}$.

It is in this sense that $\vec{1}_x$*,* $\vec{1}_y$ *and* $\vec{1}_z$ *are "orthonormal vectors in space".*

Inertial frames from the Minkowski metric

It is interesting to note the place of inertial frames within the formulation of special relativity on the Minkowski spacetime. The axiomatic given by $(\mathbb{M}_4, \boldsymbol{\eta})$ allows us to deduce the existence of a class of frames for which all observers will see light as propagating in a straight line at speed c as we see from (1.16) and (1.18). For practical purposes, they will play the exact same role as the inertial frames used to define the postulates of SR in the old-fashioned way. The crucial difference is that, now, these frames are not occupying the central place in the formulation of these postulates anymore.

It is in this sense that the axiomatic formulation of special relativity obtained by endowing the mathematical structure of $(\mathbb{M}_4, \boldsymbol{\eta})$ with the right physical interpretation removes the privileged role of inertial frames. These frames will still be useful and "special" in the sense that they allow to work with the simplest form for the coefficients describing $\boldsymbol{\eta}$ but they are not used in the definition of spacetime anymore. In this new picture, inertial frames are "just" those useful frames in which (1.13) happens to holds.

Tensors on Minkowski spacetime

The structure of $(\mathbb{M}_4, \boldsymbol{\eta})$ also naturally allows defining tensors and tensor fields on Minkowski spacetime.

Since $\mathcal{V}^4$ is a (finite dimensional) real vector space, we can define its dual space $\mathcal{V}^{4*}$. From these two vector spaces, as usual, one can construct tensors of a given rank (m, r) as multilinear maps from $\mathcal{V}^{4*} \times \cdots \times \mathcal{V}^{4*} \times \mathcal{V}^4 \times \cdots \times \mathcal{V}^4$ to $\mathbb{R}$, where $\mathcal{V}^{4*}$ appears m times and $\mathcal{V}^4$ appears r times in the Cartesian product.[1.11]

Even more interestingly, we can also define tensor fields. A tensor field of a given rank will be constructed by assignment of a tensor of this given rank for any event $P \in \mathbb{M}_4$.

What is remarkable then is that, for any $P \in \mathbb{M}_4$, the tensor assigned to P can be seen as living in the same vector space as the tensors assigned to other events. This property then allows making immediately sense of the variation of a tensor field between distinct points. This then also allows defining notions of derivation of a tensor field defined on $\mathbb{M}_4$. As we should see in section 1.3, this is a property much less obvious than it may sound that is, here, "trivial" to obtain due to the affine structure of $\mathbb{M}_4$.

Physical processes and timelike vectors

Now that the foundations of special relativity have been settled mathematically, again, it is clear that Minkowski spacetime will allow us to derive all the consequences outlined by the old-school formulation. In this respect, it would be highly unnecessary to repeat the discussion of the causal structure, has encoded in (1.15), or how one can construct the proper time of an observer. Yet, it might be useful to recall some vocabulary that is strongly related to it in the context of Minkowski spacetime.

[1.11] Remember that, in this picture, a vector $\vec{v} \in \mathcal{V}^4$ can be seen as a $(1,0)$ tensor. It is so because $\mathcal{V}^4$ will be canonically isomorphic to the dual space $\mathcal{V}^{4**}$ of its dual space $\mathcal{V}^{4*}$. Indeed, a $\vec{v}^* \in \mathcal{V}^{4**}$ consist in a linear map $\vec{v}^* : \mathcal{V}^{4*} \to \mathbb{R}$ *i.e.* a $(1,0)$ tensor. Since $\mathcal{V}^{4*}$ itself consist in the vector space of linear maps $\underline{\theta} : \mathcal{V}^4 \to \mathbb{R}$ ($(0,1)$ tensors), we can define a one to one correspondance between $\mathcal{V}^4$ and $\mathcal{V}^{4**}$ by associating any $\vec{v} \in \mathcal{V}^4$ to the unique $\vec{v}^* \in \mathcal{V}^{4**}$ such that $\forall \underline{\theta} \in \mathcal{V}^{4*},\ \vec{v}^*(\underline{\theta}) := \underline{\theta}(\vec{v}) \in \mathbb{R}$.

Since the Minkowski metric is not a scalar product, in the sense that it is not positive definite, it does not allow us to define a norm on $\mathcal{V}^4$. That being said, we have already seen that, to construct spacetime intervals, it was useful to evaluate $\boldsymbol{\eta}$ with the same 4-vector as first and second argument. In this respect, we can define the pseudo-norm of a 4-vector $\vec{x} \in \mathcal{V}^4$ as the quantity $\boldsymbol{\eta}(\vec{x}, \vec{x})$. According to the properties of the Minkowski metric, the pseudo-norm can be negative, null or positive. This then allows us to split 4-vectors into three categories. A 4-vector $\vec{x} \in \mathcal{V}^4$ will be said

1. Timelike if $\boldsymbol{\eta}(\vec{x}, \vec{x}) < 0$,
2. null (or lightlike) if $\boldsymbol{\eta}(\vec{x}, \vec{x}) = 0$,
3. spacelike if $\boldsymbol{\eta}(\vec{x}, \vec{x}) > 0$.

As one can expect, this vocabulary allows reformulating the conditions expressed by the causal structure. To be physical, a given process cannot propagate faster than the speed of light. This is related to the idea that, all along the world line describing this process, $\mathrm{d}s^2 \geq 0$. We can also formulate this idea using the above vocabulary.

A world line is a curve on $\mathbb{M}_4$. If we choose a parametrisation (not necessarily the proper time), we can describe it by means of a function

$$\mathscr{C} : \mathbb{R} \to \mathbb{M}_4 : \lambda \mapsto P(\lambda).$$

If we choose an origin O, we can equivalently describe the curve via the 4-vectors $\vec{x}(\lambda) \coloneqq \overrightarrow{OP}(\lambda)$. We can thus define the "4-velocity" along the curve as[1.12]

$$\vec{u}(\lambda) \coloneqq \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}(\lambda) \in \mathcal{V}^4.$$

This allows defining the notions of being time-, light- or space-like for curves. A curve $\mathscr{C} : \mathbb{R} \to \mathbb{M}_4$ will be called

1. Timelike if $\forall\lambda,\ \boldsymbol{\eta}(\vec{u}(\lambda), \vec{u}(\lambda)) < 0$,
2. null (or lightlike) if $\forall\lambda,\ \boldsymbol{\eta}(\vec{u}(\lambda), \vec{u}(\lambda)) = 0$,
3. spacelike if $\forall\lambda,\ \boldsymbol{\eta}(\vec{u}(\lambda), \vec{u}(\lambda)) > 0$.

One could also check that the above condition does not depend on the choice of the parametrization of the world line. We can thus say that a world line defines a physical process if and only if it is either a timelike or a null curve. The world line of a light ray should be a null curve. Also, if it is not a light ray, the world line of an object should correspond to a timelike curve.

[1.12] Note that it is the affine structure of $\mathbb{M}_4$ that allows to make sense of the limit defining the derivative with respect to λ and thus to define $\vec{u}(\lambda)$.

Note also that the definition of $\vec{x} \coloneqq \overrightarrow{OP}$ requires to fix a point $O \in \mathbb{M}_4$ as origin but that the definition of $\vec{u} \in \mathcal{V}^4$ is independent of this choice as it actually relies on a limit taken on a 4-vector computed from variations of the "position" (in spacetime) on the worldline when λ varies.

4-velocity

For that matter, usually, the name 4-velocity is saved for situations where a timelike curve is parametrised by its proper time τ. This proper time can always be defined as we previously did in the case of a timelike curve. The term 4-velocity is thus saved for

$$\vec{u}(\tau) \coloneqq \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}(\tau) \in \mathcal{V}^4, \tag{1.19}$$

where $\vec{x}(\tau) = \overrightarrow{OP}(\tau)$, represents the 4-vector associated with the event $P(\tau)$ on the particle worldline once fixed an origin O. This is why we used quotation marks above.

It is not necessary to make sense of the notion of 4-velocity from the point of view of spacetime but, for the sake of interpretation, it is interesting to note that, when expressed using an inertial frame $\left(O; \{\, \vec{1}_\mu \,\}\right)$, the timelike curve can be described using $\vec{x}(\tau) = x^\mu(\tau)\vec{1}_\mu$, with $(x^\mu(\tau)) = (ct(\tau), x(\tau), y(\tau), z(\tau))$. In this case, the components of the 4-velocity can be written as

$$(u^\mu) = \left(\gamma(v)c, \gamma(v)\frac{\mathrm{d}x}{\mathrm{d}t}, \gamma(v)\frac{\mathrm{d}y}{\mathrm{d}t}, \gamma(v)\frac{\mathrm{d}z}{\mathrm{d}t}\right), \tag{1.20}$$

where

$$\gamma(v) \coloneqq \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{||\vec{v}||^2}{c^2}}} > 0. \tag{1.21}$$

In this definition

$$\vec{v} \coloneqq v^x\vec{1}_x + v^y\vec{1}_y + v^z\vec{1}_z \coloneqq \frac{\mathrm{d}x}{\mathrm{d}t}\vec{1}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\vec{1}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\vec{1}_z \tag{1.22}$$

is a (spacelike) vector and $||\vec{v}||^2 \coloneqq \vec{v}\cdot\vec{v} = (v^x)^2 + (v^y)^2 + (v^z)^2$ is the Euclidian square norm induced by $\boldsymbol{\eta}$ on $\mathrm{Span}\left\{\, \vec{1}_x, \vec{1}_y, \vec{1}_z \,\right\}$ (see remark 1.1). Naturally, if the timelike curve is interpreted as the world line of a particle, $\vec{v}$ should be interpreted as the (spatial) velocity of the particle as measured in the inertial frame. It is interesting to note that here, formally, according to our definitions, $\vec{v} \in \mathrm{Span}\left\{\, \vec{1}_x, \vec{1}_y, \vec{1}_z \,\right\} \subset \mathcal{V}^4$ so it is, strictly speaking, a 4-vector. Nevertheless, on account for our interpretation, one could sensibly introduce the slight abuse of notation that

$$(u^\mu) = \left(\gamma(v)c, \gamma(v)\frac{\mathrm{d}x}{\mathrm{d}t}, \gamma(v)\frac{\mathrm{d}y}{\mathrm{d}t}, \gamma(v)\frac{\mathrm{d}z}{\mathrm{d}t}\right) \eqqcolon (\gamma(v)c, \gamma(v)\vec{v})\,. \tag{1.23}$$

According to (1.19), for a timelike curve, we will always have that

$$\boldsymbol{\eta}\left(\vec{u}, \vec{u}\right) = -c^2. \tag{1.24}$$

This relation can be proved true at any given τ by expressing this invariant quantity in an inertial frame in which the spatial velocity (1.22) instantly vanishes.

One should then recognise that the above definitions of time-, light- and space-like curves provide equivalent (but formally cleaner) versions of the conditions describing the causal structure of spacetime stated in terms of the infinitesimal spacetime interval $\mathrm{d}s^2$.

1.1.4 Particle Dynamics in Special Relativity

Now that we have presented Minkowski spacetime, it is interesting to note one thing : we have spent a lot of energy to emphasise that Minkowski spacetime allowed to recover all the predictions from the old formulation of special relativity but, considering the above discussion, we are still missing the "superstar" from special relativity. Indeed, no serious presentation of special relativity can escape the discussion of its most famous prediction : $E = mc^2$. So, where is it ? The answer is simple : below.

Announcement effects aside, this sarcastic remark gives us the opportunity to emphasise one important thing : So far, in this discussion, we have been interested in the structure of spacetime. In special relativity, similarly to Newtonian mechanics, spacetime is the (fixed) theatre in which physical processes take place. The main difference between the notion of spacetime in special relativity and in Newtonian mechanics lies in the structure of this spacetime (see remark 1.2). What we aim to emphasise here is then that Minkowski spacetime (just as time and space in Newtonian mechanics) provides "the language" in which we should describe physical processes according to special relativity, the *kinematics* of these physical processes. What it obviously does not encode in its definition are the laws ruling these processes, the *dynamics* describing physical processes in the language of special relativity.

Saying a word or two about how one conceives dynamical processes in the context of special relativity is the object of the next two paragraphs. Contrary to the previous part of the discussion, our aim is not to be as systematic here but to highlight some of the main conceptual steps as it should be useful later in this chapter (typically, in section 1.3 and to distinguish the physical content of sections 1.4 and 1.6).

For this presentation, we will discuss most of the conditions intrinsically on Minkowski spacetime *i.e.* without introducing a frame. That being said, when needed, we will assume to have at our disposal a given inertial frame $\big(O; \{\, \vec{1}_\mu \,\}\big)$. When expressing our relations in terms of the components of 4-vectors, we will then make use of this inertial frame, without always referring to it.

Motion of pointwise particles

The first thing we should specify is how special relativity deals with the idealised situation of a free pointwise particle. By "free" we, of course, mean "that is subject to no external physical influence". For simplicity, let us further assume that this particle as a mass $m > 0$. Such a particle will then, in all generality have to move along a timelike curve. This then always allows us to make sense of its proper time τ. We can then make use of this proper time to define the

4-velocity of the particle via (1.19). The relations (1.20–1.24) will thus also hold for any massive particle described in special relativity. They are still kinematic. To fix a dynamics, one should then impose a constraint on the form of the world line of the particle.

In special relativity, just as in Newtonian mechanics, one will assume that the motion of a free particle is a straight line. More precisely, one will impose the dynamical condition that

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau}(\tau) = 0, \ \forall\tau. \tag{1.25}$$

On account of (1.19), this relation then imposes that the world line should be found by solving

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}\tau^2}(\tau) = 0, \ \forall\tau. \tag{1.26}$$

Once studied in an inertial frame, (1.26) becomes

$$\ddot{x}^\mu(\tau) = 0, \ \forall\tau, \tag{1.27}$$

where, to lighten the notation, we introduced $\dot{x}^\mu(\tau) \coloneqq \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau)$. This indeed shows that, in the absence of an external influence, the particle will move along a straight line since we will have $x^\mu(\tau) = x^\mu(0) + \dot{x}^\mu(0)\tau$.

Note that the resolution of (1.27) really predicts that the world line will be a straight line *in spacetime*. But since, on account of (1.24), we should have $\dot{x}^0(0) \neq 0$, the parameter τ can always be eliminated in terms of the time coordinate $t = x^0(\tau)/c$ in $x(\tau)$, $y(\tau)$ and $z(\tau)$. This then allows interpreting the solution of (1.27) as describing a particle whose motion, as perceived by an inertial observer, will draw a straight line *in space* as time goes by.

4-momentum and interaction with an external force

One should then wonder how the situation would change in the presence of an external influence. To specify this point, we should introduce a new 4-vector associated to a pointwise particle, its 4-momentum

$$\vec{P} \coloneqq m\vec{u} \in \mathcal{V}^4. \tag{1.28}$$

This 4-vector can always be defined once we have determined that a given time-like curve should correspond to the world line of a given particle of mass m. In this sense, it can also be formulated at the level of the particle's kinematics.[1.13] Due to (1.24), the 4-momentum always satisfies

$$\boldsymbol{\eta}\left(\vec{P},\vec{P}\right) = -m^2c^2. \tag{1.29}$$

[1.13] But not "kinematics" in the sense of "included in the structure of Minkowski spacetime", of course, since the mass m is not part of this structure.

This 4-vector encodes useful informations on the particle. Once given an inertial frame, we can introduce the following notation for the components of the 4-momentum

$$(P^\mu) = (E/c, p^x, p^y, p^z) =: (E/c, \vec{p}) \,, \tag{1.30}$$

With the same abuse of notation as in (1.23) in the last equality. The interest of this notation is that E will be interpreted as the total energy of the particle while $\vec{p}$ will be interpreted as its (spatial) momentum as measured in the inertial frame. From (1.20) we see that

$$E = m\gamma(v)c^2, \ \vec{p} = m\gamma(v)\vec{v}. \tag{1.31}$$

As we know, the interpretation of these relations is then motivated by comparison with the Newtonian case since we get that

$$E \approx mc^2 + \frac{m}{2} \left|\left|\vec{v}\right|\right|^2, \ \vec{p} \approx m\vec{v}, \quad \text{when} \ \left|\left|\vec{v}\right|\right| \ll c.$$

Note – as this is too famous to be ignored – that it is here, in (1.31), that the relation $E_0 = mc^2$ appears as expressing the value of the energy of a body of mass m as measured in its proper frame.

Using these notations, we will get that

$$\boldsymbol{\eta}\left(\vec{P}, \vec{P}\right) = -\frac{E^2}{c^2} + \vec{p} \cdot \vec{p}, \tag{1.32}$$

so that (1.29) can be rewritten as

$$E^2 = m^2c^4 + \left|\left|\vec{p}\right|\right|^2 c^2, \tag{1.33}$$

where, again, $\left|\left|\vec{p}\right|\right|^2 := \vec{p} \cdot \vec{p}$. This is the famous Einstein's relation from special relativity. It is also interesting to quote that, just as time and space are "mixed" in Minkowski spacetime, so are the physical quantities (energy and spatial momentum) related to them. Saying this, we refer to the results from classical mechanics relating the time translational invariance of a system to the conservation of energy and, similarly for the invariance under translation in a given space direction and the conservation of the corresponding component of the momentum.

The mass m being constant, in terms of the 4-momentum, the dynamics of a free particle is thus described by the condition

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}\tau}(\tau) = 0, \ \forall \tau. \tag{1.34}$$

We thus see how such a relation encodes simultaneously conservation of energy and (spatial) momentum of the particle.

To introduce an external physical influence in this picture, one should then find a way to encode the associated physical interaction in a modification of

(1.34). This will be done by introducing for any value of τ a 4-vector force $\vec{K}\left(\vec{x}(\tau), \vec{u}(\tau), \tau\right) \in \mathcal{V}^4$ and by postulating that the equation of motion for the particle is

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}\tau}(\tau) = \vec{K}\left(\vec{x}(\tau), \vec{u}(\tau), \tau\right), \ \forall \tau. \tag{1.35}$$

Again, this might be given, or motivated, by the study of the Newtonian limit of this equation which is supposed to reduces to equations from Newtonian mechanics.

This thus leads us to an important conclusion : in special relativity, as in Newtonian mechanics, the motion of a free particle is a straight line (in spacetime). In this context, and according to (1.35), the presence of an external interaction should be detected by means of the observation of deviations from this linear motion. This is the procedure that allows saying that a given physical interaction will be detectable by means of a force.

A famous and important example is given by the case of a particle of mass m and electric charge q interacting with an external electromagnetic field. In this case, in an inertial frame, one has that

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}(\tau) = q\ F^\mu{}_\nu\left(x^\alpha(\tau)\right) u^\nu(\tau), \tag{1.36}$$

where $F^\mu{}_\nu$ is the Faraday tensor (see below). The right-hand side of this equality encodes the well-known Lorentz force from electromagnetism. From this relation, one can thus (at least in principle) detect and measure the electromagnetic field by studying the motion of – a sufficiently huge number of – particles with different charge-to-mass ratios. This is, in a sense, the motivating example that led to the relation (1.35) and to the associated procedure of encoding an interaction in a force.

1.1.5 Field Dynamics in Special Relativity

The notion of dynamics in special relativity does not reduce to the question of the motion of pointwise particles. Indeed, to be complete, one should also expect to be able to describe the laws ruling the behaviour of the physical interactions in the language of special relativity. [1.14]

Electromagnetism

Again, the motivational and fundamental example in this respect is electromagnetism. Maxwell's equations for electromagnetism

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}, \\ \vec{\nabla}\cdot\vec{B} = \vec{0}, \qquad \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}, \end{cases} \tag{1.37}$$

[1.14] The fact that this thesis exists utlimately rely on the fact that the situation is much more complicated when it comes to the description of the gravitational interaction but let us keep this question aside for a tiny more pages.

where ρ represents the charge density of the sources producing the electromagnetic field, $\vec{J}$ the density of current of these sources, ε_0 the permittivity of vacuum, μ_0 the permeability of vacuum (with $\mu_0\varepsilon_0 = 1/c^2$), $\vec{E}$ the electric field and $\vec{B}$ the magnetic field, can be encoded on Minkowski spacetime as

$$\begin{cases} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu & (1.38a) \\ \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 & (1.38b) \end{cases},$$

where

$$(J^\mu) := (\rho c, J^x, J^y, J^z) \tag{1.39}$$

are the components of the current density 4-vector,

$$(F_{\mu\nu}) := \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & B^z & -B^y \\ E^y/c & -B^z & 0 & B^x \\ E^z/c & B^y & -B^x & 0 \end{pmatrix} \tag{1.40}$$

are the components of the Faraday tensor and where indices in (1.38) were lowered and raised, when necessary, using the coefficients $\eta_{\mu\nu}$ from (1.13).[1.15] As we know, the second the relation in (1.38) can also be used to ensure the existence of a 4-vector of components A^μ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.41}$$

Before going further, let us remark that, as we can see from (1.40), the Faraday tensor is antisymmetric. This antisymmetry encodes a crucial physical property of electromagnetism. Indeed, because of this antisymmetry, if one contracts (1.38a) with a partial derivative ∂_ν, one automatically gets (since partial derivatives commute with each other) that

$$\partial_\nu J^\nu = 0. \tag{1.42}$$

If this equation is crucial, it is because it encodes the conservation of the electric charge in electromagnetism. This can be seen by using (1.39) in (1.42). This will indeed give

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{1.43}$$

This automatic conservation of the electric charge on account of (1.38) (or (1.37)) is the main consequence of Maxwell's work. It is indeed its famous addition of the term $\mu_0\varepsilon_0\partial\vec{E}/\partial t$ to (1.37) that allowed the laws of electromagnetism to automatically lead to this conclusion. Putting the historical line upside down, it is also this addition that allows rewriting (1.37) as (1.38) in Minkowski spacetime.

[1.15] Once again, this well known procedure of raising and lowering indices with the metric should be restated more formally in section 1.3.

The aim of this paragraph is obviously not to give a review on electromagnetism. We should thus stop here our description of the properties of Maxwell's equations but, for the sake of later discussions, let us simply further comment quickly on the above relations.

First of all, it is no surprise that (1.37) admit a nice reformulation when expressed in terms of quantities formulated on Minkowski spacetime since the properties of Maxwell's formulation of electromagnetism are precisely at the origin of the discovery of special relativity as we briefly recalled at the beginning of this section.

Next, it is interesting to see that, just as we did for Minkowski spacetime, instead of starting from (1.37) and using (1.39) and (1.40) to define a vector field and an antisymmetric tensor field on Minkowski spacetime so that (1.37) can be rewritten as (1.38), we could give a formulation of electromagnetism by fixing an inertial frame $\big(O; \{\, \vec{1}_\mu \,\}\big)$, starting from an antisymmetric tensor field of components $F_{\mu\nu}$ and a 4-vector field of components J^μ and fixing the dynamics of the system by means of (1.38). Equivalently, we could also have started with a 4-vector field of components A^μ and another 4-vector field of components J^μ and defined the dynamics of the system by means of (1.38a) with $F_{\mu\nu}$ defined by (1.41); in which case (1.38b) is an identity coming from (1.41). In either case, with this more axiomatic view, we change the objects appearing as primary in the construction so that the ":" in (1.39) and (1.40) should go on the other side of the equation. In this case, these relations thus define the charge density, current density, electric and magnetic fields as experienced by an observer at rest in the inertial frame.

Finally, we should note that, in the above formulation, we have introduced an inertial frame who plays a role in the writing of the field equations. We did this for convenience and to be able to connect to (1.37) but we should emphasise that, to really be able to see them as relations on Minkowski spacetime, we should be able to rewrite (1.38) as equations between tensor fields (here 4-vector fields) *i.e.* we should be able to remove the use of the inertial frame in the formulation of the law. To do this, we should be able to make sense of the derivatives appearing in (1.38) and (1.41) so that they define the components of a tensor field on Minkowski spacetime. We will not do it explicitly here since it would lead to mainly duplicate constructions that we should present in section 1.3 in more general situations but we should stress that it is indeed completely possible. We would then obtain a version of Maxwell's electrodynamics intrinsically formulated on Minkowski spacetime.

Energy-momentum tensor

There is one last quantity whose importance was first revealed in special relativity that we should recall here : the energy-momentum tensor. This is a tensor that can be attributed to a physical system to which we can associate notions of energy and momentum. We will here review two fundamental examples.

The first one consists in a massive free particle of mass m. This is the system that we have presented in section 1.1.4. In this section, we saw that the particle could be attributed a 4-momentum that basically encodes (or generalises) the usual notions of energy and momentum. This can also be done by means of a tensor. If we write the world line of the particle as $\mathscr{C} : \mathbb{R} \to \mathbb{M}_4 : \tau \mapsto P(\tau)$ with, in a given inertial frame, $\overrightarrow{OP}(\tau) = \vec{x}(\tau) = x^\mu(\tau)\vec{1}_\mu$ and define the 4-velocity $\vec{u}(\tau) = u^\mu(\tau)\vec{1}_\mu$ by (1.19), the components of the energy-momentum tensor of the particle are given by

$$T^{\mu\nu}_{\mathrm{p}}\left(x^\alpha\right) := m \int_{\mathscr{C}} u^\mu(\tau) u^\nu(\tau)\ \delta^4\left(x^\alpha - x^\alpha(\tau)\right) \mathrm{d}\tau, \tag{1.44}$$

where the lower script in the left-hand side stands for "particle". What this relation encodes is the fact that the particle transports in spacetime some energy and some momentum that are entirely located on its world line.

The physical interest of the energy-momentum tensor in this case is that a little bit of calculation shows that the equation of motion for the particle (1.25) is equivalent to the condition

$$\partial_\mu T^{\mu\nu}_{\mathrm{p}}\left(x^\alpha\right) \equiv 0,\ \forall x^\alpha, \tag{1.45}$$

With $\partial_\mu = \partial/\partial x^\mu$. Consequently, just as (1.25) encoded the conservation of the energy and momentum of the particle (1.34), so does (1.45).

With only this example, one might have the erroneous impression that (1.45) is just a more convoluted way of writing (1.25). It is, of course, more subtle than that.

As we said, we can ascribe an energy-momentum tensor to any system with energy and momentum. Following what we did earlier in this section, we should illustrate this in the case of electromagnetism. The study of Maxwell's equations (with or without use of Minkowski spacetime) reveals that the electric and magnetic field transport some energy and some momentum. This is beautifully illustrated by the study of electromagnetic waves. To encode the energy and momentum carried out by the electromagnetic fields in terms of Minkowski spacetime concepts, one uses the energy-momentum tensor defined from the Faraday tensor and Minkowski metric as

$$T^{\mu\nu}_{\mathrm{EM}} := \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu}\right), \tag{1.46}$$

where $\eta^{\mu\nu}$ are the components of the inverse of (1.13) and the lower script "EM" stands for "electromagnetic". According to Maxwell's equations (1.38), the energy-momentum tensor satisfies

$$\partial_\mu T^{\mu\nu}_{\mathrm{EM}} = J^\alpha\ F_\alpha{}^\nu. \tag{1.47}$$

In particular, far away from the sources ($J^\alpha = 0$), this relation reduces to $\partial_\mu T^{\mu\nu}_{\mathrm{EM}} = 0$ which encodes the conservation of the energy and momentum carried by electromagnetic waves. We thus see how, similarly to (1.45), (1.47) encodes conservation of energy and momentum in the case of the electromagnetic field.

In fact, more than that, for a pointwise particle of mass m and charge q (*i.e.* a particle that satisfies (1.36)) one can define a 4-vector field current density given by

$$J^{\mu}_{\mathrm{p}}\left(x^{\alpha}\right) \coloneqq q \int_{\mathscr{C}} u^{\mu}(\tau)\delta^{4}\left(x^{\alpha} - x^{\alpha}(\tau)\right) \mathrm{d}\tau. \tag{1.48}$$

In this case, a straightforward calculation will show that, if (1.36) and (1.38) are satisfied, one will have that

$$\partial_{\mu}\left(T^{\mu\nu}_{\mathrm{p}} + T^{\mu\nu}_{\mathrm{EM}}\right) \equiv 0, \tag{1.49}$$

which corresponds to the conservation of the energy and momentum of the system in interaction composed of the charged particle and the electromagnetic field.

This thus further reinforces the intimacy between the energy-momentum tensor and the processes of exchange of energy and momentum.

As a final example, we should also mention that, if we can associate energy and momentum to a single pointwise particle, so can we for a system composed of several particles. One can thus associate to the system a total 4-momentum given by the sum of the individual momenta of each particle. In this case, in the absence of any exterior influences, special relativity postulates that the total 4-momentum is conserved (that is (1.34) where $\vec{P}$ is the total 4-momentum). This situation can thus also be described in terms of the total energy-momentum tensor of the system, given by the sum of the energy-momentum tensors of each particle via (1.45) if $T^{\mu\nu}_{\mathrm{p}}$ is understood as the total energy-momentum tensor of the system.

In the limit of a continuous distribution of matter in spacetime, one will obtain the description of a relativistic fluid. The total energy-momentum tensor can thus be obtained by "integration of (1.44) over the world lines of each particle composing the fluid". In the case of a perfect fluid, this total energy-momentum tensor will assume a particularly simple form

$$T^{\mu\nu}_{\mathrm{fl}} \coloneqq P\ \eta^{\mu\nu} + \left(\rho + \frac{P}{c^{2}}\right) u^{\mu}u^{\nu}, \tag{1.50}$$

where u^{μ} are the components of the fluid's 4-velocity, ρ is the mass-energy density, P the pressure of the fluid and where the lower script "fl" stands for "fluid". To describe the dynamics of a system defined by such an energy-momentum tensor, similarly to what is done for the system of particles, one will impose that

$$\partial_{\mu}T^{\mu\nu}_{\mathrm{fl}}\left(x^{\alpha}\right) \equiv 0,\ \forall x^{\alpha}, \tag{1.51}$$

and close the system by imposing a relation between ρ and P of the form

$$\rho = \rho\left(P\right). \tag{1.52}$$

Such a relation is called an equation of state and aims to take effective count of all the internal interactions in the fluid. One thus sees that, in this context, the conservation of the energy-momentum tensor is even thought of as the field equation encoding the dynamics of the system.

Note, again, that all the relations in this paragraph have been written, for simplicity, by means of an inertial frame but that it is possible to reformulate these relations as tensor equations on Minkowski spacetime.

1.2 Gravity and Inertia : a Question of Principle

When it comes to the discussion of very generic physical principles, one can quickly start to feel disarmed. As we already sketched in the introduction chapter, the formulation of precise physical ideas usually requires a well-established mathematical formalism to make complete sense. In this context, the "primary" physical principles guiding the choice of the formalism must then be presented (at least in part) prior to adopt a precise formalism and one is then much more exposed to the risk of discussing "wavy hand" arguments. An extra side effect of this is that different authors may then use similar terminologies to refer to different concepts or, the other way around, use different terminologies to refer to similar concepts.

That being said, these principles will remain crucial to root physical arguments as long as their formulation is strong enough to overcome this lack of, say, mathematical preciseness.

As a consequence, this word of caution being delivered, we will still run the risk of devoting this section to a succinct review of two important principles at the core of modern theories in physics. The first one is a very general principle that governs all physical theories. The second one is more precisely related to the study of the gravitational interaction and pervades the construction of all modern theories of gravity (especially general relativity).

We present the following discussion to clearly fix our terminology for the rest of this chapter. This aims to avoid ambiguities when referring to those principles in latter discussions. It should also serve as a solid motivation for the introduction of quite some mathematical tools in the next section. We will then, of course, not be exhaustive in reviewing the possible different approaches on these subjects.

1.2.1 Principle of General Covariance

The term "principle of general covariance" refers to the physical principle specifying how physical laws should be built to ensure that different observers will always be able to reconcile their respective observations. In fact, more precisely, at the very core of the principle there is the idea that, in the definition of physical systems and the formulation of the laws of physics, any direct reference to an observer is superfluous.

Nature is really a thing

To state it differently, the principle of general covariance is the idea that "no matter if there is someone to do it or not, Nature is really something that exists and that can be described" raised to the rank of a scientific postulate.

Of course, to work with these physical systems, especially to make sense of measurements, one will quickly need to introduce observers in the picture and we do not expect different observers to imperatively obtain the same results when following similar measurement procedures from their respective point of view (this idea would make little sense). The idea of the principle is that, by their measurements, different observers will access different ways to "look at the same things" but that the laws of physics should be able to talk about "the thing itself" prior to the introduction of any measurement procedure.

This idea that what one could call the physical reality should make sense independently of the ways we use to look at it is, in a sense, equivalent to the idea that there cannot be any meaningful notion of "the best way to look at a system". In other words, there cannot be any way to look at a system that is fundamentally built in the laws of physics as "the right one".[1.16]

The principle of general covariance is then the main motivation to aim to get rid of the prominent role played by inertial frames in the formulation of classical mechanics and special relativity. Anticipating a bit what we should do in the next section, this is the principle that urges us to describe physical quantities using tensors and to express the laws of physics as equations between tensors.

In the context of gravity, the principle of general covariance also plays a role to help in the specification of how laws of physics valid on Minkokwsi spacetime (*i.e.* in the absence of gravity) should transform in the presence of gravity. We will come back to this in the next paragraph.

General covariance, a meta-principle ?

Before doing so, let us comment on the status of the principle. The principle of general covariance is more a meta-principle, or a philosophical mindset, that an actual physical principle in the sense that, in itself, it does not give any constraint on the laws of physics that one can formulate. It is not a principle that constrain what the laws of physics should contain but what they should look like. Our laws should involve tensors, but which ones and how, the principle does not say, so with only this principle one can still construct theories that predicts pretty much anything. Even worse, any theory originally formulated using a preferred set of frames can, in principle, be rewritten in a tensorial form.

The interest of this more geometrical, or intrinsic, formulation of the laws of physics is then unrelated to the actual content of the laws but it argues in favour of a search of simplicity for their formulation. Here, like for the search of a grand unified theory (see introduction chapter), the interest of this æsthetic

[1.16] Some ways to look at a system might be more convenient than others for practical purposes (we will recover this idea several times in the following) but none should be fundamentally necessary to make sense of the laws of physics.

consideration can be justified by the fact that, across the XX$^{\text{th}}$ century, the most successful theories have been the ones that happen to admit a simple and concise formulation in terms of tensorial equations – it is in particular the case of electromagnetism, as we saw, or GR, as we should see.

Also, note that this inability of the principle to really constrain the laws of physics themselves does not necessarily discredit its core idea that, to think of physical processes, an explicit reference to a given observer should not be necessary. This aim of explicitly getting rid of the special role of some observers in the formulation of a theory is thus still relevant regarding this idea. Even if it is hopeless to attempt to use it as a way to decide between rival theories on its own, it can thus be useful as a safeguard in the construction of a new theory when coupled to a search for minimal hypothesis.

1.2.2 Equivalence Principle

Generic considerations

The term "equivalence principle" can be used to refer to several related but non-equivalent ideas. In any case, it aims to talk about a fundamental (if not defining) property of the gravitational interaction. The formulation of this principle is then quite tricky as it first requires to specify a bit what is the physical process called "gravity" without relying on a specific way to model it.

To address this preliminary question very informally, gravity is the word used to refer to "the physical process that makes stuffs fall". The term "physical process" itself could require clarifications but we should assume here that our reader will understand it in the usual sense of "an interaction", see introduction chapter. We could then also say that the word gravity refers to "the interaction that causes things to fall". In this acceptation, "gravity" and "gravitational interaction" are synonyms and we shall use them interchangeably.

This characterisation is, of course, very very (very!) laconic but the use of the verb "to fall" already acknowledge for an important fact : our first experience of what we want to call gravity occurs when we try to understand why it happens that objects systematically tend to reach the surface of the earth if nothing prevents them from doing it.

To have a chance to progress in the formulation of our principle, we need to have a look at more precise ways to make sense of this fact. In particular, we need to have a look at how this process is described in the context of Newtonian gravity. Now, as we said, if our principle aims to be fundamental, it cannot be rooted in a specific model for the gravitational interaction; especially if we aim to use it to find a theory that will extend the results of Newtonian gravity. That being said, one should still be able to rely on the idea that, on account of its undeniable experimental successes in everyday experiments, Newtonian gravity should contain at least part of the "truth" about the gravitational interaction. It thus can – and, in the absence of any better way to do it, should – be used to help us identify the crucial properties of gravity.

Newtonian gravity and Weak equivalence principle

In Newtonian gravity, as we mentioned in the introduction chapter, gravity is described by the law of universal gravitation which states that two massive material points of (gravitational) mass m_1 and m_2 mutually attract each other by means of the force

$$\vec{F} = \frac{\mathcal{G} m_1 m_2}{r^2} \vec{1}_r, \tag{1.53}$$

where $\mathcal{G} \approx 6.67 \times 10^{-11}$N m^2 Kg^{-2} is Newton's constant of gravitation, r is the distance between the bodies and $\vec{1}_r$ represents the unitary vector directed along the line joining the two points and pointing towards the attracting body (*i.e.* for the force experienced by the particle of mass m_1 due to the presence of the mass m_2, $\vec{1}_r$ is directed towards m_2).

Locally (*i.e.* on a small scale) this law of universal gravitation also implies that the force explaining the free fall of a body (seen as a material point) of gravitational mass m_{G} on the surface of a body of gravitational mass M seen as a sphere of radius R was given by

$$\vec{F} = -m_{\text{G}} g \vec{1}_z = m_{\text{G}} \vec{g}, \tag{1.54}$$

where $\vec{1}_z$ here represents the direction perpendicular to the surface of the body of mass M pointing "upward" and $g \coloneqq \mathcal{G}M/R^2$. For a body falling on the surface of the earth, we find $g \approx 9.81$m s^{-2}.

This gravitational force then dictates the motion of the freely falling body by means of Newton's second law[1.17]

$$\vec{F} = m_{\text{I}} \vec{a}, \tag{1.55}$$

where $\vec{a}$ represent the acceleration of the body and m_{I} its inertial mass.

The core of the equivalence principle is then based on the fundamental properties of gravity revealed by the Newtonian formulation. First of all, gravity is a universal force. Any body will attract any other body by means of (1.53). But, already in Newtonian mechanics, the universal nature of gravity is deeper than that. Indeed, the most important property of gravity concerns the gravitational mass of bodies.

This is a well-known property of Newtonian physics but, for the sake of the discussion, let us emphasise that the gravitational and inertial mass of a body enjoy *a priori* very different status in the theory. On one hand, the inertial mass is a fundamental property of the body that measures its resistance to the action of a force in the sense that, no matter what the external force applied, it is always this same quantity m_{I} that must enter (1.55). On the other hand, the gravitational mass is *a priori* much less fundamental in the sense that m_{G} is merely the coupling constant fixing to which extent the body interact with the exterior gravitational field produced by an attracting body. Stated in this way, which is just a rephrasing of the definition of each quantity, there is no

[1.17] Remember that this law is utlimately formulated in an inertial frame.

obvious reason why these two quantities should be related to each other but, as we know, this is where gravity hides its first big surprise : they always are.

Again, this is not a defining property coming from the Newtonian formulation but experience suggests (with extremely high precision) that the gravitational mass of a body is always proportional to its inertial mass and, much more importantly, that the ratio between gravitational and inertial mass is a universal quantity *i.e.* it is the same for any body, independently of any other property it may have. Since m_{I} and m_{G} both have the same dimensions (they have units of mass), one can always get this ratio equal to 1 by choosing appropriate units. In this context, what we meant by talking of extremely high precision is that the most recent results available that we are aware of at the moment we write this sentence argue in favour of a dimensionless ratio

$$\frac{m_{\mathrm{G}}}{m_{\mathrm{I}}} = 1 \pm 10^{-15}, \tag{1.56}$$

within a 1σ statistical uncertainty (see [Touboul et al., 2017]).

A first precise version of the equivalence principle is then to postulate that gravity is such that we always have

$$m_{\mathrm{G}} = m_{\mathrm{I}}. \tag{1.57}$$

This version is usually called, in modern terminology, the weak equivalence principle.

Stated in this way, the principle settles in the Newtonian formulation of gravity as it involves concepts whose meaning comes from this theory. One may also wonder how to really test it *i.e.* one would wonder if it is possible to really measure directly those two quantities to compare them. That being said, it can easily be reformulated to refer instead to more universal notions such as the motion of a body. Indeed, when used within (1.54), (1.57) allows (1.55) to be written as

$$\vec{a} = \vec{g}. \tag{1.58}$$

While the left hand-side dictates the motion of the body, the right hand-side expresses the local manifestation of the gravitational field. On account for (1.58), we can thus give a reformulation of the equivalence principle by stating that

"Locally, all bodies fall in the same way in a gravitational field",

where the term "locally" should be understood as "in regions of space and on time intervals for which variations of the gravitational field can be neglected". In (1.58), this corresponds to the idea that $\vec{g}$ can be regarded as uniform; that is, constant.[1.18]

[1.18] Note that here, we may have presented things upside-down in the sense that it is actually this relation (1.58) that suggests how one can test (1.57) by searching for any noticeable difference in the acceleration of different test bodies during a free fall submitted to strictly identical test conditions. This is typically how one obtains results as (1.56).

Other experimental procedures rely, for example, on the study of torsion balances but, in any case, the experimental procedure is based on the study of acceleration rather than a mere estimation of the mass.

Actually, (1.58) also suggests another key property of gravity revealed by the Newtonian approach. To understand it, we should re-emphasise that this relation is a rewriting of Newton's second law (1.55) that uses (1.57). It then talks about the acceleration of a body as measured in the class of inertial frames. The study of how (1.55) transforms when looked at in frames that do not belong to this class (*i.e.* non-inertial frames) to include the so-called inertial effects then reveals that the effects of a uniform gravitational field can always be removed by considering (1.55) in the class of non-inertial frames whose relative acceleration with respect to the class of inertial frames is precisely given by $\vec{g}$. In particular, this will be the case in the proper frame of the freely falling body; that is, in the referential "attached to it". We thus obtain another reformulation of the principle as

"Locally, the effects of a uniform gravitational field can be suppressed in a uniformly accelerated reference frame."

This formulation might be seen as a bit more tied to the Newtonian formulation of gravity compared to the previous one since it refers to Newtonian concepts or, more precisely, to the special role of inertial frames. Indeed, the term "uniformly accelerated" implicitly refers to an acceleration defined relatively to those frames. It is nevertheless expressing a crucial idea of the equivalence principle : the universality of gravity makes it intrinsically tied to inertial effects.

Removing inertial frames from the Weak equivalence principle ?

In an attempt to state it differently, one could say that

"The universality of gravity makes it (locally) indistinguishable from effects that might be attributed to the relative point of view of an observer."

In a sense, this last way of looking at the principle mixes it with the principle of general covariance in that, by saying that gravity is "indistinguishable" from effects relative to observers viewpoint, one adds to the previous version of the principle the idea that there should thus not be any physically meaningful way of finding the referentials that measures "the" gravitational field as opposed to the referentials in which it is, totally or partially, suppressed by inertial – *i.e.* relative – effects. This is a (arguably reasonable and definitely interesting) way to take into account the necessity to remove the reference to inertial frames demanded by the principle of general covariance but it is then important to note that this does not compulsorily come as a logical consequence of the formulations of the weak equivalence principle given above. Basically, this consists in removing the role played by inertial frames in distinguishing gravity from inertial effects by postulating that it cannot make any sense, physically speaking, to make such a distinction.

This principle, which also seems to us to deserve the name of equivalence principle, is thus subtly distinct from the weak equivalence principle. Baring the risk to repeat ourselves, it is an evolution of the weak equivalence principle that tries to incorporate the principle of general covariance in the most drastic way possible. This version of an equivalence principle clearly played an important role in Einstein's reflections on the subject when developing general relativity.

Special relativity and Einstein equivalence principle

Coming back on the weak equivalence principle, this idea of a tight link between measuring gravity and studying accelerated frames led to another important development of the equivalence principle when considered in the context of special relativity.

Indeed, in SR the notion of inertial frame is still present. It takes a slightly different form than in Newtonian mechanics since the spacetime structure is different (see remark 1.2) but it plays mainly the same role. It refers to the class of referentials that cohere to the (rigidly fixed) structure of Minkowski spacetime. In other words, it refers to coordinate systems for which the Minkowski metric assumes the form (1.13).

One can thus still make sense of the notion of a uniformly accelerated frame as in the usual Newtonian meaning of the term. As we know, gravity is not included in the formulation of special relativity but, when imported into the framework of the theory, the weak equivalence principle suggests that its effects should locally correspond to what would be observed by a uniformly accelerated observer. For a pedagogical discussion on the motion of a uniformly accelerated observer in SR, the interested reader should look at chapter 6 of [Misner et al., 1973]. As a French-speaking writer (which is thus also a French reading learner), let us also slip that readers familiar with French could also have a look at chapter 12 of [Semay and Silvestre-Brac, 2016].

For now, let us focus on the conclusions and implications of this study. As emphasised in section 1.1, the heart of special relativity lies in the structure of Minkowski spacetime and thus in the Minkowski metric. The most important conclusion is then that spacetime, as perceived by an accelerated observer, will present a metric that does not assume the form (1.13) but instead depends on the coordinates of the spacetime point at which we aim to study it. Of course, in this case, this is purely induced by the fact that we are explicitly observing spacetime from an "unsuitable" point of view; in other words, the metric is still the Minkowski metric, we are just observing it from a non-inertial frame.[1.19] But the weak equivalence principle then suggests that, locally, this should make no difference with the physical effects of a gravitational field.

In other words, this brings in the idea that

"In a framework compatible with special relativity, gravity should manifest itself at the level of the spacetime metric."

Actually, to state it more carefully, one should phrase this as

[1.19] This is similar to what happens when considering the Minkoswki metric in an inertial frame but using curvilinear coordinates. The components of the metric will depend on the spacetime (curvilinear) coordinates but the metric itself is still the same; as are all of its properties.

"In a framework compatible with special relativity, gravity should manifest itself in the spacetime structure",

since, clearly, in special relativity, the spacetime structure is specified by its metric (and by the assumption that Minkowski spacetime, as Newtonian spacetime, as the mathematical structure of an affine space) but this does not mean that one cannot come with more subtle ways to play with what we call the spacetime structure in more general situations. We will definitely come back to this in section 1.3 but we have a few more things to say about the equivalence principle beforehand.

First, strictly speaking, the above assertions are not what one would dub the equivalence principle of special relativity (or the relativistic weak equivalence principle, or whatever) but they are actually very close to what one calls the Einstein equivalence principle. Actually, the above idea is sort of a preliminary version of the Einstein equivalence principle in the following sense : from the study of uniformly accelerated observers we can say a bit more than the mere fact that gravity should be searched at the level of the spacetime structure. We can be more precise in the way it should do it. In a similar fashion, we could say that we can be more precise with what we mean by "a framework compatible with special relativity".

If the study of the uniformly accelerated observer leads to the idea that gravity will be part of the spacetime factory, we should also consider the reciprocal of the equivalence. Gravity will "produce" a non-Minkoswkian spacetime but, locally, one should still be able to remove these effects by observing spacetime in the referential of a freely falling observer. So, gravity should influence the spacetime structure but in such a way that spacetime still formally looks like Minkowski spacetime when looked at on sufficiently small regions.

Now, again, we have to be a bit more precise with what we physically mean by "looking at spacetime". Obviously, one does not look at spacetime with a magnifying glass. To reveal the spacetime structure, one needs to perform experiments. We obtained the weak equivalence principle by considering the motion of test particles in a gravitational field in Newtonian mechanics. To obtain a version of the equivalence principle that will be compatible with special relativity, one could do the same and postulate that the physical trajectory followed by particles only subject to the gravitational interaction should cohere the trajectory of free particles in special relativity (*i.e.* the straight lines) when observed locally in the frame of a freely falling observer.

Actually, Einstein went one step further. Indeed, we certainly want the above consideration on the motion of particles freely falling to hold in a way that is compatible with special relativity instead of Newtonian mechanics but, following Einstein, we should ask more than that as physics does not reduce to objects freely falling. If one can really remove the gravitational field by going into the appropriate frame, this feature should reveal itself in the study of each and every possible physical process (other than gravity itself, by construction). The Einstein equivalence principle then states that

"Locally, there must always be a class of referentials for which the

(non-gravitational) laws of physics assume their special relativistic form."

Here again, the term "locally" is crucial and refers to the idea that this remarkable property is only expected to hold in small enough regions of spacetime where the effects of the gravitational field can be considered as uniform. Yet, it is also useful to emphasise that, if this property should only be true locally, it should still hold around any point of spacetime.

The fact that the principle does only apply to non-gravitational laws of physics can seem redundant as there is no special relativistic description of gravity but this also includes the idea that the procedure to hide gravity should only make sense as long as no significant gravitational process is involved in the system under study. For example, when considering the motion of objects, we would like to have objects "small enough" so that their internal gravitational processes can be neglected. This principle would then not apply to big stars, planets, ... More importantly, this idea that the principle does not apply to gravitational laws of physics themselves then leaves some freedom in the way one would understand the term "spacetime structure" and how one would precisely encode gravity in it.

Remark 1.2. *Adopting a modern point of view, one could say that inertial frames in Newtonian mechanics correspond to the class of frames that fits the structure of the Newtonian spacetime. This spacetime is an affine space, call it* $\mathbf{N}^4$*, such that*

$$\mathbf{N}^4 \approx \mathbb{R}_{\mathbf{t}} \times \mathbb{E}^3,$$

where $\mathbb{E}^3$ *denotes the 3-dimensional Euclidian space and* $\mathbb{R}_{\mathbf{t}}$ *is as a one-dimensional affine space isomorphic to the real numbers that encodes the Newtonian absolute time; see chapter 12 of [Misner et al., 1973] (French familiar readers should also consider chapter 6 of [Spindel, 2001] which admittedly inspired this remark). This structure then encodes the absoluteness of simultaneity and the fact that spaces of simultaneity are all isomorphic to Euclidian space. These are the defining properties of the spacetime of Newtonian physics.*

One can then construct coordinates fitting this structure using a linear mapping $t : \mathbf{N}^4 \to \mathbb{R}_{\mathbf{t}}$*, encoding the absolute time. This mapping ascribes a "date"* $t_P := t(P)$ *to any event* $P \in \mathbf{N}^4$ *(this date can be regarded as a real number by further choosing a time scale and a time origin by means of a mapping from* $\mathbb{R}_{\mathbf{t}}$ *to* $\mathbb{R}$*). This then also ascribes to a given* P *his space of simultaneity* $\mathbb{E}^3_{t_P} := t^{-1}(t_P) \approx \mathbb{E}^3$*. One can then locate* P *in his space of simultaneity by endowing this space with Cartesian coordinates.*

In this rather formal approach, this way of constructing coordinates defines what will be interpreted as the inertial frames. This will indeed make sense. This way of ascribing coordinates to a point will not be unique but, if we look at the subset of transformations of $\mathbf{N}^4$ *that preserves this spacetime structure, we will precisely recover the group of Galilean transformations that relates inertial frames to one another in Newtonian physics.*

In the sense of this technical remark, we can then say that the inertial frames in Newtonian mechanics and in special relativity refer to the same concept as *inertial frames in special relativity will be constructed just as we did here in Newtonian spacetime but with respect to a different spacetime structure – the one of Minkowski spacetime – fixed by the Minkowski metric* $\boldsymbol{\eta}$ *instead of the absolute time* t*.*

Strong equivalence principle

There is actually a more restrictive version of the equivalence principle, known as the "strong equivalence principle". This version of the principle basically aims to modify the Einstein equivalence principle to include gravitational processes. It could be stated as

"Locally, there must always be a class of referentials for which references to the gravitational field can be removed from the description of physical systems."

As in all the other equivalence principles, the locality condition remains fundamental. The strong equivalence principle then aims to apply even in systems where internal gravitational processes play an important role (see the examples above) but, for this condition to make sense, one should only apply the principle to regions of spacetime where also these internal effects can be seen as constant.[1.20]

Stated in this way, the principle may almost look like a slight reformulation of the Einstein equivalence principle but this idea to promote the application of the principle to systems which include gravitational effects as an important part of their description is in fact much more restrictive regarding how one can possibly implement gravity in a theoretical framework. For example, a direct consequence of the strong equivalence principle is that Newton's constant of gravitation $\mathcal{G}$ should assume the same value everywhere on spacetime. Indeed, since the principle should apply locally but around any point in spacetime, variations of the parameters entering gravitational laws cannot be allowed. This then rules out the possibility to promote this constant to a field of the theory (whose dynamics would be constrained by the modern-day observations on Newton's constant but anyway) and is thus in direct contradiction with the formulation of theories of gravity such as the Brans-Dicke theory.

More generally, the strong equivalence principle seems to automatically rule out the possibility to even formulate many kinds of alternative theories of gravity. It has been claimed that this principle actually implies that gravity should be entirely geometrical in such a way that we cannot consider theories including fields other than the metric for the description of gravity. Apparently, some authors have even claimed that general relativity is the only theory of gravity that can possibly satisfy this principle.

In any case, on account for the subject of this thesis, such a principle should clearly be regarded as too restrictive for our discussion and we will not elaborate more on it.

[1.20] For example, one does not expect the principle to be applicable on portions of the systems big enough for tidal effects to be measurable. This type of gravitational effect, when sizable enough to be taken into account, should not be erasable by a mere change of frame.

On the equivalence between the equivalence principles

To close this presentation, let us emphasise one more thing : schematically speaking, through this discussion, we have constructed different versions of what is called the equivalence principle by successively reformulating ideas to add more content (*i.e.* more restrictions) in the formulation of the principle. Schematically, or presentation can be summarised as

$$\text{Newt.physics} \xhookrightarrow{m_{\mathrm{G}}=m_{\mathrm{I}}} \text{Weak} \xhookrightarrow[-Newt.physics]{+Special\ Relativity} \text{Einstein} \xhookrightarrow[to\ gravity]{+apply} \text{Strong.}$$

In this diagram, the "$\hookrightarrow$" symbols represent our argumentation but does not represent a proof nor a real piling of the concepts present at each step. In particular, even though the strong equivalence principle is by construction more restrictive than the Einstein equivalence principle which might itself be seen as more restrictive than the weak equivalence principle (if we consider that this last principle applies to Newtonian physics which can be seen as a "small velocity approximation of special relativity"), **we do not have**

$$\text{Strong} \Longrightarrow \text{Einstein} \Longrightarrow \text{Weak},$$

especially if one aims to understand the "$\Rightarrow$" symbols as logical implications. The different equivalence principles are inevitably related to each other from the way they were constructed but saying that one version is more restrictive than one other does not simply mean that it precisely corresponds to the previous one to which we have added more requirements. The successive reformulations of the ideas and their applications in different conceptual frameworks prevent such a direct way of relating the different versions of the equivalence principle.

In the rest of this text, when mentioning "the" equivalence principle, we will most likely refer to the Einstein equivalence principle. We should nevertheless stay very attentive to the part of the above discussion in which we raised the technical and conceptual difficulty of combining this with the principle of general covariance to remove the special role played by inertial frames in the formulation of the principle even in special relativity. The path followed to overcome this difficulty might make the difference between the road guiding to general relativity (section 1.4) and the one leading to alternative (but practically equivalent) formulations of a relativistic theory of the gravitational interaction (section 1.6).

1.3 Mathematical Playground

The above discussion raised the necessity to eliminate the special role of inertial observers and suggested relating gravity to the very structure of spacetime. An extremely powerful way to achieve this double goal can be found in the tools of differential geometry.

Before entering the precise formulation of general relativity, or any other relativistic theory of gravity, we will thus first review the mathematical tools that allow for a precise formulation of their ideas.

Before going further, let us emphasise that the aim of the discussion below is not to be mathematically complete or to talk about all the details and subtleties regarding the objects we present; this can be found in other places (such as, again, [Wald, 1984, Carroll, 1997] or [Kobayashi and Nomizu, 1996]). Here, while trying to preserve a decent level of formalism, we rather choose to emphasise the very tangible intuitions behind the abstract formalism of differential geometry and to focus on conceptual issues rather than technical ones.

1.3.1 Differential Manifold

With our physicist's dream of describing the gravitational interaction in terms of the spacetime structure in mind, our very first task is to find a satisfactory way to describe what spacetime could be. This might clearly be a question of abyssal depth from a philosophical point of view... From a more technical viewpoint, what we want from spacetime at the very least is to be able to perform space and time measurements on it. This certainly raises the need to introduce a metric as a way to formalise space and time "rulers" but let us not go too far too fast. Prior to introduce "measuring equipments", we should introduce the nature of our measurements and the ways by which we can look at spacetime.

Topological manifold

In this regard, it is reasonably natural to assert that the measure of any physical property of a system should lead to a bunch of real numbers. Let us call n the number of real numbers[1.21] necessary to perform our measures on space and time. In other words, the measure of "where and when" something happens should be describable in terms of an element of $\mathbb{R}^n$. So at this point, mathematically, spacetime should at least be a set whose elements (called events) can be mapped individually to elements of $\mathbb{R}^n$.

This can obviously be achieved if we assume a global correspondence between our spacetime and $\mathbb{R}^n$ *i.e.* if we suppose that it must be possible to univoquely map *at once* any possible event to a single n-uplet.[1.22] A more minimal but richer way to proceed nevertheless is to assume that this only has to be true *locally*. This can be motivated by several basic ideas. First of all, when studying physical systems, more than just performing a couple of independent measurements of a given property of this system, one aims to capture its evolution; that is, how this property evolves in space and time. To achieve this task, it is generically unnecessary to relate the system to the entire universe.

[1.21] Regarding a measurement of space+time properties one could *a priori* expect 4 numbers to be necessary and sufficient but, from both a mathematical and physical perspective, it can be useful to keep an open mind and perform the discussion simply assuming that there is a fixed but arbitrary (natural) number of (real) numbers necessary to describe this kind of measurements.

[1.22] This is the case in Minkowski spacetime for example.

By this we mean that, as long as physical interactions are local (that is that the physical quantities needed to describe the interaction at a given point only have to be evaluated at that specific point) and that their intensity decreases sufficiently fast has one moves away from the source, it should be enough to be able to relate the situation at one event to the situation at "neighbouring" ones to capture its evolution. The second idea is that, unless some very specific or extreme conditions are met, one will expect this evolution to be "smooth". In other words, we will want to capture the idea that if we only slightly change the event considered, the physical quantities will only slightly change as well; again this will only require to describe the system "around" a given event and not necessarily on all the spacetime.

In a mathematical framework, this idea of something being "around", "next to", "in the neighbourhood of" something else is captured by the notion of a topology. If we want to be able to apply those concepts to describe the spacetime dependence of physical phenomena, we then have to assume that our spacetime is a topological space (a set equipped with a topology *i.e.* a set for which we have defined open subsets; see remark 1.4). In addition, connecting to our first requirement, this topology should be such that, locally, our spacetime behaves as $\mathbb{R}^n$ so that, around a given point, these notions of neighbourhood and of continuous variations reduces to the ones we are used to. We then arrive at the mathematical notion of a real topological manifold of dimension n.

Basically, a real topological manifold of dimension n, $\mathcal{M}$, is precisely a topological space (see remark 1.3) that looks like $\mathbb{R}^n$ around any point. This is formalised by requiring that one can cover $\mathcal{M}$ with a set $\{ U_i \}$ of open sets such that any U_i is homeomorphic to an open subset of $\mathbb{R}^n$; *i.e.* there must exist homeomorphisms[1.23] $\Phi_i : U_i \to \Phi_i(U_i) \subset \mathbb{R}^n$. The collection of the U_i's together with the corresponding homeomorphisms Φ_i, $\{ (U_i, \Phi_i) \}$, is called an atlas on $\mathcal{M}$. The couple (U_i, Φ_i) is called a chart. It might formally be seen as a local coordinate system on $\mathcal{M}$.

Note that the consistency of the fact that the topological properties of $\mathcal{M}$ locally reduces to that of $\mathbb{R}^n$ is ensured by the fact that, given an atlas on $\mathcal{M}$, for any two open sets U_i and U_j in the atlas that overlap with each other ($U_i \cap U_j \neq \emptyset$), the function $\Phi_i \circ \Phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ (called a transition function) is a homeomorphism in the usual sense on $\mathbb{R}^n$.

Remark 1.3. *To be precise, for the consistency of our definition of a topological manifold, we should say that $\mathcal{M}$ should be a Hausdorff and second countable topological space.*

A Hausdorff space is a topological space with one important extra property : given any points $p, q \in \mathcal{M}$ distinct from each other, there must exist a neighbourhood V of p and a neighbourhood W of q such that $V \cap W = \emptyset$. This property, that is not automatically verified for any topological space, intuitively

[1.23] An homeomorphism is a function that is continuous, invertible and whose inverse is also a continuous function. In the realm of topology, homeomorphisms are isomoprhisms of topological spaces in the sense that they provide a one to one correspondance between the elements on both side while being compatible with the topological structures.

states that the topology on $\mathcal{M}$ is "smart enough" to distinguish the points on $\mathcal{M}$. This intuition can be more formally captured by the fact that a topological space is a Hausdorff space if and only if any singleton $\{ p \} \subset \mathcal{M}$ is equal to the intersection of all closed neighbourhoods of the corresponding $p \in \mathcal{M}$.

A second countable space is a topological space for which there exist a countable family of open sets $\{ O_i \}_{i=0}^{\infty}$ with the property that any open set U can be obtained from a union of elements of this countable family i.e. *$U = \cup_{j=0}^{\infty} O_j$ for some O_j's in the family. Intuitively, this condition states that the topological structure of our set can be obtained from "not too many" open sets.*

*At a very intuitive level, we could see the interest of these conditions for our physical spacetime as coming from the idea that our notion of neighborhood should be "thin enough" to distinguish different events but able to do it in a "minimal way" (*i.e. *with at most a countable number of neighbourhoods). The importance of these conditions at a more formal level is more technical and we will not discuss it here (see remark 1.4) even though we had to mention the conditions for completeness.*

Remark 1.4. *It is worth noting that a topological space is, by definition, a set endowed with a topology. Without recalling the details of the definition of a topology here, we would like to mention one important point : there is always more than one way to define a topology on a given (non-empty) set. This means that, if one is only given a set, defining a topology requires to make a choice.*

This may lead to a natural question in the context of this discussion : what topology should we choose for our spacetime ? A topology being one of the first structures that we do want for our spacetime, this question is as natural as important. It should thus be acknowledged that, despite its importance, there is no answer to this question that reaches a general consensus.

The reason for this deficit of consensus could deserve a more detailed discussion but we mostly wanted to mention it here to emphasise that, if the notion of a topology is necessary before almost anything else, among the physics community this is mostly viewed as a preliminary step in the construction of the *structure that we are interested in : a differential structure.*

For this reason, in the following, we will assume the topological structure of our spacetime to be fixed and to have all the necessary properties to allow us to make sense of the concepts that we need to introduce. We will thus avoid references to topological questions unless it is of primary importance.

Differential manifold

This mathematical structure is already very rich but this is still not enough for what we want from a physical spacetime. Indeed, when studying physical systems, more than the ability to describe the spacetime dependence of different quantities, physicists aims to capture the laws ruling this dependence thanks to specific equations. Stated in an intuitive and quite informal way, what these equations do, in one sense or another, is relating the "infinitesimal" variations of a physical attribute to the behaviour of what sources it. This may need a more in-depth discussion but let us merely suggest here, for the sake of intuition, that this structure might be seen as a reflection of the fact that one distinguishes the presence of an interaction from its absence by the fact that some quantity, that would not change otherwise, is changing due to this interaction. In any case, the ability to talk about the "rate of change" of a quantity is not automatically ensured by the topological structure. To fit this new notion in our formalism we need a differential structure. In other words, we need the link between our spacetime manifold and $\mathbb{R}^n$ to be sufficiently strong to ensure that we can talk about derivatives of a function in the same way as one would do on $\mathbb{R}^n$. We are then left with the concept of a real differential manifold of dimension n.

Cutting a long story short, a real differential manifold of dimension n, $\mathcal{M}$, is simply a real topological manifold of dimension n for which it is also possible to do differential calculus. This is ensured formally by endowing our topological manifold with an atlas $\{ (U_i, \Phi_i) \}$ for which the transition functions $\Phi_i \circ \Phi_j^{-1}$ are diffeomorphisms[1.24] in the usual sense on $\mathbb{R}^n$. In this case, one calls $\{ (U_i, \Phi_i) \}$ a differentiable atlas.

Note that, technically, such a structure might only allow differentiating functions once. Mathematically speaking, if one needs to differentiate more than once, one needs to impose a more restrictive conditions on the transition functions. Generally, in physics, in order to not bother too much about this, we will automatically assume that the transition functions $\Phi_i \circ \Phi_j^{-1}$ define $\mathcal{C}^\infty$-diffeomorphisms, leading to a $\mathcal{C}^\infty$ (or smooth) differential manifold. We will also do as if we were only dealing with $\mathcal{C}^\infty$-functions. In this way, we are assured that we can differentiate things as much as needed.

Before continuing, let us quickly comment on an important point : in the presentation we gave so far, the choice of an atlas seems to be part of the structure of the differential (smooth, ...) manifold. This might be seen as a problem. Indeed, one does not want the differential (smooth, ...) structure defined on $\mathcal{M}$ to depend on a "preferred choice of coordinate systems" but this would, in a sense, be the case if we defined the structure on $\mathcal{M}$ in terms of an atlas without a bit of extra care. Let us do our best to clarify this point.

An atlas $\{ (U_i, \Phi_i) \}$ gives a way to express the local correspondence between $\mathcal{M}$ and $\mathbb{R}^n$. For a topological manifold, the existence of an atlas follows from the topological structure already present on $\mathcal{M}$. For a differential manifold the atlas (and more precisely the transition functions) is (are) used to make sense of the differentiable structure on $\mathcal{M}$.

Given a differentiable atlas, one will say that a chart $\left(\tilde{U}, \tilde{\Phi}\right)$ (that does not belong to the atlas *a priori*) is compatible with the atlas if for all U_i in the atlas such that $\tilde{U} \cap U_i \neq \emptyset$, one has that $\tilde{\Phi} \circ \Phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism[1.25]. In other words, a chart is compatible with an atlas if the set obtained by adding

[1.24] A diffeomorphism is a function that is differentiable, invertible and whose inverse is also a differentiable function. In a sense, diffeomorphisms are the natural way to define isomorphisms of differentiable structures.

[1.25] This is the condition for differential manifolds. Similar notions apply to different types of manifolds. As an example, for a smooth manifold, one would require $\tilde{\Phi} \circ \Phi_i^{-1}$ to be smooth. This generalizes straightforwardly to other types of manifolds.

this chart to the atlas is still an atlas. This is where the problem lies. A chart compatible with a given atlas is a chart that "could have been in the atlas" but *a priori* does not. We then arrive at a situation where the charts in the atlas are, in a sense, privileged over all (equivalent) possible choices to make sense of the differential structure. There is, nevertheless, a class of atlases that allows to avoid this feature : maximal atlases. A maximal differentiable atlas is defined as a differentiable atlas $\{ (U_i, \Phi_i) \}$ such that for any chart $\big(\tilde{U}, \tilde{\Phi}\big)$ compatible with the atlas, one has that $\big(\tilde{U}, \tilde{\Phi}\big) \in \{ (U_i, \Phi_i) \}$. Intuitively, this corresponds to the idea that all the charts that are "acceptable" have been taken into account. A maximal differentiable atlas (resp. a maximal smooth atlas) will then unambiguously define a differentiable (resp. smooth) manifold structure on $\mathcal{M}$.

One will then generically define a differentiable (smooth, ...) manifold in the same way as what we sketched above but with the explicit requirement to use a maximal differentiable (smooth, ...) atlas. In this way, the structure defined on $\mathcal{M}$ is "purged" from preferred choices of coordinate systems.

Another, equivalent, way to do this is to define the structure on the manifold by means of an equivalence class of atlases; the equivalence relation being given by the property that two atlases are related if and only if each chart of one atlas is compatible with the other atlas. These equivalence classes are in one-to-one correspondence with maximal atlases. Intuitively, this alternative way to proceed corresponds to the idea of expressing the manifold structure with only a few charts (*i.e.* with an atlas that is not necessarily maximal) while keeping in mind that others are equally valid.

As a final note on this subtlety, we thus see that *a priori* there might be more than one way to define a differential (smooth, ...) manifold from a given topological manifold and that the number of non-equivalent ways to do it is given by the number of maximal differentiable (smooth, ...) atlases.

In the following, when we will deal with charts, these will always be assumed to belong to the maximal atlas that defines the differential manifold.

1.3.2 Natural Structures on a Differential Manifold

To summarise our discussion so far, the very central ingredient for our construction of a formal (classical) description of spacetime is thus a smooth real differential manifold of dimension n. We tried to emphasise in which sense this structure is at the same time very minimal from a physical point of view but already pretty rich mathematically speaking. To further emphasise this last statement, let us comment on the impressive quantity of structures that naturally arises once given a differential manifold :

Coordinate systems

As we have already quoted, the choice of a chart (U, Φ) really corresponds to the choice of a local coordinate system on $\mathcal{M}$. To see this more explicitly, let us just recall that an element $v = \big(v^1, \cdots, v^n\big) \in \mathbb{R}^n$ is, by definition, entirely characterised by its components and that these components can be obtained individually using the functions

$$\mathfrak{X}^\mu : \mathbb{R}^n \to \mathbb{R} : \left(v^1, \cdots, v^{\mu-1}, v^\mu, v^{\mu+1}, \cdots, v^n\right) \mapsto v^\mu$$

for each value of $\mu = 1, \cdots, n$. Given a chart (U, Φ), we can define the functions

$$x^\mu := \mathfrak{X}^\mu \circ \Phi : U \to \mathbb{R}.$$

Any $p \in U$ can then be represented in the chart as

$$\Phi(p) = \left(x^1(p), \cdots, x^n(p)\right) = (x^\mu(p)) \in \mathbb{R}^n.$$

It is then equivalent to specify Φ itself or the set of functions $\{ x^\mu \}_{\mu=1,\cdots,n}$. This then really corresponds to the intuitive notion of a choice of coordinate. In the following, unless it is notationally more convenient to use Φ, we will use the notation $\{ x^\mu \}$ to denote a coordinate system, according to the usual usage in the physics literature, with the above definition in mind. We will also allow ourselves the slight misuse of notation $\Phi(p) = (x^\mu(p)) =: (x^\mu)$, *i.e.* we will allow ourselves to drop the reference to p when writing its components in a local coordinate system, which should be clear from context.[1.26]

Directional derivative of functions

On a differential manifold $\mathcal{M}$, one can naturally make sense of the notion of differentiability and smoothness for real-valued functions $f : \mathcal{M} \to \mathbb{R}$ and for curves "drawn on $\mathcal{M}$" $\mathscr{C} : \mathbb{R} \to \mathcal{M}$. This will be done by requiring that the representation of these objects in any local coordinate system (U, Φ) is a differentiable (resp. smooth) function in the usual sense on $\mathbb{R}^n$; these representations being, respectively,

$$\boldsymbol{f} := f \circ \Phi^{-1} : \mathbb{R}^n \to \mathbb{R} \text{ and } \boldsymbol{\mathcal{C}} := \Phi \circ \mathscr{C} : \mathbb{R} \to \mathbb{R}^n \tag{1.59}$$

Thanks to this, one can define a notion of directional derivative at a given point $p \in \mathcal{M}$ for the real-valued functions. This is done by studying the derivative of $f \circ \mathscr{C} : \mathbb{R} \to \mathbb{R}$. Assuming, without loss of generality, that $\mathscr{C}(0) = p$, one will define the derivative of f in the direction of $\mathscr{C}$ at point p as the quantity $(f \circ \mathscr{C})'(0) \in \mathbb{R}$, where the $'$ denote the derivative.

This really corresponds to the usual idea of a directional derivative. Indeed, given any coordinate system $\{ x^\mu \}$ defined in a neighbourhood of p, we can introduce the representations of f and $\mathscr{C}$ in this system of coordinate as defined above. We will then have $f \circ \mathscr{C} = \boldsymbol{f} \circ \boldsymbol{\mathcal{C}}$ and the usual chain rule on $\mathbb{R}^n$ will give

$$(f \circ \mathscr{C})'(0) = (\boldsymbol{\mathcal{C}}^\mu)'(0)\ \boldsymbol{\partial}_\mu \boldsymbol{f}[\Phi(p)],$$

[1.26] For example, when dealing with a function $f : \mathcal{M} \to \mathbb{R}$ and coordinate system (U, Φ), we can define the representation of f in the coordinate system as $\boldsymbol{f} := f \circ \Phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ and we will then have that for any $p \in U$, $f(p) = \boldsymbol{f}[\Phi(p)] = \boldsymbol{f}\,(x^\mu(p))$ but, unless we want to stress the nature of the objects for a specific reason, we will simply note $f(p) = \boldsymbol{f}[\Phi(p)] =: f\,(x^\mu)$, ignoring the distinction between f and $\boldsymbol{f}$ and between p, $\Phi(p)$ and x^μ. Again, this should lead to no confusion when refering to the context.

where $\boldsymbol{\partial}_\mu$ is the usual partial derivative defined on $\mathbb{R}^n$. This corresponds to the directional derivative of $\boldsymbol{f}$ at point $\Phi(p) \in \mathbb{R}^n$ in the direction of the vector $\boldsymbol{\mathcal{C}}'(0) \in \mathbb{R}^n$ which is the vector tangent to the curve $\boldsymbol{\mathcal{C}}$ at $\Phi(p) = \boldsymbol{\mathcal{C}}(0)$. In other words, one can write that, given any coordinate system defined in a neighbourhood of p,

$$(f \circ \mathscr{C})'(0) = \nabla_{\boldsymbol{\mathcal{C}}'(0)} \boldsymbol{f}|_{\Phi(p)}, \tag{1.60}$$

where the right-hand side uses the usual definition of directional derivative known on $\mathbb{R}^n$. That being said, note that $(f \circ \mathscr{C})'(0)$ does not require to choose a coordinate system to be defined.

Tangent vectors

This directional derivative allows defining intrinsically the notion of a vector tangent to $\mathcal{M}$ at p thanks to the "velocity of the curve at p". Formally, the vector tangent to a curve $\mathscr{C}$ at $p = \mathscr{C}(0) \in \mathcal{M}$ will be defined as the real-valued (linear) differential operator, noted $\vec{v}$, whose action on a real-valued function f defined in a neighbourhood of p is given by

$$\vec{v}(f) \coloneqq (f \circ \mathscr{C})'(0). \tag{1.61}$$

On account of the above observation, one will usually write

$$\nabla_{\vec{v}} f|_p \coloneqq \vec{v}(f) \tag{1.62}$$

and call this the derivative of f at point p in the direction of $\vec{v}$, blurring a bit the reference to the curve $\mathscr{C}$ used for the definition. Note in passing that this is not harmful since, according to the above procedure, many different curves will define the same vector $\vec{v}$ so it does not matter which one is used for the calculation. This gives rise for any $p \in \mathcal{M}$ to the tangent space to $\mathcal{M}$ at p, noted $T_p\mathcal{M}$, which is the n dimensional real vector space of all the tangent vectors to $\mathcal{M}$ at p.

Covectors and tensors

As always, once given a real vector space, we can consider its dual space, that is the set of real-valued linear functions defined on this vector space. This then allows defining the cotangent space at $p \in \mathcal{M}$, noted $T_p^*\mathcal{M}$, as the dual of $T_p\mathcal{M}$. The elements of $T_p^*\mathcal{M}$ will be called 1-forms or covectors.

With those two vector spaces at hand, one can further define tensors at $p \in \mathcal{M}$ as multilinear maps from $T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M} \times T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}$ to $\mathbb{R}$ with the appropriated number of $T_p^*\mathcal{M}$ and $T_p\mathcal{M}$ factors depending on its rank.

Vector fields

This construct also naturally allows defining vector fields on $\mathcal{M}$. These will be defined by picking a vector at each point of a domain $U \subseteq \mathcal{M}$. Note that given a vector field $\vec{v}$ and $f : \mathcal{M} \to \mathbb{R}$, one can define a function $\nabla_{\vec{v}} f : \mathcal{M} \to \mathbb{R}$ by

$$\left(\nabla_{\vec{v}} f\right)(p) \coloneqq \vec{v}|_p(f) = \nabla_{\vec{v}|_p} f|_p, \tag{1.63}$$

where $\vec{v}|_p$ (often abbreviated as $\vec{v}_p$) denote the evaluation of $\vec{v}$ at p. One can then define the notion of differentiability (resp. smothness) of a vector field by requiring that for any differentiable (smooth) function f, the function $\nabla_{\vec{v}} f$ is differentiable (smooth) in the above sense. The set of all smooth vector fields on $\mathcal{M}$ will be noted $\Gamma\left(T\mathcal{M}\right)$.

The observation that, if $\vec{v}$ is a vector field and f a real-valued function, $\vec{v}(f)$ is again a real-valued function allows defining an important operation for vector fields called the Lie bracket, or the commutator, of two vector fields. If $\vec{v}, \vec{w} \in \Gamma\left(T\mathcal{M}\right)$, the commutator of $\vec{v}$ and $\vec{w}$ is another vector field, denoted $[\vec{v}, \vec{w}]$, defined via the following relation : for any $f : \mathcal{M} \to \mathbb{R}$, we have that

$$[\vec{v}, \vec{w}]\left(f\right) \coloneqq \vec{v}\left(\vec{w}(f)\right) - \vec{w}\left(\vec{v}(f)\right). \tag{1.64}$$

One can easily check that this relation really defines a vector field. This then defines a bilinear and antisymmetric operation $[\cdot, \cdot] : \Gamma\left(T\mathcal{M}\right) \times \Gamma\left(T\mathcal{M}\right) \to \Gamma\left(T\mathcal{M}\right)$. This operation allows to define the difference between deriving a function in the direction of a vector field $\vec{w}$ and then deriving the result in the direction of $\vec{v}$ and the opposite.

Covector fields and tensor fields

Obviously, one can do a similar construction with covectors and define covector fields by picking a covector at each point of a domain $U \subseteq \mathcal{M}$. Given a covector field $\underline{\theta}$, one will then define a notion of smoothness by requiring that for any $\vec{v} \in \Gamma\left(T\mathcal{M}\right)$, the function $\underline{\theta}\left(\vec{v}\right) : \mathcal{M} \to \mathbb{R}$ defined via $\left[\underline{\theta}\left(\vec{v}\right)\right](p) \coloneqq \underline{\theta}|_p\left(\vec{v}|_p\right)$ is smooth. The set of all smooth covector fields on $\mathcal{M}$ is often denoted $\Gamma\left(T^*\mathcal{M}\right)$. The same can also be done for tensors of any ranks.[1.27]

Given a smooth function $f : \mathcal{M} \to \mathbb{R}$, we can also construct a smooth covector field, noted $\mathrm{d}f$ and called the differential of f, via the condition that, for any p where it makes sense,

$$\mathrm{d}f|_p\left(\vec{v}\right) \coloneqq \vec{v}(f) \in \mathbb{R} \tag{1.65}$$

for any $\vec{v} \in T_p\mathcal{M}$.

Coordinate basis

By nature, vectors, covectors and tensors are defined intrinsically on $\mathcal{M}$. They do not need a choice of coordinate to be defined. Nevertheless, for practical uses, it is generally convenient to relate these notions to a choice of coordinate.

Given a local coordinate system $\{\, x^\mu \,\}$, one can define n vector fields, noted $\frac{\vec{\partial}}{\partial x^\mu}$, $\vec{\partial}_\mu$ or simply ∂_μ for $\mu = 1, \cdots, n$, by fixing that for any smooth $f : \mathcal{M} \to \mathbb{R}$,

[1.27] All these informations can also be encoded using the language of fibre bundles within the tangent-, cotangent- and tensor-bundle by "attaching" to any point $p \in \mathcal{M}$ the corresponding tangent-, cotangent- or tensor-space. These constructs being naturally endowed with a structure of differential manifold built on that of $\mathcal{M}$, they can also provide natural ways to formalise the notion of continuity and differentiability for the vector-, covector- and tensor-fields.

we have

$$\partial_\mu|_p(f) \coloneqq \boldsymbol{\partial}_\mu \boldsymbol{f}[\Phi(p)] \in \mathbb{R}. \tag{1.66}$$

It is easy to prove that $\partial_\mu \in \Gamma(T\mathcal{M})$ and that $\{\partial_\mu\}_{\mu=1,\cdots,n}$ form a basis of $T_p\mathcal{M}$ for any p where they are defined. This basis is called the natural basis associated with the coordinate system $\{x^\mu\}$; it basically provides the set of vector fields that act as the usual partial derivative operation when we study our objects in this coordinate system. In the following, we will usually note such a basis as $\{\partial_\mu\}$, dropping the explicit reference to the dimension of the manifold.

Note that, according to the above convention, we should note $\partial_\mu(f)$ as $\nabla_{\partial_\mu} f$. This is usually abbreviated as $\nabla_\mu f$. Note also that, since the usual partial derivatives $\boldsymbol{\partial}_\mu$ on $\mathbb{R}^n$ commute with each other, we will naturally have that

$$[\partial_\mu, \partial_\nu] \equiv 0, \quad \forall \mu, \nu = 1, \cdots, n \tag{1.67}$$

for any coordinate system.

We can remark that, within this basis, a vector $\vec{v} = v^\mu\ \partial_\mu|_p \in T_p\mathcal{M}$ is the vector tangent to a curve $\mathscr{C}$ at $p = \mathscr{C}(0)$ if and only if we have that $v^\mu = (\boldsymbol{\mathcal{C}}^\mu)'(0)$. This further reinforces the idea that $\vec{v}$ is "the speed of the curve at p".

In fact, more generally, one can write the vector tangent to a curve $\mathscr{C}$ at any given point $\mathscr{C}(\lambda)$ as the unique vector $\vec{v}_\lambda \in T_{\mathscr{C}(\lambda)}\mathcal{M}$ such that for any $f : \mathcal{M} \to \mathbb{R}$ defined in a neighbourhood of the curve we have

$$\vec{v}_\lambda(f) = (f \circ \mathscr{C})'(\lambda). \tag{1.68}$$

This provides a smooth assignation of vectors in terms of λ. Stated differently, this gives a vector field that is defined only on the points lying on $\mathscr{C}$. In this case, using the natural basis of any given coordinate system, we will have that $\vec{v}_\lambda = v^\mu(\lambda)\ \partial_\mu|_{\mathscr{C}(\lambda)}$, where each $v^\mu : \mathbb{R} \to \mathbb{R}$ defines a smooth function of λ such that $v^\mu(\lambda) = (\boldsymbol{\mathcal{C}}^\mu)'(\lambda)$. Since $\boldsymbol{\mathcal{C}}^\mu \coloneqq \mathfrak{X}^\mu \circ \boldsymbol{\mathcal{C}} = x^\mu \circ \mathscr{C}$, using our little misuse of notation, the above relation is usually written as $v^\mu(\lambda) = (x^\mu)'(\lambda)$ or as

$$v^\mu(\lambda) = \dot{x}^\mu(\lambda), \tag{1.69}$$

where the $'$ is replaced by a $^{\bullet}$ to further lighten the notation.

Integral curves of a vector field

This observation also suggests the following : given a vector field $\vec{V} \in \Gamma(T\mathcal{M})$ defined in a neighbourhood of a point $p \in \mathcal{M}$, there is a unique curve through p such that the vector tangent to the curve at any point $\mathscr{C}(\lambda)$ is given by the value of $\vec{V}$ at this point, noted $\vec{V}|_{\mathscr{C}(\lambda)}$ or simply $\vec{V}_{\mathscr{C}(\lambda)}$. This curve is called the integral curve of $\vec{V}$ trough p. This comes from the fact that, once written in any coordinate basis $\{\partial_\mu\}$, the requirement that $\vec{V}|_{\mathscr{C}(\lambda)} = \vec{v}_\lambda$ for all λ, where $\vec{v}_\lambda$ denote the vector tangent to $\mathscr{C}$ at $\mathscr{C}(\lambda)$, reduces to

$$\dot{x}^\mu(\lambda) = V^\mu(x^\alpha(\lambda)). \tag{1.70}$$

This is a system of n first order differential equations defined by the components of $\vec{V}$ in the coordinate basis. Once given a set of initial conditions $x^\mu(0) = x_0^\mu$, this system possesses a unique solution. This solution precisely describes the coordinates $x^\mu(\lambda) \coloneqq (x^\mu \circ \mathscr{C})(\lambda)$ of the points on a curve $\mathscr{C}$ whose tangent vector at $\mathscr{C}(\lambda)$ is $\vec{V}|_{\mathscr{C}(\lambda)}$ and such that $\mathscr{C}(0) = p$ with $p \in \mathcal{M}$ the unique point such that $x^\mu(p) = x_0^\mu$. In the following, we will note this curve, the integral curve of $\vec{V}$ trough p, as $\mathscr{C}_p^{\vec{V}}$.

Dual basis of a coordinate basis

Once given a coordinate system, we can also define the n covector fields $\mathrm{d}x^\mu$ defined as the differential of the functions x^μ defining the coordinate system. By construction, we will have that given a p for which the coordinate system makes sense,

$$\mathrm{d}x^\mu|_p\,(\vec{v}) \coloneqq \vec{v}\,(x^\mu) = v^\mu \tag{1.71}$$

for any $\vec{v} = v^\mu\ \partial_\mu|_p \in T_p\mathcal{M}$. In particular, on their domain of definition,

$$\mathrm{d}x^\mu|_p\,(\partial_\nu|_p) = \delta^\mu{}_\nu, \quad \forall \mu, \nu = 1, \cdots, n, \tag{1.72}$$

which proves that $\{\,\mathrm{d}x^\mu\,\}_{\mu=1,\cdots,n}$ provides, for each point p where it is defined, the dual basis of $\{\,\partial_\mu\,\}_{\mu=1,\cdots,n}$ at p.

This also means that for any smooth function $f : \mathcal{M} \to \mathbb{R}$, we will have that

$$\mathrm{d}f|_p = \boldsymbol{\partial}_\mu \boldsymbol{f}[\Phi(p)]\ \mathrm{d}x^\mu|_p, \tag{1.73}$$

which further motivates to call $\mathrm{d}f$ the differential of f as its components in the basis associated with the coordinate system are those of the differential of its representation in this coordinate system. In the following, we will usually note the dual of a coordinate basis as $\{\,\mathrm{d}x^\mu\,\}$, dropping the explicit reference to the dimension of the manifold.

All of this – and many more actually[1.28] but we will stop here for the sake of conciseness of our discussion – comes automatically as a consequence of the structure of differential manifold. This brings in a natural way the differential calculus known for functions in $\mathbb{R}^n$ to these more general constructs.

1.3.3 Parallel Transport

Interestingly, what does not arise as naturally as one might have expected with the structure of a differential manifold is a way to make sense of the variation of vector fields (or other tensorial fields) between distinct points. This comes from the more fundamental fact that there is no built-in way to compare the vectors in $T_p\mathcal{M}$ with those of $T_q\mathcal{M}$ if p and q are distinct points of $\mathcal{M}$. Each tangent space is strictly defined at one point and one point only. For the physicists we

[1.28] such as all the exterior calculus on the manifold, the exterior derivative, the Lie derivative, . . .

are, this is a problem one needs to solve. Indeed, we already emphasised how important it was, to formulate our physical laws, to be able to make sense of variations of physical quantities and (most of) those physical quantities, as we know, are precisely encoded using tensors.

There are different ways to solve this problem (see remarks 1.6 and 1.7 at the end of this discussion) but probably the most tangible one, if we think of the problem as coming from the fact that we cannot compare vectors "at a distance", is to formalise the idea of "displacing a vector from one point to another without altering it" so that we can compare vectors defined at different points by simply bringing them at the same point. This physically sounded idea is mathematically formalised by the notion of parallel transport.

One way to reformulate our problem in a more mathematical way is to remark that, given any two points $p, q \in \mathcal{M}$, $T_p\mathcal{M}$ and $T_q\mathcal{M}$ will be finite dimensional real vector spaces of the same dimension[1.29]. As a consequence, we know from linear algebra that those two are isomorphic as vector spaces. Yet, this isomorphism is not unique and there is no canonical way to exhibit one here. In other words, one way to solve our problem would be to find a way to "consistently" pick such an isomorphism for any pair of points p and q – and use this dictionary to translate vectors at p in the language of $T_q\mathcal{M}$ so that comparing them make sense – but this will automatically require us to make a choice among many possible ones. The parallel transport is then an additional structure that we will put on top of our differential manifold $\mathcal{M}$.

Also, this might not be completely trivial from an intuitive point of view but it is important to note that, in general, to define how to transport a vector from one point p to another point q it is not enough to just specify those points. One also needs to specify the entire path followed to join p to q. This might be seen as the idea that vectors transported on a manifold are somehow similar to the packages one gets delivered at home. The state of the package at the arrival does not only depend on where it came from but also on how it was handled along the road...

Silly comparisons aside, what a parallel transport operation does is the following : given a curve

$$\mathscr{C} : \begin{cases} \mathbb{R} \to \mathcal{M} \\ \lambda \mapsto P_\lambda \end{cases}$$

that passes through a point $p \in \mathcal{M}$ (without loss of generality, we can assume that the curve is such that $p = P_0$), the parallel transport provides an isomorphism between $T_p\mathcal{M}$ and $T_{P_\lambda}\mathcal{M}$ for any value of $\lambda \in \mathrm{Dom}\,(\mathscr{C}) \subseteq \mathbb{R}$ in a way that is "smooth for both $\mathscr{C}$ and λ". Note that this notion of "smoothness" when one changes the curve is not trivial to define but let us leave that aside for now.

Given a vector $\vec{v} \in T_p\mathcal{M}$, let us denote by $\vec{v}_{||\mathscr{C}}\,(P_\lambda)$ the result of his parallel transport to $T_{P_\lambda}\mathcal{M}$ along the curve $\mathscr{C}$. If we have a vector $\vec{w} \in T_q\mathcal{M}$, we can now make sense of the difference between $\vec{v}$ and $\vec{w}$ by computing $\vec{v}_{||\mathscr{C}}\,(P_\lambda) - \vec{w}$ with

[1.29] $\dim\,(T_p\mathcal{M}) = n = \dim\,(T_q\mathcal{M})$

λ such that $P_\lambda = q$. Let us emphasise that the result will then, by construction, depend on $\mathscr{C}$.

Since the parallel transport gives us vector spaces isomorphisms, in particular, it should be linear, meaning that for all curve $\mathscr{C}$ through p and all values of the descriptive parameter λ, we have

$$\begin{cases} (\alpha \vec{v})_{||\mathscr{C}}(P_\lambda) = \alpha\ \vec{v}_{||\mathscr{C}}(P_\lambda)\,, \\ (\vec{u}+\vec{v})_{||\mathscr{C}}(P_\lambda) = \vec{u}_{||\mathscr{C}}(P_\lambda) + \vec{v}_{||\mathscr{C}}(P_\lambda)\,, \end{cases} \qquad \forall \vec{u}, \vec{v} \in T_p\mathcal{M}, \forall \alpha \in \mathbb{R}. \tag{1.74}$$

Also, all the parallel transport operations being isomorphisms, they must be invertible. The inverse of the parallel transport operation along a curve $\mathscr{C}$ from $p = P_0$ to $q = P_{\lambda_0}$ will be realised by the parallel transport from q to p along the curve $-\mathscr{C}$ that correspond to the same curve as $\mathscr{C}$ but run backward with descriptive parameter $\tilde{\lambda}$ such that $P_{\tilde{\lambda}=0} = q$ and $P_{\tilde{\lambda}=\lambda_0} = p$.[1.30]

With this notion at hand, we can now make sense of the variation of a vector field $\vec{v} \in \Gamma(T\mathcal{M})$ between two points p and q when displaced along a curve $\mathscr{C}$ joining them. One can make the comparison on $T_q\mathcal{M}$ by computing $\vec{v}_q - (\vec{v}_p)_{||\mathscr{C}}(q)$. Alternatively, we can also do the comparison on $T_p\mathcal{M}$ via $(\vec{v}_q)_{||-\mathscr{C}}(p) - \vec{v}_p$.[1.31]

Up to now, the notion of parallel transport that we defined does only apply to vectors. This means, in particular, that at this point, we still have no way to talk about the variation of covectors. Indeed, just as it was the case for vectors, there is no natural way to compare elements of $T_p^*\mathcal{M}$ and $T_q^*\mathcal{M}$ if p and q are distinct points on $\mathcal{M}$. Yet, the situation is very similar to what we just discussed in the sense that for all $p \in \mathcal{M}$, $T_p^*\mathcal{M}$ is a real vector space with $\dim\left(T_p^*\mathcal{M}\right) = n$. This observation motivates the definition of a parallel transport operation for covector that will behave similarly to that for vectors, exhibiting a vector space isomorphism between two cotangent spaces at different points once given a curve joining those points. Given a point $p \in \mathcal{M}$, a curve $\mathscr{C}$ through p and $\underline{\theta} \in T_p^*\mathcal{M}$, similarly to what we did above, let us denote the result of the parallel transport of $\underline{\theta}$ along $\mathscr{C}$ by $\underline{\theta}_{||\mathscr{C}}(P_\lambda)$. It is interesting to note that this parallel transport of covectors could *a priori* be an operation completely independent to what we did for vectors.

That being said, since covectors have a natural action on vectors, one can relate the parallel transport operations for vectors and covectors in a pretty

[1.30] Note that the choice of the notation $-\mathscr{C}$ is motivated by the fact that if $\vec{V}$ represent a vector field tangent to $\mathscr{C}$, *i.e.* if $\mathscr{C} = \mathscr{C}_p^{\vec{V}}$, we will have that $-\mathscr{C} = \mathscr{C}_q^{-\vec{V}}$.

[1.31] The properties of the parallel transport ensure a consistent link between the two notions. Namely, we have that

$$\left[(\vec{v}_q)_{||-\mathscr{C}}(p) - \vec{v}_p\right]_{||\mathscr{C}}(q) = \vec{v}_q - (\vec{v}_p)_{||\mathscr{C}}(q)\,.$$

natural way by requiring that given any point $p \in \mathcal{M}$, curve $\mathscr{C}$, vector $\vec{v} \in T_p\mathcal{M}$ and covector $\underline{\theta} \in T_p^*\mathcal{M}$ one has

$$\left[\underline{\theta}_{||\mathscr{C}}(P_\lambda)\right]\left(\vec{v}_{||\mathscr{C}}(P_\lambda)\right) := \underline{\theta}(\vec{v}) \in \mathbb{R}. \tag{1.75}$$

Condition (1.75) might be seen as a way to extend the notion of parallel transport for vectors to covectors since, parallel transport being linear, this condition completely determines a way to parallel transport covectors once given one for vectors.

Following the same line, we can define a notion of parallel transport for tensors of any rank since the space of tensors of a given variance at a point $p \in \mathcal{M}$ is always a finite dimensional real vector space. Once again it is technically possible to define this operation in an autonomous way for each type of tensor. Nevertheless it is in general much more natural to determine this operation from that known to vectors and covectors by asking that

$$\begin{gathered}\left[\mathcal{T}_{||\mathscr{C}}(P_\lambda)\right]\left(\underline{\theta}^{(1)}{}_{||\mathscr{C}}(P_\lambda), \cdots, \underline{\theta}^{(m)}{}_{||\mathscr{C}}(P_\lambda), \vec{v}^{(1)}_{||\mathscr{C}}(P_\lambda), \cdots, \vec{v}^{(r)}_{||\mathscr{C}}(P_\lambda)\right) := \\ \mathcal{T}\left(\underline{\theta}^{(1)}, \cdots, \underline{\theta}^{(m)}, \vec{v}^{(1)}, \cdots, \vec{v}^{(r)}\right) \in \mathbb{R}\end{gathered} \tag{1.76}$$

for a tensor $\mathcal{T}$ of rank (m, r).

In any case, once equipped with this parallel transport, we can finally make sense of the variation of vector-, covector- and generic tensor- fields between distinct points on the manifold given that we specify a path joining the points.

1.3.4 Covariant Derivative and Affine Connection

In the sense described above, a parallel transport gives us a way to evaluate changes in vector fields via finite differences. If we want to be able to talk about the "rate of change" of a vector field or, to state it differently, to dispose of a notion of directional derivative similar to the one we have for functions, there is still a bit of work to do.

Thanks to the parallel transport, we can now give sense to the covariant derivative of a vector field $\vec{v} \in \Gamma(T\mathcal{M})$ along another vector field $\vec{V} \in \Gamma(T\mathcal{M})$ at a point p. This will be defined as

$$\nabla_{\vec{V}}\vec{v}|_p := \lim_{\varepsilon \to 0} \frac{(\vec{v}_{P_\varepsilon})_{||-\mathscr{C}}(p) - \vec{v}_p}{\varepsilon}, \tag{1.77}$$

where $\mathscr{C} = \mathscr{C}_p^{\vec{V}}$.

The parallel transport being linear according to (1.74), this means that, given a curve $\mathscr{C}$ through $p \in \mathcal{M}$, the action of the parallel transport along $\mathscr{C}$ on any vector of $T_p\mathcal{M}$ is completely determined by its action on a basis of $T_p\mathcal{M}$. From this fact, we can derive very important properties of the covariant derivative (1.77).

Let us imagine that we have a set of smooth vector fields $\{\vec{e}_{(a)}\}_{a=1,\cdots,n}$ which form a basis of $T_q\mathcal{M}$ for any point $q \in \mathcal{M}$ where they are defined. The result of the parallel transport of one of those vector fields at a given point can then always be expressed in terms of the basis they form at that final point

$$\left(\vec{e}_{(a)}|_p\right)_{||\mathscr{C}}(P_\lambda) =: \mathcal{E}_a^b(\mathscr{C};\lambda)\,\vec{e}_{(b)}|_{P_\lambda}. \tag{1.78}$$

Under our assumption that the parallel transport must be "smooth in $\mathscr{C}$ and λ", we will then be allowed to "Taylor expand" $\mathcal{E}_a^b(\mathscr{C};\lambda)$, so to say, leading to

$$\left(\vec{e}_{(a)}|_p\right)_{||\mathscr{C}}(P_\varepsilon) =: \left(\delta_a^b - \varepsilon\,\omega_{ac}^b\,V^c\right)\vec{e}_{(b)}|_{P_\varepsilon} + \mathcal{O}(\varepsilon^2), \tag{1.79}$$

where $\vec{V} = V^a\vec{e}_{(a)}$ is the tangent to $\mathscr{C}$ (evaluated at point p in the above equation) and ω_{ac}^b might be seen as the coefficients of the linear expansion of $\mathcal{E}_a^b(\mathscr{C};\lambda)$, *i.e.* they correspond to coefficients characterising the action of the parallel transport at first order.

Pay attention to the position of the indices in the coefficients ω_{ac}^b in (1.79). It is important since, as we should see, there is, in general, no reason why these coefficients should be symmetric under the exchange of their lower indices. One other reason for the importance of this remark is that different authors may use a different convention (writing ω_{ca}^b where we write ω_{ac}^b and vice versa). One should thus always be careful when comparing formulas from different sources.

Assuming that our definition of the smoothness of the parallel transport is robust enough to ensure that (1.79) is well defined, we will immediately obtain from (1.77)

$$\nabla_{\vec{V}}\vec{e}_{(a)}|_p = \omega_{ac}^b\,V^c\,\vec{e}_{(b)}|_p, \tag{1.80}$$

and, more generally, we get that for an arbitrary smooth vector field $\vec{v} \in \Gamma(T\mathcal{M})$

$$\nabla_{\vec{V}}\vec{v} = V^c\left(\vec{e}_{(c)}\left(v^b\right) + \omega_{ac}^b\,v^a\right)\vec{e}_{(b)}, \tag{1.81}$$

where $\vec{v} = v^a\vec{e}_{(a)}$ and where all the quantities on the right-hand side of the equation are evaluated at the point where we want to compute the left-hand side. The relations (1.79 – 1.81) call for several comments.

1. First, remark that (1.81) establishes two important properties of the covariant derivative. Firstly, the operation is linear with respect to the vector field $\vec{v}$ that we derive <u>and also</u> with respect to the vector field $\vec{V}$ along which the derivative is taken. Secondly, to compute the covariant derivative of $\vec{v}$ along $\vec{V}$ at a point $p \in \mathcal{M}$, one only needs to know $\vec{V}|_p$.[1.32] Thanks to this property, it is possible to consider the derivative of a vector field along a vector (not a vector field !) at a given point, using (1.81) as definition of this operation (or (1.77) directly, using any curve $\mathscr{C}$ for which the tangent vector at $p \in \mathcal{M}$ is $\vec{V} \in T_p\mathcal{M}$).

[1.32] This is not true for $\vec{v}$ since its components in the basis, v^b (which will define functions of the spacetime points), are acted upon the differential operator $\vec{e}_{(c)}|_p$.

2. Thanks to these properties, we see that the covariant derivative actually defines an application.

$$\nabla : \begin{cases} \Gamma\left(T\mathcal{M}\right) \times \Gamma\left(T\mathcal{M}\right) \to \Gamma\left(T\mathcal{M}\right) \\ \left(\vec{v}, \vec{V}\right) \mapsto \nabla(\vec{v}, \vec{V}) \coloneqq \nabla_{\vec{V}} \vec{v} \end{cases} \tag{1.82}$$

that

   (a) is bilinear,

   (b) satisfy

$$\nabla_{(f\vec{V})} \vec{v} = f \nabla_{\vec{V}} \vec{v}$$

   for any $\vec{v}, \vec{V} \in \Gamma\left(T\mathcal{M}\right)$ and $f : \mathcal{M} \to \mathbb{R}$,

   (c) satisfy a Leibnitz rule for its first argument.
   That is, given $\vec{v}, \vec{V} \in \Gamma\left(T\mathcal{M}\right)$ and $f : \mathcal{M} \to \mathbb{R}$,

$$\nabla_{\vec{V}}(f\vec{v}) = f\nabla_{\vec{V}}\vec{v} + \nabla_{\vec{V}}(f)\ \vec{v} = f\nabla_{\vec{V}}\vec{v} + \vec{V}(f)\ \vec{v}.$$

3. It is also important to comment on the coefficients ω^b_{ac}. They are, in a sense, the key ingredient of (1.81). They define what is called a linear connection on $\mathcal{M}$ and are then called connection coefficients.[1.33] We should also mention, to connect with the title of this section, that in general the terms “linear connection” and “affine connection” are used indistinctly. Strictly speaking, these terms refer to different objects, but these are in one-to-one correspondence with each other, justifying the misuse of language.

   Since (1.81) is valid in any basis, we can specify the connection coefficients in any basis. It is then important to emphasise that, given two bases $\{\,\vec{e}_{(a)}\,\}$ and $\{\,\vec{e}'_{(b)}\}$, the corresponding connection coefficients ω^b_{ac} and ω'^b_{ac} will NOT be related to each other by the transformation law of a $(1,2)$-tensor if the change of basis between $\{\,\vec{e}_{(a)}\,\}$ and $\{\,\vec{e}'_{(b)}\}$ is point dependent.

   If the bases are related by the point dependent relation $\vec{e}'_{(a)}|_p = \Lambda_a{}^b(p)\,\vec{e}_{(b)}|_p$ and the corresponding dual basis by $\underline{\theta}'^{(a)}|_p = \Lambda^a{}_b(p)\ \underline{\theta}^{(b)}|_p$, we will have that

$$\omega'^b_{ac} = \Lambda^b{}_k\ \omega^k_{ld}\ \Lambda_a{}^l\ \Lambda_c{}^d - \Lambda_a{}^l\ \Lambda_c{}^d\ \vec{e}_{(d)}\left(\Lambda^b{}_l\right). \tag{1.83}$$

   From a physicist’s point of view, this is, of course, not a surprise since the connection coefficients provide the piece that ensures that

$$V^c v^b_{;c} \coloneqq V^c\left(\vec{e}_{(c)}\left(v^b\right) + \omega^b_{ac}\ v^a\right) \tag{1.84}$$

[1.33] A fully satisfactory definition of a linear connection would require to introduce it on the frame bundle of $\mathcal{M}$ but, in order to circumvent the introduction of this notion, we will deliberately omit it here and only discuss it in terms of the coefficients ω^b_{ac}.

transforms as the components of a vector. The inhomogeneous term in the transformation law of ω^b_{ac} cancels the corresponding inhomogeneous term in the transformation law of $\vec{e}_{(c)}\left(v^b\right)$ that arises from the fact that $\vec{e}_{(c)}$ acts as a differential operator on v^b (see remark 1.5).

4. It is interesting to note that we have carefully expressed (1.81) using a completely arbitrary basis of vector fields $\{\,\vec{e}_{(a)}\,\}_{a=1,\cdots,n}$. In particular, given a coordinate system $\{\,x^\mu\,\}$, one can, of course, use the basis $\{\,\partial_\mu\,\}_{\mu=1,\cdots,n}$ associated with this coordinate system. In this case, the connection coefficients will conventionally be written as $\Gamma^\rho_{\mu\nu}$ and called the Christoffel's symbols and (1.81) will assume the familiar form

$$\nabla_{\vec{V}}\vec{v} = V^\nu\left(\partial_\nu v^\rho + \Gamma^\rho_{\mu\nu}\ v^\mu\right)\partial_\rho. \tag{1.85}$$

In this case, it is usual to abbreviate $\nabla_{\partial_\mu}\vec{v}$ as $\nabla_\mu\vec{v}$ just as we did for derivatives of functions.

Note on the way that, if one wants, it is also possible to mix the basis $\{\,\vec{e}_{(a)}\,\}$ and $\{\,\partial_\mu\,\}$ and to write, for example,

$$\nabla_{\vec{V}}\vec{v} = V^\nu\left(\partial_\nu v^b + \omega^b_{a\nu}\ v^a\right)\vec{e}_{(b)}. \tag{1.86}$$

The form $\omega^b_{a\nu}$ of the connection coefficients is usually referred to as the spin connection coefficients in the physics literature; these are the ones used to express $\nabla_\nu\vec{v}$ in an arbitrary basis $\{\,\vec{e}_{(a)}\,\}$. This is the kind the coefficients used when one has to express the covariant derivative of spinors (hence the name).

Note also that relation (1.83) and the corresponding inverse relation can be used to relate the Christoffel's symbols $\Gamma^\rho_{\mu\nu}$ to the generic connection coefficients ω^b_{ac} or to the spin connection coefficients $\omega^b_{a\mu}$. If we write $\vec{e}_{(a)} = e_a{}^\mu\ \partial_\mu$ and $\underline{\theta}^{(a)} = e^a{}_\mu\ \mathrm{d}x^\mu$, we get

$$\omega^b_{ac} = e^b{}_\rho\ \Gamma^\rho_{\mu\nu}\ e_a{}^\mu\ e_c{}^\nu - e_a{}^\alpha\ e_c{}^\beta\ \partial_\beta e^b{}_\alpha, \tag{1.87}$$

conversely

$$\Gamma^\rho_{\mu\nu} = e_b{}^\rho\ \omega^b_{ac}\ e^a{}_\mu\ e^c{}_\nu + e_b{}^\rho\ \partial_\nu e^b{}_\mu, \tag{1.88}$$

and, since $\omega^b_{a\nu} := \omega^b_{ac}\ e^c{}_\nu$, (1.88) immediately gives $\Gamma^\rho_{\mu\nu}$ in terms of $\omega^b_{a\mu}$ while (1.87) simplify as

$$\omega^b_{a\nu} = e^b{}_\rho\ \Gamma^\rho_{\mu\nu}\ e_a{}^\mu - e_a{}^\alpha\ \partial_\nu e^b{}_\alpha. \tag{1.89}$$

5. Following the way we introduced the objects, the connection coefficients encode local informations on the parallel transport and, according to (1.81), this information entirely specifies the notion of covariant derivative as defined in (1.77). To complete our discussion, it is very important to mention that these three data (linear connection, covariant derivative,

parallel transport) form, in fact, a "Trinity". They all provide equivalent ways to encode the whished additional structure on $\mathcal{M}$.

The relation (1.81) already strongly suggests that giving a set of coefficients ω^b_{ac} with the correct transformation law will univoquely define a covariant derivative and conversely. At the same time, (1.79) indicates that given a parallel transport operation, one can obtain a set of coefficients ω^b_{ac} with the correct transformation law. And, even though this is not trivial to establish, we can go the other way around as well and fully define a notion of parallel transport from the connection coefficients.

In addition, it is also equivalent to specify the covariant derivative for vectors by means of an application $\nabla : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ with the properties described below (1.82). If we take this as an axiomatic definition of the covariant derivative of vectors, we will be able to conclude that, for any given basis $\{\vec{e}_{(a)}\}$, there must exist coefficients ω^b_{ac} such that (1.81) hold. And these coefficients will then have the expected transformation law.

These equivalences are important to emphasise since different authors might prefer one formulation over the others and use seemingly unrelated definitions of the covariant derivative or appear as if they disagree with each other on which concept comes first. All of these would nevertheless encode the same well-defined operation that is, basically, a directional derivative for vector fields.

This is a common situation in the realm of differential geometry to have different objects which appear to encode the same structure or different ways to represent the same object. As another example, a linear connection can be represented in terms of the coefficients $\omega^b_{a\mu}$ but this information can also conveniently be encoded by means of a set of differential forms $\omega^b{}_a$ whose components in the basis $\{\mathrm{d}x^\mu\}$ are precisely given by the coefficients $\omega^b_{a\mu}$. In general, all the different points of view collapse (by construction) when written in components. The distinctions might then, at first, appear as secondary for the purpose of physicists. We nevertheless strongly disagree with such a viewpoint since the different ways to land on a concept will in general reveal different aspects of the notion and trigger complementary (yet distinct) ideas. Also, mastering the different ways to formulate the "same thing" can avoid one to feel lost when reading a text written with "unusual" conventions. That being said, let us come back to our subject.

Remark 1.5. *In a naive attempt, one could have tried to define the variation of a vector field $\vec{v} \in \Gamma(T\mathcal{M})$ in the direction of a vector $\vec{V} \in T_p\mathcal{M}$ by fixing a basis $\{\vec{e}_{(a)}\}$ and then computing the quantity $V^c\vec{e}_{(c)}\left(v^b\right)\vec{e}_{(b)}$. This attempt would correspond to the idea of defining the variation of the vector field in terms of the variation of its components with respect to a fixed basis. This is conceptually unsatisfying for at least two reasons. First, this procedure would imply the specification of a privileged basis in contradiction with the spirit of differential geometry <u>and</u> the physical idea of the non-existence of a preferred reference* *frame to express the laws of physics. Second, provided two such notions of variations are introduced (given two different bases), the link between the two notions would not have been covariant. In this respect, the introduction of the connection coefficients might be seen as a natural way to circumvent these problems. The introduction of a set of connection coefficients ω^b_{ac} does require to fix a basis first but this is done in such a way that the notion of covariant derivative itself does not, even if we were to use* (1.81) *as its definition.*

Here again, as for the parallel transport, we have developed a notion of covariant derivative that is *a priori* only applicable to vectors up to now. Obviously, as soon as we have a notion of parallel transport defined for covectors, the above procedure can be applied identically, leading to a notion of covariant derivative of a covector field $\underline{\sigma} \in \Gamma\left(T^*\mathcal{M}\right)$ along a vector field $\vec{V} \in \Gamma\left(T\mathcal{M}\right)$, noted $\nabla_{\vec{V}}\underline{\sigma}$. The definition will be the same as (1.77) but with the replacement $\vec{v} \to \underline{\sigma}$.

As above, if we give ourselves a set of smooth covector fields $\{\,\underline{\theta}^{(a)}\,\}_{a=1,\cdots,n}$ which form a basis of $T^*_p\mathcal{M}$ for any $p \in \mathcal{M}$ on which they are defined, we can obtain an expression for the covariant derivative of a covector field similarly to what we just did. If we assume, for simplicity, that $\{\,\underline{\theta}^{(a)}\,\}_{a=1,\cdots,n}$ is the dual basis of a basis $\{\,\vec{e}_{(a)}\,\}_{a=1,\cdots,n}$ and that we express $\vec{V}$ in that basis, we obain

$$\nabla_{\vec{V}}\underline{\sigma} = V^c\left(\vec{e}_{(c)}\left(\sigma_a\right) + \kappa^b_{ac}\;\sigma_b\right)\underline{\theta}^{(a)}, \tag{1.90}$$

where κ^b_{ac} are the coefficients characterising the parallel transport of covectors at first order, similarly to what we did in (1.79). If we impose condition (1.75) to hold, using $\underline{\theta}^{(a)}\left(\vec{e}_{(b)}\right) = \delta^a{}_b$, we will get that

$$\kappa^b_{ac} = -\omega^b_{ac} \tag{1.91}$$

and (1.90) will assume the usual form

$$\nabla_{\vec{V}}\underline{\sigma} = V^c\left(\vec{e}_{(c)}\left(\sigma_a\right) - \omega^b_{ac}\;\sigma_b\right)\underline{\theta}^{(a)}. \tag{1.92}$$

Once again, one can look at (1.92) in coordinate basis $\{\,\partial_\mu\,\}$ and $\{\,\mathrm{d}x^\mu\,\}$ and get the familiar form

$$\nabla_{\vec{V}}\underline{\sigma} = V^\nu\left(\partial_\nu\sigma_\mu - \Gamma^\rho_{\mu\nu}\;\sigma_\rho\right)\mathrm{d}x^\mu, \tag{1.93}$$

or mix the basis and get

$$\nabla_{\vec{V}}\underline{\sigma} = V^\mu\left(\partial_\mu\sigma_a - \omega^b_{a\mu}\;\sigma_b\right)\underline{\theta}^{(a)}. \tag{1.94}$$

As for the covariant derivative defined for vector fields, we see that the covariant derivative is linear in $\vec{V}$ and that $\nabla_{\vec{V}}\underline{\sigma}|_p$ only depends on $\vec{V}|_p$. We can also establish easily that the covariant derivative of covectors satisfies the same properties that those displayed below (1.82) provided we use the replacement $\vec{v} \to \underline{\sigma}$ in the formulas. Also, here again, one will introduce the notation $\nabla_\mu\underline{\sigma}$ as a shortcut for $\nabla_{\partial_\mu}\underline{\sigma}$.

Now that we are equipped with a notion of covariant derivative for both vector fields and covector fields, we can easily repeat the procedure for tensor fields of any ranks and make sense of $\nabla_{\vec{V}}\mathcal{T}$ for generic tensor fields via the same definition as (1.77) but where $\vec{v} \to \mathcal{T}$. If we assume (1.76) to hold, the parallel transport of tensors will result from those of vectors and covectors and we will find that

$$\begin{aligned} \nabla_{\vec{V}}\mathcal{T} = V^c\Big(&\vec{e}_{(c)}\left(\mathcal{T}^{a_1\cdots a_m}{}_{b_1\cdots b_r}\right) \\ &+ \omega^{a_1}_{ac}\ \mathcal{T}^{aa_2\cdots a_m}{}_{b_1\cdots b_r} + \cdots + \omega^{a_m}_{ac}\ \mathcal{T}^{a_1\cdots a_{m-1}a}{}_{b_1\cdots b_r} \\ &- \omega^{b}_{b_1 c}\ \mathcal{T}^{a_1\cdots a_m}{}_{bb_2\cdots b_r} - \cdots - \omega^{b}_{b_r c}\ \mathcal{T}^{a_1\cdots a_m}{}_{b_1\cdots b_{r-1}b}\Big) \\ &\vec{e}_{(a_1)} \otimes \cdots \otimes \vec{e}_{(a_m)} \otimes \underline{\theta}^{(b_1)} \otimes \cdots \otimes \underline{\theta}^{(b_r)} \quad , \end{aligned} \tag{1.95}$$

for a tensor field of type (m, r). Of course, one can obtain the equivalent of (1.95) in terms of $\Gamma^{\rho}_{\mu\nu}$ if one express $\nabla_{\vec{V}}\mathcal{T}$ in a natural basis or in terms of the spin connection coefficients $\omega^{b}_{a\mu}$ if one simply expresses $\vec{V}$ in a natural basis in (1.95).

Again, from (1.95) we see that $\nabla_{\vec{V}}\mathcal{T}$ is linear in $\vec{V}$, depend only on $\vec{V}|_p$ when evaluated at $p \in \mathcal{M}$ and satisfy the properties expressed below (1.82) if we replace $\vec{v} \to \mathcal{T}$ in the formulas. As always, $\nabla_\mu\mathcal{T}$ will be used as a shortcut for $\nabla_{\partial_\mu}\mathcal{T}$.

We then obtain an operator ∇ that takes a vector field as first argument, a tensor field as second argument, that can be applied to any type of tensor field and that satisfies the following properties

(a) If $\mathcal{T}$ is a tensor field of type (m, r), so is $\nabla\left(\vec{V}, \mathcal{T}\right)$, for any $\vec{V} \in \Gamma(T\mathcal{M})$. For simplicity of the notation, we can abbreviate $\nabla\left(\vec{V}, \mathcal{T}\right)$ by $\nabla_{\vec{V}}\mathcal{T}$ when needed.

(b) When given a $(0, 0)$ tensor field (*i.e.* a scalar function $f : \mathcal{M} \to \mathbb{R}$) as second argument, ∇ coincide with the usual directional derivative, *i.e.*

$$\nabla\left(\vec{V}, f\right) = \vec{V}(f).$$

(c) ∇ is bilinear.

(d) For all $f : \mathcal{M} \to \mathbb{R}$, all $\vec{V} \in \Gamma(T\mathcal{M})$ and all tensor field $\mathcal{T}$ (no matter his rank),

$$\nabla_{f\vec{V}}\mathcal{T} = f\nabla_{\vec{V}}\mathcal{T}.$$

(e) ∇ satisfy a Leibnitz rule for its second argument.

That is, for all $f : \mathcal{M} \to \mathbb{R}$, all $\vec{V} \in \Gamma(T\mathcal{M})$ and all tensor field $\mathcal{T}$ (no matter his rank),

$$\nabla_{\vec{V}}(f\mathcal{T}) = f\nabla_{\vec{V}}\mathcal{T} + \nabla_{\vec{V}} f\ \mathcal{T} = f\nabla_{\vec{V}}\mathcal{T} + \vec{V}(f)\ \mathcal{T}.$$

(f) ∇ commute with every contraction applied to its second argument $\mathcal{T}$.

(g) ∇ satisfy a Leibnitz rule over the tensor product.

If $\mathcal{T}$ and $\mathcal{S}$ are tensor fields (possibly of different ranks), we have

$$\nabla_{\vec{V}}(\mathcal{T} \otimes \mathcal{S}) = \nabla_{\vec{V}}(\mathcal{T}) \otimes \mathcal{S} + \mathcal{T} \otimes \nabla_{\vec{V}}(\mathcal{S}),$$

for all $\vec{V} \in \Gamma(T\mathcal{M})$.

All these properties follow almost immediately from (1.95), except maybe (b) which is merely a consistency condition with the notion of directional derivative already defined for functions (independently of any covariant derivative operation) so that we can really see all our covariant derivatives as coming from ∇. Interestingly, similarly to what we already discussed, we could have worked the other way around and defined a covariant derivative axiomatically as an operation ∇ satisfying (a)–(g); in this way, we would immediately have a notion of covariant derivative that can be applied to any vector-, covector- and tensor-fields. If we do so, properties (a)–(e) will ensure, for any given basis $\{\vec{e}_{(a)}\}$, the existence of coefficients ω^b_{ac} such that (1.81) hold for vector fields and also the existence of coefficients κ^b_{ac} such that (1.90) hold for covector fields. Finally, properties (f) and (g) will give us (1.91) and (1.95). Once again, we would then obtain that the operation ∇ is entirely determined (for any tensor field) once given the connection (via the coefficients ω^b_{ac}).

We should also remark that the covariant derivative of a (m, r)-tensor field $\mathcal{T}$, as expressed by (1.95), always allows defining a $(m, r+1)$-tensor field $\nabla\mathcal{T}$ whose components $(\nabla\mathcal{T})^{a_1\cdots a_m}{}_{b_1\cdots b_r c}$ are defined by identification via the relation

$$\nabla_{\vec{V}}\mathcal{T} =: V^c\,(\nabla\mathcal{T})^{a_1\cdots a_m}{}_{b_1\cdots b_r c}\,\vec{e}_{(a_1)} \otimes \cdots \otimes \vec{e}_{(a_m)} \otimes \underline{\theta}^{(b_1)} \otimes \cdots \otimes \underline{\theta}^{(b_r)}. \quad (1.96)$$

This procedure then defines an operator, usually also denoted ∇, that maps (m, r)-tensor fields to $(m, r+1)$-tensor fields by sending $\mathcal{T}$ to $\nabla(\mathcal{T}) \coloneqq \nabla\mathcal{T}$. This operator is known as the covariant differential. To state it differently, the covariant differential is defined as the operator that maps (m, r)-tensor fields to $(m, r+1)$-tensor fields such that the contraction of the last index of the obtained $\nabla\mathcal{T}$ with a vector field $\vec{V}$ produces $\nabla_{\vec{V}}\mathcal{T}$. Loosely speaking, the covariant differential can be seen as a covariant derivative whose direction has not yet been specified. Once again, this new tool can give raise to an equivalent way of defining a covariant derivative. Similarly to the previous formulations, this operation fully specifies (and is fully specified by the choice of) a connection. Following the common use, we will take the habit to use the abbreviation.

$$(\nabla\mathcal{T})^{a_1\cdots a_m}{}_{b_1\cdots b_r c} =: \nabla_c \mathcal{T}^{a_1\cdots a_m}{}_{b_1\cdots b_r}.$$

Before closing this section, we should comment on an important (yet deliberate) flaw of the above presentation. Here, we have chosen to introduce the notions in an intuitive way. We started from a semi-formal definition of the parallel transport, emphasising how this "physically" solved the problem of defining the variation of vector fields and how this naturally led to the covariant derivative as an extension of the concept of directional derivative to vector fields. Using this material we saw how, in any local basis, the covariant derivative was entirely specified by means of the coefficients ω^b_{ac}. In a more axiomatic approach, people should usually go the other way around. One will start by a proper definition of a linear connection. This notion of linear connection will then allow us to define the notion of parallel transport; in particular, this formalism will allow us to define in a simple way the "smoothness" of the parallel transport that we deliberately kept fuzzy here. With these proper definitions at hand, one will then define the covariant derivative using (1.77). For a detailed presentation of this construction, see [Kobayashi and Nomizu, 1996].

Nevertheless, this will not change the key fact that needs to be emphasised here : these data will all encode different aspects of the same machinery. One will still have that a linear connection will induce connection coefficients ω^b_{ac} with the expected transformation law and that, on the contrary, a set of coefficients ω^b_{ac} with the correct transformation law defines a unique linear connection. One will also have that, given a $\nabla : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ as in (1.82), there is a unique linear connection for which this defines the associated covariant derivative. And so on and so forth. See [Kobayashi and Nomizu, 1996] (vol. 1, chap. 3, §7) for detailed proofs of these equivalences. This will then constitute a firmer basis to mathematically formulate the concepts[1.34] but will not alter the intuitive content nor the validity of any of the properties stated in the above, say heuristic, discussion.

In the following, we will assume that we have endowed our manifold with a covariant derivative which can be applied to any kind tensor field and refer to this structure in terms of the corresponding linear connection (that we will describe via its coefficients ω^c_{ab}) with your favourite viewpoint on what should be the definition in the above discussion in mind.

Remark 1.6. *Once given a differential manifold, one can always define totally antisymmetric tensor fields of type $(0, r)$, for any $r \leq n$.*[1.35] *These are called differential forms of degree r or r-forms. According to this definition, covector fields are 1-forms and, perhaps abusing a bit the notation, functions can be considered as 0-forms. For a given $r \leq n$, the set of r-forms is a vector space (actually a subspace of the vector space of tensor fields of type $(0, r)$). One can*

[1.34] At least, it is the case for the concept of linear connection.
[1.35] Due to the total antisymmetry, the cases $r \geq n$ are all trivial.

also consider all differential forms (independently of their degree) "at once" by constructing the direct sum of the vector spaces of r-forms for all $r \leq n$.

The study of differential forms is a fruitful branch of differential geometry[1.36] *and, once again, can be done intrinsically on a manifold without need of any extra structure. For the seek of this remark, we want to recall two key operations that can be defined on differential forms.*

First, given an r-form $\underline{\sigma}$ and an s-form $\underline{\theta}$ (for $r, s \leq n$), one can always construct a $(r+s)$-form by means of the so-called wedge product, denoted $\wedge$. The idea is that the $(r+s)$-form $\underline{\sigma} \wedge \underline{\theta}$ will be defined from the antisymmetric part of the tensor product $\underline{\sigma} \otimes \underline{\theta}$ (with a normalisation factor). Of course, for this to define a non-trivial object, one should have $r+s \leq n$.

Second, one can define a notion of derivation for differential forms. This is the exterior derivative d. *In a nutshell, the exterior derivative will increase the degree of a differential form by one. If $\underline{\sigma}$ is an r-form, $\mathrm{d}\underline{\sigma}$ is a $(r+1)-form$. One way to define the action of* d *is to say that if, in a given coordinate system $\{ x^\mu \}$, the components of the r-form $\underline{\sigma}$ in the natural basis associated to this coordinate system are $\sigma_{\mu_1 \cdots \mu_r}$, the components of $\mathrm{d}\underline{\sigma}$ are $(\mathrm{d}\sigma)_{\mu_1 \cdots \mu_r \mu} := \partial_{[\mu} \sigma_{\mu_1 \cdots \mu_r]}$. Of course, one should check that this definition is independent of the coordinate system chosen to write the previous equation.*

From this definition, we see that, taking functions as 0*-forms,* $\mathrm{d}f$ *is really the differential of f as previously defined;* i.e. $\mathrm{d}f|_p(\vec{v}) = \nabla_{\vec{v}} f$ *for all $\vec{v} \in T_p\mathcal{M}$. From this, one can also get that, given an r-form $\underline{\sigma}$ and an s-form $\underline{\theta}$, we have* $\mathrm{d}\left(\underline{\sigma} \wedge \underline{\theta}\right) = \mathrm{d}\underline{\sigma} \wedge \underline{\theta} + (-1)^r \underline{\sigma} \wedge \mathrm{d}\underline{\theta}$ *and that* $\mathrm{d}\left(\mathrm{d}\underline{\sigma}\right) \equiv 0$ *for any differential form. In fact, the exterior derivative can also be defined axiomatically in terms of these 3 properties.*

In any case, the exterior derivative provides a notion of derivation for differential forms independently of the presence or not of a connection. This is a canonical operation. That being said, it should be clear that this does not yet solve the problem raised at the beginning of this section to find a meaningful notion of derivation for any tensor fields as it only applies to differential forms and not to arbitrary tensor fields.

Remark 1.7. *It is possible to define a notion of derivation for tensor fields of any rank (m, r) using only the native structure of a differential manifold. More precisely, given a tensor field $\mathcal{T}$ of type (m, r) and a vector field $\vec{V} \in \Gamma(T\mathcal{M})$, we can define a notion of variation of $\mathcal{T}$ along the flow associated to the integral curves of $\vec{V}$. This is the Lie derivative of $\mathcal{T}$ along $\vec{V}$, noted $\mathcal{L}_{\vec{V}}\mathcal{T}$.*

Without providing a complete definition here (since we will make little explicit use of this notion in the following), the idea is to use the fact that to any vector field $\vec{V}$ one can always locally assign a flow thanks to the solutions of the system (1.70)*. This flow can be used to define a way (somehow similar to, but distinct from, the parallel transport) to transport the value of the tensor field $\mathcal{T}$ from a point to another along this flow. This then allows comparing the value of the tensor field $\mathcal{T}|_p$ at one point $p \in \mathcal{M}$ to its value $\mathcal{T}|_q$ at another point*

[1.36] Among other things, it allows to define integration on a manifold and to obtain a generalisation of Stokes theorem.

$q \in \mathcal{M}$ (sufficiently close from p) by transporting $\mathcal{T}|_q$ at p via the flow. Again, similarly to the covariant derivative, one can then define the Lie derivative from an appropriate limit.

The Lie derivative will have some properties similar to the covariant derivative. As previously, these properties can be used as a way to define the Lie derivative axiomatically. We will spare ourselves the complete list of these properties here but let us merely mention that, seeing functions as $(0,0)$-tensor fields, one can define the Lie derivative of a function and that this will coincide with the usual derivative defined for functions; i.e. *for any $\vec{V} \in \Gamma(T\mathcal{M})$ and any smooth $f : \mathcal{M} \to \mathbb{R}$, we have that $\mathcal{L}_{\vec{V}} f = \nabla_{\vec{V}} f = \vec{V}(f)$.*

The Lie derivative then gives a way to measure (and/or to define) the variation of a tensor field $\mathcal{T}$ along a vector field $\vec{V}$ by means of the measure of its variation along the flow associated to $\vec{V}$. As outlined here, this procedure is, in a sense, similar to the covariant derivative (especially if we present it as coming from the parallel transport operation) but we should highlight two key differences :

First, as we said, the Lie derivative is defined only thanks to the differential structure of the manifold. Unlike the covariant derivative, no extra structure is required.

Second, unlike the covariant derivative, the Lie derivative can only be defined along vector fields. Indeed, the construction sketched here in terms of the flow associated to a vector field does not make sense for a vector defined only at one point.

To conclude this remark, it is interesting to further compare the three notions of derivative that we have one a differential manifold : the Lie derivative $\mathcal{L}$, the exterior derivative d *(see remark 1.6) and the covariant derivative ∇. One very interesting point is that these notions give three natural ways to generalise to (some types of) tensor fields the natural notion of derivation known to functions on $\mathbb{R}^n$ that already generalises naturally to functions on $\mathcal{M}$. Indeed, all three notions can be applied to functions $f : \mathcal{M} \to \mathbb{R}$ and, in this case, their actions coincide with the usual notion of directional derivative. Nevertheless, in general, their action will differ on more general tensor fields.*

It is also interesting to see that, out of the three notions, two can be defined as soon as one have a differential manifold while the third one requires an extra structure. This property is also, in a sense, part of the interest of the covariant derivative ∇. Since its definition requires to make a choice, it is conceivable that this quantity can – loosely speaking – encode some degree of freedom in the sense that it is possible to first define a differential manifold and then to constrain the connection (and/or the covariant derivative) by means of some conditions.

In the following, we will continue to mostly consider the properties of the covariant derivative. As we should see, these are the keys to encode the gravitational interaction in the structure of our spacetime.

1.3.5 Parallel Transported Tensor Fields and Geodesics

Parallel transported tensor fields

We saw that the notion of covariant derivative allows us to make sense of a notion of directional derivative for tensor fields. We also saw that this notion is closely related to the notion of parallel transport.

Using these ideas, we can express the idea of a tensor field being unchanged when transported along a curve in terms of the covariant derivative. Given a vector field $\vec{V} \in \Gamma(T\mathcal{M})$ and a tensor field $\mathcal{T}$ we will say that $\mathcal{T}$ is parallel transported along $\vec{V}$ if

$$\nabla_{\vec{V}} \mathcal{T} = 0. \tag{1.97}$$

In other words, we require that the derivative of $\mathcal{T}$ in the direction of $\vec{V}$ vanishes at any point where this is defined.

This notion, as one can expect, is closely related to the intuitive notion of a tensor field being unchanged in terms of the parallel transport operation. If we have a parallel transport operation, once given a curve $\mathscr{C}$ and a tensor field $\mathcal{T}$ that is defined at least for any points of $\mathscr{C}$, the idea of $\mathcal{T}$ being unchanged when moved along $\mathscr{C}$ would be that

$$\forall \lambda,\ \left(\mathcal{T}|_{P_0}\right)_{||\mathscr{C}}(P_\lambda) = \mathcal{T}|_{P_\lambda}. \tag{1.98}$$

In a sense, (1.97) is the infinitesimal version of this requirement. Clearly, (1.97) will automatically hold if we have a vector field $\vec{V}$ for which (1.98) is satisfied on $\mathscr{C}_p^{\vec{V}}$.

If we state it carefully enough, the converse is also true. The link between (1.97) and (1.98) is similar to the one relating a vector field to its integral curves. First of all, we need to formalise the idea of derivation along a curve for tensor fields that would only be defined along the points of this curve. Once given a linear connection, we can always do this by taking inspiration of (1.95). Given a curve $\mathscr{C}$ and a vector field $\vec{V}$ and (m, r)-tensor field $\mathcal{T}$ that are both *a priori* defined only along $\mathscr{C}$ – we should emphasise this by writing them as $\vec{V}_\lambda$ and $\mathcal{T}_\lambda$ – we define

$$\begin{aligned}
\nabla_{\vec{V}_\lambda} \mathcal{T}_\lambda \coloneqq \Big[& \dot{\mathcal{T}}^{a_1 \cdots a_m}{}_{b_1 \cdots b_r}(\lambda) + V^c(\lambda) \Big(\omega^{a_1}_{ac}\ \mathcal{T}^{a a_2 \cdots a_m}{}_{b_1 \cdots b_r}(\lambda) \\
& + \cdots + \omega^{a_m}_{ac}\ \mathcal{T}^{a_1 \cdots a_{m-1} a}{}_{b_1 \cdots b_r}(\lambda) \\
& - \omega^{b}_{b_1 c}\ \mathcal{T}^{a_1 \cdots a_m}{}_{b b_2 \cdots b_r}(\lambda) - \cdots - \omega^{b}_{b_r c}\ \mathcal{T}^{a_1 \cdots a_m}{}_{b_1 \cdots b_{r-1} b}(\lambda) \Big) \Big] \\
& \vec{e}_{(a_1)} \otimes \cdots \otimes \vec{e}_{(a_m)} \otimes \underline{\theta}^{(b_1)} \otimes \cdots \otimes \underline{\theta}^{(b_r)} \quad ,
\end{aligned} \tag{1.99}$$

where the basis vectors and connection coefficients are all evaluated at point $\mathscr{C}(\lambda)$. (1.99) then consistently gives another (m, r)-tensor field defined only along $\mathscr{C}$; independently of the basis chosen to write down the definition. This definition is based on the linear connection and clearly inspired by (1.95) in the sense that for a vector field $\vec{V} \in \Gamma(T\mathcal{M})$ and a (m, r)-tensor field $\mathcal{T}$ both defined in a neighbourhood of a curve $\mathscr{C}$, (1.99) will correspond to $\nabla_{\vec{V}}\mathcal{T}|_{\mathscr{C}(\lambda)}$ if we set down $\vec{V}_\lambda := \vec{V}|_{\mathscr{C}(\lambda)}$ and $\mathcal{T}_\lambda := \mathcal{T}|_{\mathscr{C}(\lambda)}$. In fact, if a parallel transport operation is defined, we could obtain (1.99) using (1.77) (where $\vec{v} \to \mathcal{T}_\lambda$, $\vec{V} \to \vec{V}_\lambda$ and $\vec{V}_\lambda$ denote the tangent to the curve $\mathscr{C}$) since the limit on the right-hand side only require to evaluate the objects along points of a given curve.

Now, on account of (1.99), given a curve $\mathscr{C}$ whose tangent vector at $\mathscr{C}(\lambda)$ is written as $\vec{v}_\lambda$, we would say that a tensor field $\mathcal{T}_\lambda$ defined on the curve is constant if

$$\forall\lambda,\ \nabla_{\vec{v}_\lambda}\mathcal{T}_\lambda = 0. \tag{1.100}$$

According to (1.99), once given a basis of vector fields defined in a neighbourhood of the curve, this condition will give rise to a set of first order differential equations on the components of $\mathcal{T}_\lambda$ defined by the connection coefficients and the components of $\vec{v}_\lambda$. If we give an initial condition by fixing a tensor at point $\mathscr{C}(0)$, this system of equations will possess a unique solution that will define along $\mathscr{C}$ a tensor field constant in the above sense.

We could then say that a tensor field $\mathcal{T}$ defined in a neighbourhood of a curve $\mathscr{C}$ is constant along the curve if the solution $\mathcal{T}_\lambda$ to the system arising from (1.100) with initial condition $\mathcal{T}_0 = \mathcal{T}|_{\mathscr{C}(0)}$ is such that $\forall\lambda,\ \mathcal{T}_\lambda = \mathcal{T}|_{\mathscr{C}(\lambda)}$. This last condition corresponds to (1.98) in the sense that the procedure outlined here to construct $\mathcal{T}_\lambda$ corresponds to the definition of $(\mathcal{T}|_{P_0})_{||\mathscr{C}}(P_\lambda)$ that one could give when the linear connection (or the covariant derivative) is given beforehand and the parallel transport operation derived from it.

Geodesics

Following this discussion, we can define a notion of "straight line". In the context of differential geometry, such a curve is called a geodesic. Intuitively, a geodesic is really a curve that is "as straight as possible" in the sense that its tangent vector is constant along the curve. More precisely, a curve $\mathscr{C} : \mathbb{R} \to \mathcal{M}$ whose tangent vector at point $\mathscr{C}(\lambda)$ is $\vec{v}_\lambda$ is a geodesic if

$$\forall\lambda,\ \nabla_{\vec{v}_\lambda}\vec{v}_\lambda = 0. \tag{1.101}$$

According to (1.99), once given a basis of vector fields $\{\,\vec{e}_{(a)}\,\}$, a curve will be a geodesic if the components of its tangent vector $v^a(\lambda)$ satisfies.

$$\dot{v}^a(\lambda) + \omega^a_{bc}\big(\mathscr{C}(\lambda)\big)\ v^b(\lambda)v^c(\lambda) = 0,\ \forall\lambda,\ \forall a = 1,\cdots,n. \tag{1.102}$$

If we introduce a coordinate system $\{\,x^\mu\,\}$ we can also write that a curve $\mathscr{C}$ whose points have coordinates $x^\mu(\lambda)$ is a geodesic if and only if

$$\ddot{x}^\mu(\lambda) + \Gamma^\mu_{\nu\rho}\big(x^\alpha(\lambda)\big)\ \dot{x}^\nu(\lambda)\dot{x}^\rho(\lambda) = 0,\ \forall\lambda,\ \forall\mu = 1,\cdots,n\ , \tag{1.103}$$

where we have used (1.69). We then arrive at a system of second order differential equations defined by the Christoffel symbols of the chosen connection.

From (1.103), using the usual results from the theory of differential equations, we see that given a point $p \in \mathcal{M}$ and a vector $\vec{v} \in T_p\mathcal{M}$, in a neighbourhood of p there is always a unique geodesic $\mathscr{C}$ such that $\mathscr{C}(0) = p$ and $\vec{v}_0 = \vec{v}$.

Note that this is again coherent with the notion of covariant derivative defined above in the sense that the integral curves of a vector field $\vec{V} \in \Gamma(T\mathcal{M})$ are geodesics if and only if $\nabla_{\vec{V}}\vec{V} = 0$.

This discussion also stresses that, on a differential manifold, if the notion of the speed along a curve can always be defined, we need a connection to define the acceleration via the right-hand side of (1.101). In this sense, a geodesic is really a curve for which the acceleration vanishes.

1.3.6 Curvature and Torsion

Once given a differential manifold $\mathcal{M}$ endowed with a linear connection ω^b_{ac} and the corresponding covariant derivative ∇, we finally have a way to talk about the rate of change of tensor fields (including functions, vector fields and covector fields). We are thus very close to have the perfect mathematical machinery to capture the notion of a (classical) spacetime. The only thing we miss yet is a notion of causality. But before introducing this notion, *i.e.* before finally bringing a metric into the picture, we shall comment on some important consequences of the introduction of a linear connection. This new structure added on top of our manifold plays a central role in our construction, as outlined in the previous paragraph, and, despite its coefficients ω^b_{ac} not defining the components of a tensor field, there are two extremely important tensor fields associated to it. These two tensor fields are what we want to discuss now.

One way to introduce these objects (again, among several others) is to focus on two important properties of the covariant derivative of vector fields.

Curvature

First of all, it is important to remark that, unlike the derivative operators ∂_μ associated with coordinate systems, covariant derivatives usually do not commute with each other. To see this, one can compute $\nabla_{\vec{V}}\nabla_{\vec{W}}\vec{v} - \nabla_{\vec{W}}\nabla_{\vec{V}}\vec{v}$ for $\vec{v}, \vec{V}, \vec{W} \in \Gamma(T\mathcal{M})$.

If we fix a basis of vector fields $\{\vec{e}_{(a)}\}$ and use (1.81) to perform the calculation, we will find that

$$\nabla_{\vec{V}}\nabla_{\vec{W}}\vec{v} - \nabla_{\vec{W}}\nabla_{\vec{V}}\vec{v} = \nabla_{[\vec{V},\vec{W}]}\vec{v} + v^d V^a W^b R^c{}_{dab}\vec{e}_{(c)}, \tag{1.104}$$

where

$$R^c{}_{dab} \coloneqq \vec{e}_{(a)}\left(\omega^c_{db}\right) - \vec{e}_{(b)}\left(\omega^c_{da}\right) + \omega^c_{ka}\omega^k_{db} - \omega^c_{kb}\omega^k_{da} - \left[\vec{e}_{(a)}, \vec{e}_{(b)}\right]^l \omega^c_{dl}. \tag{1.105}$$

Since the left-hand side and the first term of the right-hand side of (1.104) are manifestly vector fields, so must be the last term of the right-hand side. Consequently, due to the form of this term, the coefficients $R^c{}_{dab}$ must define the components of a $(1,3)$-tensor field. This tensor field will be denoted $\mathbf{R}$ and called the curvature tensor of the connection. By construction, it satisfies $R^c{}_{dab} = -R^c{}_{dba}$.

From this definition, we see that the curvature encodes the obstruction for the commutator[1.37] of covariant derivatives $\left[\nabla_{\vec{V}}, \nabla_{\vec{W}}\right] = \nabla_{\vec{V}}\nabla_{\vec{W}} - \nabla_{\vec{W}}\nabla_{\vec{V}}$ to be equal to the covariant derivative along the commutator $\nabla_{[\vec{V},\vec{W}]}$. Geometrically, this tensor field encodes the obstruction for the parallel transport to be independent of the curve used. This is usually seen by considering the changes appearing on a vector when it is parallel transported around an infinitesimal closed curve, back to the point where it started. The interested reader can refer to [Penrose, 2005] for more details on this point. In this sense, the curvature encodes local informations on how much our parallel transport differs from the usual one known on $\mathbb{R}^n$. We should also mention in passing that the curvature is also responsible from the fact that nearby geodesics tend to deviate from each other – this is encoded in the so-called geodesic deviation equation, see [Carroll, 1997, Wald, 1984] for more details.

As always, we can play with the components of $\mathbf{R}$ in different types of basis. If we choose to express $\vec{V}$ and $\vec{W}$ in a natural basis $\{\,\partial_\mu\,\}$ in (1.104), to connect the result of our computation to the computation of $\nabla_\mu\nabla_\nu\vec{v} - \nabla_\nu\nabla_\mu\vec{v}$, we will obtain a mixed indices version of $\mathbf{R}$

$$R^c{}_{d\mu\nu} = \partial_\mu\omega^c_{d\nu} - \partial_\nu\omega^c_{d\mu} + \omega^c_{k\mu}\omega^k_{d\nu} - \omega^c_{k\nu}\omega^k_{d\mu}, \tag{1.106}$$

where we have used (1.67) to kill the last term of (1.105). Also, if we were to use only a natural basis, we would recover the usual formula

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\sigma\nu} - \partial_\nu\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\sigma\nu} - \Gamma^\rho_{\beta\nu}\Gamma^\beta_{\sigma\mu}. \tag{1.107}$$

Note that, if we think of the spin connection coefficients $\omega^b_{a\mu}$ as the components of the connection forms $\omega^b{}_a$ that we mentioned before, (1.106) suggests an alternative definition of the curvature in terms of a $(1,1)$-tensor valued 2-form[1.38], also denoted $\mathbf{R}$. Indeed, using the tools of the exterior calculus on a manifold, we can write (1.106) as

$$\mathbf{R}^c{}_d \coloneqq \mathrm{d}\omega^c{}_d + \omega^c{}_k \wedge \omega^k{}_d, \tag{1.108}$$

where "d" and "$\wedge$" respectively denote the exterior derivative and wedge product of differential forms (see remark 1.6) and where each of the $\mathbf{R}^c{}_d$ is a 2-form such that the law of transformation of this bunch of 2-forms is that of the components of a $(1,1)$-tensor. This description of the curvature as a $(1,1)$-tensor valued 2-form can also be captured by writing (1.104) as

$$\mathbf{R}\left(\vec{V},\vec{W}\right)\vec{v} \coloneqq \nabla_{\vec{V}}\nabla_{\vec{W}}\vec{v} - \nabla_{\vec{W}}\nabla_{\vec{V}}\vec{v} - \nabla_{[\vec{V},\vec{W}]}\vec{v}, \tag{1.109}$$

[1.37] Here, obviously, the word "commutator" stands for the commutator of two operators and should not be mistaken with the commutator of vector fields.

[1.38] Remember that a 2-form is a totaly antisymmetric $(0,2)$-tensor field, see remark 1.6.

where $\mathbf{R}$ is understood as this tensor valued 2-form. Note that, when written using a coordinate basis, (1.108) will give $\frac{1}{2}$ of (1.106).

In the same spirit as what we discussed for the covariant derivative, the relations (1.105), (1.108) and (1.109) might be seen as equivalent ways to define the notion of curvature for a linear connection. Once carefully written in components using natural basis $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$, all these definitions will reduce to (1.107).

Ricci tensor

If we see the curvature as a $(1,3)$ tensor field, we can always use it to define a $(0,2)$ tensor field known as the Ricci tensor. This tensor field will be defined in terms of its components in a given basis R_{ab} via the following contraction of the curvature tensor

$$R_{ab} \coloneqq R^c{}_{acb}. \tag{1.110}$$

Remark that, at this point, the Ricci tensor does not present any noteworthy symmetry under the exchange of its indices.

Torsion

The second important tensor field that we should introduce arises when one realises that, if $\vec{V},\vec{W} \in \Gamma(T\mathcal{M})$, both $\nabla_{\vec{V}}\vec{W}$ and $\nabla_{\vec{W}}\vec{V}$ make sense (see remark 1.8). Here, contrarily to the previous case, there is *a priori* no reason why we should expect these two quantities to be equal. Nevertheless, it is fruitful to compare these by computing $\nabla_{\vec{V}}\vec{W} - \nabla_{\vec{W}}\vec{V}$.

As before, if we fix a basis of vector fields $\{\vec{e}_{(a)}\}$ and use (1.81) to perform the calculation, we will find that

$$\nabla_{\vec{V}}\vec{W} - \nabla_{\vec{W}}\vec{V} = \left[\vec{V},\vec{W}\right] + V^a W^b T^c{}_{ab}\vec{e}_{(c)}, \tag{1.111}$$

where

$$T^c{}_{ab} \coloneqq \omega^c_{ba} - \omega^c_{ab} - \left[\vec{e}_{(a)},\vec{e}_{(b)}\right]^c. \tag{1.112}$$

This time again, the left-hand side and the first term of the right-hand side of (1.111) being manifestly vector fields, so must be the last term of the right-hand side and, due to the form of this term, the coefficients $T^c{}_{ab}$ must define the components of a $(1,2)$-tensor field. This tensor field will be denoted $\mathbf{T}$ and called the torsion tensor of the connection. By construction, it satisfies $T^c{}_{ab} = -T^c{}_{ba}$.

From this definition, we see that the torsion encodes an obstruction for the antisymmetric part of the covariant derivative of vector fields to reduce to the notion of commutator for vector fields. Geometrically, this tensor field encodes an obstruction for infinitesimal parallelograms to be closed curves. We might give a more detailed formulation of this statement but we will not as it will not be of great importance for the rest of our discussion. We simply want to emphasise that torsion encodes another property of our parallel transport operation that makes it different from the usual one known on $\mathbb{R}^n$ and that this notion really corresponds to the intuition that our manifold has been (locally) "twisted". Here again, the interested reader can refer to [Penrose, 2005] for more details on this point.

As always, we can play with the components of $\mathbf{T}$ in different types of basis. If we introduce a natural basis $\{ \partial_\mu \}$ and its dual basis $\{ \mathrm{d}x^\mu \}$ and write the basis $\{ \vec{e}_{(a)} \}$ and its dual basis $\{ \underline{\theta}^{(a)} \}$ as $\vec{e}_{(a)} = e_a{}^\mu \partial_\mu$ and $\underline{\theta}^{(a)} = e^a{}_\mu \mathrm{d}x^\mu$, we can obtain[1.39] a mixed indices version of $\mathbf{T}$ in terms of the spin connection coefficients

$$T^c{}_{\mu\nu} = \partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu + \omega^c_{b\mu} e^b{}_\nu - \omega^c_{b\nu} e^b{}_\mu. \tag{1.113}$$

Also, if we use a natural basis $\{ \partial_\mu \}$ to express anything in our computations, we will recover the well-known relation

$$T^\rho{}_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = -2\ \Gamma^\rho_{[\mu\nu]}, \tag{1.114}$$

again, using (1.67) to kill the last term in (1.112).

Similarly to what we just did with the curvature, (1.113) suggests an alternative definition of the torsion as a vector valued 2-form, also denoted $\mathbf{T}$. Indeed, using the tools of the exterior calculus on a manifold, we can define n 2-forms

$$\mathbf{T}^c \coloneqq \mathrm{d}\underline{\theta}^{(c)} + \omega^c{}_b \wedge \underline{\theta}^{(b)}, \tag{1.115}$$

whose components in the natural basis $\{ \mathrm{d}x^\mu \}$ are thus equivalent to (1.113), and such that this set of 2-forms transforms as the components of a vector. This point of view is also captured if we write (1.111) as

$$\mathbf{T}\left(\vec{V}, \vec{W}\right) \coloneqq \nabla_{\vec{V}} \vec{W} - \nabla_{\vec{W}} \vec{V} - \left[\vec{V}, \vec{W}\right], \tag{1.116}$$

where $\mathbf{T}$ is understood as this vector valued 2-form. Again, pay attention that, when written using a coordinate basis, (1.115) will actually give $\frac{1}{2}$ of (1.113).

Note that some authors might define the torsion with the opposite sign (this convention allows us to avoid the "$-$" in (1.114)) so this is something one should be careful with.

Here again, like for the curvature, the relations (1.112), (1.115) and (1.116) might be seen as equivalent ways to define the notion of torsion for a linear connection. Once carefully written in components using natural basis $\{ \partial_\mu \}$ and its dual basis $\{ \mathrm{d}x^\mu \}$, all these definitions will reduce to (1.114).

The definitions (1.115) for the torsion and (1.108) for the curvature are often referred to as the first and second Cartan structure equation, respectively.

Remark 1.8. *The fact that we can make sense of both $\nabla_{\vec{V}} \vec{W}$ and $\nabla_{\vec{W}} \vec{V}$ for $\vec{V}, \vec{W} \in \Gamma(T\mathcal{M})$ is specific to the fact that we consider the covariant derivative of vector fields.*

[1.39] The computation will require to use the relations $e^a{}_\mu e_b{}^\mu = \delta^a_b$, $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$ and the components of $\left[\vec{e}_{(a)}, \vec{e}_{(b)}\right]$ written as

$$f^c_{ab} \coloneqq \left[\vec{e}_{(a)}, \vec{e}_{(b)}\right]^c = -e_a{}^\mu e_b{}^\nu \left(\partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu\right).$$

For a covector field $\underline{\sigma} \in \Gamma(T^*\mathcal{M})$, for example, $\nabla_{\vec{V}}\underline{\sigma}$ make sense but this is not the case for a quantity like "$\nabla_{\underline{\sigma}}\vec{V}$" unless we find a way to univoquely assign a vector field to any covector field. This is possible in principle since $\forall p \in \mathcal{M}, \dim(T_p\mathcal{M}) = \dim(T_p^*\mathcal{M})$ but there is no canonical way to find an isomorphism between a vector space and its dual so this would require us to make a choice i.e. *to introduce an additional structure on $\mathcal{M}$. We will not do it but let us merely mention that, in general, such a structure that allows finding an isomorphism between some abstract vector spaces of dimension n "attached" to each point $p \in \mathcal{M}$ and the corresponding tangent space $T_p\mathcal{M}$ is called a soldering.*

This can be done for covector fields (or more abstract vector spaces with the right dimension) but this will then be impossible for generic tensor fields $\mathcal{T}$ since the space of tensors of a given rank at $p \in \mathcal{M}$ usually has a bigger dimension than $T_p\mathcal{M}$, by construction. Hence, we cannot make sense of something like "$\nabla_{\mathcal{T}}\vec{V}$" as we do for $\nabla_{\vec{V}}\mathcal{T}$. This observation naturally extend to more abstract vector spaces that one would "attach" to each point $p \in \mathcal{M}$ if the dimension of these vector spaces is different from $\dim(T_p\mathcal{M}) = n$.

As a corollary to this discussion, it is worth noting that, if one can pretty easily generalise the notion of connection and covariant derivative to more abstract objects than tensor fields (typically to fields taking value in abstract vector spaces), one will in general not be able to define a notion of torsion in this generalisation. It is then also worth noting that this problem does not apply to the curvature which can always be defined, via (1.109) *for example, even for connections for which the $\vec{v}$ in this equation will be a field valued in an abstract vector space.*[1.40]

To conclude this discussion, we should emphasise that, independently of how we represent these objects, curvature and torsion are ultimately properties of the connection. We have not defined any metric on our manifold so far !

In the following, when referring to curvature and torsion, unless explicitly stated otherwise, we will think of them as tensor fields defined by their components via (1.105) and (1.112) (or (1.107) and (1.114)) even though the alternative formulations in terms of differential forms are equally interesting.

1.3.7 Bianchi Identities

Curvature and torsion being properties of the same object, the linear connection, it is not surprising that they are somehow related to each other. This relation is encapsulated in the so-called Bianchi identities.

Once given a linear connection, the associated curvature and torsion satisfy the identities

$$\nabla_{[\mu|}T^{\alpha}{}_{|\rho\sigma]} + T^{\alpha}{}_{\lambda[\mu|}\,T^{\lambda}{}_{|\rho\sigma]} \equiv R^{\alpha}{}_{[\mu\rho\sigma]} \tag{1.117}$$

[1.40] This fact is at the core of the construction of gauge theories but this goes beyond our subject.

and

$$\nabla_{[\mu|} R^{\alpha}{}_{\beta|\rho\sigma]} + R^{\alpha}{}_{\beta\lambda[\mu|} \, T^{\lambda}{}_{|\rho\sigma]} \equiv 0, \tag{1.118}$$

respectively known as the first and second Bianchi identity or simply (collectively or individually) as the Bianchi identity. In these equations, following the common convention, we used square brackets to denote a total antisymetrization over all the indices inside the brackets except those who are placed between two vertical bars (see notations and conventions at the beginning of this thesis). Also, as it should be clear from the notation, we expressed these identities in a coordinate basis. Of course, since these equations relate components of tensor fields, they should remain valid in an arbitrary basis.

These relations can be verified via a straightforward and tedious calculation using the expressions (1.107) and (1.114). They can also be obtained in quite a direct way using the Cartan structure equations (1.108) and (1.115). Thanks to the properties of the exterior derivative and the wedge product, one obtains that

$$\mathrm{d}\mathbf{T}^{c} + \omega^{c}{}_{b} \wedge \mathbf{T}^{b} \equiv \mathbf{R}^{c}{}_{b} \wedge \underline{\theta}^{(b)} \tag{1.119}$$

and

$$\mathrm{d}\mathbf{R}^{c}{}_{d} + \omega^{c}{}_{k} \wedge \mathbf{R}^{k}{}_{d} - \mathbf{R}^{c}{}_{k} \wedge \omega^{k}{}_{d} \equiv 0. \tag{1.120}$$

Written in this way, these are identities between 3-forms. Once expressed in a natural basis $\{\, \mathrm{d}x^{\mu} \,\}$ and contracted in the appropriate way to involve only the coefficients (1.107) and (1.114), identity (1.119) (resp. (1.120)) reduces to (1.117) (resp. (1.118)).

1.3.8 Vanishing of Curvature or Torsion

As we just saw, generic linear connections will present both curvature and torsion but, for the sake of the theories of gravity that we should present right after, let us comment here on important properties of linear connections that does present either only curvature or only torsion.

Vanishing torsion

The situation that most physicists are more familiar with is when the torsion vanishes. In this case, using a generic basis of vector fields $\{\, \vec{e}_{(a)} \,\}$, the antisymmetric part of the connection coefficients is captured by the commutator of the basis vectors

$$\omega^{c}_{ab} - \omega^{c}_{ba} = -\left[\vec{e}_{(a)}, \vec{e}_{(b)}\right]^{c} \tag{1.121}$$

according to (1.112) or, most famously, using a coordinate basis $\{\, \partial_{\mu} \,\}$, the Christoffel symbols are symmetric for their two lower indices

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} \tag{1.122}$$

according to (1.114).

The vanishing of torsion also imposes restrictions on the curvature tensor. This might be seen from the Bianchi identities. Indeed, the first Bianchi identity reduces to

$$R^{\alpha}{}_{[\mu\rho\sigma]} \equiv 0, \tag{1.123}$$

which, in this case, is known as the algebraic Bianchi identity. This relation reduces the number of independent components of the curvature tensor.[1.41] Remark that, written in this way, the constraint applies to the components of $\mathbf{R}$ in a coordinate basis but, of course, this quantity being a tensor, the above relation should hold in any basis.

The second Bianchi identity, in turn, reduces to

$$\nabla_{[\mu|} R^{\alpha}{}_{\beta|\rho\sigma]} \equiv 0, \tag{1.124}$$

which, in this context, is referred to as the differential Bianchi identity.[1.42]

We should also note that, applying the appropriate contractions, (1.123) implies the following relation for the Ricci tensor

$$R_{ab} - R_{ba} = R^{c}{}_{cab}. \tag{1.125}$$

Also, let us stress that, at this point, there is no reason why $R^{c}{}_{cab}$ should vanish.

In the following, when dealing with this type of connection, we will call it a torsion-free connection.

Vanishing curvature

For connections where the curvature vanishes instead of the torsion, the situation is very different. We already mentioned that the curvature of a connection is responsible for the path dependence of the parallel transport. If the curvature vanishes, we will then obtain a linear connection for which the associated parallel transport gives the same result along any curve.

More precisely, in this case, given two points $p, q \in \mathcal{M}$ sufficiently close to each other[1.43], the parallel transport will provide an isomorphism between $T_p\mathcal{M}$ and $T_q\mathcal{M}$ that only depends on p and q. In other words, if p and q are any two close enough points, we will have that

$$\vec{v}_{||\mathscr{C}}(q) = \vec{v}_{||\tilde{\mathscr{C}}}(q) \tag{1.126}$$

[1.41] Note that this constraint arises by imposing vanishing torsion on a connection that can have one. So this constraint only apply to situations where the torsion can be defined. This should be clear if one sees (1.117) as arising from (1.119) since this last relation can only be derived if the notion of torsion makes sense. As a consequence, this constraint will not exist in more generic constructs where only curvature can be defined (see remark 1.8).

[1.42] Unlike the previous identity, this identity will always apply, even when torsion cannot be defined. Again, this should be clear by considering it as arising from (1.120) (see also remark 1.8).

[1.43] What we are about to describe does only work locally in general, unless we impose some extra conditions on the topology of our manifold. A more detailled discussion on this point would require to discuss under which conditions the frame bundle of the manifold is trivial. We will skip such a discussion here. We could just mention that manifolds for which this construction can be done globally are called parallelizable manifolds.

for any $\vec{v} \in T_p\mathcal{M}$ and any two curves $\mathscr{C}$ and $\tilde{\mathscr{C}}$ from p to q.

Using this property, given any point $p \in \mathcal{M}$, we can find, in a neighbourhood of p, a basis of vector fields $\{\, \vec{e}^{\star}_{(a)} \,\}$ such that the corresponding connection coefficients $\overset{\star}{\omega}{}^{b}_{ac}$ identically vanishes. This will be done in the following way :

Let $\{\, \vec{e}_{(a)_p} \,\}$ denote a fixed basis of $T_p\mathcal{M}$. Then, for any point q close enough from p, define

$$\vec{e}^{\star}_{(a)}|_q \coloneqq \left(\vec{e}_{(a)_p}\right)_{||\mathscr{C}}(q) \tag{1.127}$$

using any curve $\mathscr{C}$ from p to q. We can always do such a construction once given a parallel transport but the vanishing of the curvature ensures that we will here obtain a smooth basis of vector fields $\{\, \vec{e}^{\star}_{(a)} \,\}$ whose definition is independent of the curve $\mathscr{C}$. This is the important part in the above construction. Indeed, using (1.126) and the fact that parallel transporting a vector along a curve $\mathscr{C}_1$ from p to q and then parallel transporting the result via a curve $\mathscr{C}_2$ from q to $\tilde{q}$ will correspond to the parallel transport from p to $\tilde{q}$ via the curve $\mathscr{C}_1 + \mathscr{C}_2$ defined by joining the two curves[1.44] (as long as the obtained curve is still differentiable at the level of the junction), we will find that for any two points $q, \tilde{q} \in \mathcal{M}$ where this makes sense

$$\left(\vec{e}^{\star}_{(a)}|_q\right)_{||\tilde{\mathscr{C}}}(\tilde{q}) = \vec{e}^{\star}_{(a)}|_{\tilde{q}} \tag{1.128}$$

for any curve $\tilde{\mathscr{C}}$ from q to $\tilde{q}$. Note that, in the computation, we will have to use (1.127) with a curve $\mathscr{C}$ such that $\mathscr{C} + \tilde{\mathscr{C}}$ is still a differentiable curve. The fact that we can always choose $\mathscr{C}$ in this way once again require the path independence of the parallel transport and hence the vanishing of the curvature.

From this property, using (1.77), we will then have that $\forall a = 1, \cdots, n$ and $\forall \vec{V} \in \Gamma(T\mathcal{M})$,

$$\nabla_{\vec{V}} \vec{e}^{\star}_{(a)} = \vec{0}. \tag{1.129}$$

Consequently, (1.80) will give us that

$$\overset{\star}{\omega}{}^{b}_{ac} \equiv 0. \tag{1.130}$$

The fact that there are bases such that (1.130) holds dictates the form of the connection coefficients in any basis. Indeed, given such a basis $\{\, \vec{e}^{\star}_{(a)} \,\}$, for any other basis $\{\, \vec{e}_{(a)} \,\}$ defined in the same neighbourhood of p, (1.83) will give us that

$$\omega^b_{ac} = -\Lambda_a{}^l\; \vec{e}_{(c)}\left(\Lambda^b{}_l\right), \tag{1.131}$$

where $\vec{e}_{(a)}|_q = \Lambda_a{}^b(q)\; \vec{e}^{\star}_{(b)}|_q$ and $\Lambda_a{}^l \Lambda^b{}_l = \delta_a{}^b$. In addition, if we introduce a natural basis $\{\, \partial_\mu \,\}$ and write $\vec{e}^{\star}_{(a)} = \overset{\star}{e}_a{}^\mu \partial_\mu$ and $\partial_\mu = \overset{\star}{e}{}^a{}_\mu \vec{e}^{\star}_{(a)}$, we have that,

[1.44] Given two curves $\mathscr{C}_1$ and $\mathscr{C}_2$, parametrized such that $\mathscr{C}_1 : [0,1] \to \mathcal{M}$ and $\mathscr{C}_2 : [0,1] \to \mathcal{M}$ and such that $\mathscr{C}_1(1) = \mathscr{C}_2(0)$, we define

$$\mathscr{C}_1 + \mathscr{C}_2 : \begin{cases} [0,1] \to \mathcal{M} \\ \lambda \mapsto \begin{cases} \mathscr{C}_1(2\lambda) & \text{if } 0 \le \lambda \le 1/2 \\ \mathscr{C}_2(2\lambda - 1) & \text{if } 1/2 \le \lambda \le 1 \end{cases} \end{cases}.$$

according to (1.88), the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \overset{\star}{e}_b{}^{\rho}\partial_\nu \overset{\star}{e}{}^b{}_\mu. \tag{1.132}$$

Finally, following the same line, the spin connection coefficients can be written as

$$\omega^b_{a\mu} = -\Lambda_a{}^l\,\partial_\mu \Lambda^b{}_l. \tag{1.133}$$

Even though the above construction can only be done locally in general, note that it can be done around any given point. Thus, we can apply the above expressions for the connection coefficients "globally" provided we patch everything carefully enough when choosing our local basis. Also, one can prove via the linearity of the parallel transport that any two basis constructed using the procedure (1.127) will be related to each other by constant coefficients on the overlap of their respective domain of definition. As a consequence, the form of the above formulas will, consistently, be the same given any basis satisfying (1.130).

More than this, using (1.133), it is straightforward to prove that two arbitrary bases will produce the same expression for the connection coefficients if and only if they are related to each other by a constant matrix of transformation.

Note also that, here, we have derived the properties of the connection from the condition that $\mathbf{R} \equiv 0$ using its link to the parallel transport. Conversely, once given several bases of vector fields $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ carefully patched everywhere on a manifold, the condition (1.129) fully specify a linear connection for which (1.130) will hold in the specified basis and this immediately implies that $\mathbf{R} \equiv 0$ as one can see from (1.105).

All of this then draw a one-to-one correspondence between a connection such that $\mathbf{R} \equiv 0$ and a pair composed of bases of vector fields and equivalence classes[1.45] of point dependent invertible matrices (both carefully patched everywhere on the manifold). We can thus use this property as a way to define such connections if we want to.

Since this type of connection allows constructing local bases in which (1.130) holds, the situation looks very similar to what we know from Cartesian coordinates on Euclidean space (or on Minkowski spacetime) – except that we do not yet have the idea of orthonormality. It is then important to stress a key difference between the two situations : for a generic linear connection such that $\mathbf{R} \equiv 0$, (1.130) will never hold in any coordinate basis as soon as $\mathbf{T} \not\equiv 0$. To see this, it is sufficient to realise that, if the torsion does not identically vanish, a basis $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ for which (1.130) holds will be such that $[\vec{e}^{\,\star}_{(a)}, \vec{e}^{\,\star}_{(b)}] \not\equiv 0$ according to (1.112). Consequently, such a basis cannot correspond to a coordinate basis since we know that a coordinate basis $\{\,\partial_\mu\,\}$ will satisfy (1.67). On the contrary, if $\mathbf{T}$ also vanishes, it is possible to prove that the basis satisfying (1.130) will locally correspond to coordinate basis. It is so since, in general (*i.e.* independently of the presence or not of a connection), we have that a basis of vector

[1.45] The equivalence relation being that two point dependent invertible matrices are equivalent if they are related by a point independent invertible matrix.

fields such that $\left[\vec{e}_{(a)}, \vec{e}_{(b)}\right] \equiv 0$ locally corresponds to a coordinate basis.[1.46] In this context, we then see that the torsion really encodes the impossibility for (1.130) to hold in coordinate basis.

Before going further, we should also mention that for this type of geometrical structure, the interpretation of the geodesics as straight lines in the usual sense of the term is reinforced by (1.130). Indeed, choosing a basis $\{\, \vec{e}^{\,\star}_{(a)} \,\}$ for which this holds, we see that the geodesic equation (1.102) reduces to

$$\dot{v}^{a}\left(\lambda\right) = 0. \tag{1.134}$$

We then get that a geodesic is a curve whose tangent vector has constant components with respect to the basis $\{\, \vec{e}^{\,\star}_{(a)} \,\}$. Once again, the subtlety with what we know from Euclidian (or Minkowski) space is that this relation does not hold in a coordinate basis.

Finally, it is important to note that the vanishing of the curvature will also have an imprint on the torsion tensor via the Bianchi identities but that the situation is different from the previous case. The first Bianchi identity, (1.117), will give

$$\nabla_{[\mu|} T^{\alpha}{}_{|\rho\sigma]} + T^{\alpha}{}_{\lambda[\mu|}\, T^{\lambda}{}_{|\rho\sigma]} \equiv 0, \tag{1.135}$$

while the second Bianchi identity, (1.118), becomes trivial. This means that, here, there is only a differential identity but no algebraic one that would reduce *a priori* the number of independent components of the torsion.

In the following, when dealing with this type of connection, we will call it a curvature-free, or flat, connection. This type of connection is also often called a Weitzenböck connection. The fact that, on a manifold equipped with a Weitzenböck connection, the parallel transport of vectors is independent of the path followed (between close enough points[1.47]) allows for a well define generic comparison of vectors at distinct points on the manifold. This property is known as teleparallelism; the prefix "tele-" coming from the Greek and meaning "distant".

Remark 1.9. *The situation here is distinct from the one criticised in remark 1.5. Property* (1.129)*, or equivalently* (1.130)*, may indeed seem to point out to a (class of) "privileged basis" such that we define the variation of a vector field from the variation of its components with respect to that basis. Nevertheless, the situation is subtly different.*

What we do is choosing a linear connection and this, as discussed, gives rise to a covariant notion of the variation of a vector field (or any tensorial field) in the direction of another vector or vector field. We just choose a linear

[1.46]This can be seen thanks to properties of differential forms if one realizes that

$$\left(\left[\vec{e}_{(a)}, \vec{e}_{(b)}\right] = 0,\ \forall a, b = 1, \cdots, n\right) \Leftrightarrow \left(\mathrm{d}\underline{\theta}^{(a)} = 0,\ \forall a = 1, \cdots, n\right),$$

where $\{\, \underline{\theta}^{(a)} \,\}$ is the dual basis of $\{\, \vec{e}_{(a)} \,\}$.

[1.47]Unless the manifold is paralellizable; in which case we can do this for any pair of points.

connection such that "by accident" (or by construction) there are some basis for which (1.130) *holds. This property indeed implies that the variation of a vector field, encoded by its covariant derivative, is entirely described by the variation of its components with respect to such a basis. In this sense, the corresponding bases are "privileged" by the fact that calculations should be easier to use those bases but this does not mean that these bases are privileged* per se *as the only ones for which the notion of variation makes sense. They are, at most, just more convenient than others for practical purposes.*

To phrase it more carefully : basis satisfying (1.129) *are special, because linked to a property of the connection, but not privileged.*

1.3.9 Metric, Tetrads and Local Lorentz Transformations

After these – soon proved useful – digressions on properties of linear connections and the associated curvature and torsion, we can finally come to the introduction of the last key ingredient of our description of a physical spacetime : a metric. From our knowledge of what happens in Minkowski spacetime, we know that a metric is the necessary tool to introduce a causal structure on our spacetime and that, to do this, the metric should not be positive definite so that we can distinguish space-, time- and light-like vectors (see section 1.1).

Metric

From the mathematical perspective, this will be, again, an additional structure that we put on top of our manifold. This will be done by smoothly defining for each $p \in \mathcal{M}$ an application

$$\boldsymbol{g}|_p : \begin{cases} T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \\ (\vec{u}, \vec{v}) \mapsto \boldsymbol{g}|_p\,(\vec{u}, \vec{v}) \end{cases} \tag{1.136}$$

that is

1. bilinear :
$$\forall \vec{u}_1, \vec{u}_2, \vec{v} \in T_p\mathcal{M}, \forall \alpha, \beta \in \mathbb{R},$$
$$\boldsymbol{g}|_p\,(\alpha\vec{u}_1 + \beta\vec{u}_2, \vec{v}) = \alpha\ \boldsymbol{g}|_p\,(\vec{u}_1, \vec{v}) + \beta\ \boldsymbol{g}|_p\,(\vec{u}_2, \vec{v})$$
and the same for the second argument,

2. symmetric :
$$\forall \vec{u}, \vec{v} \in T_p\mathcal{M},\ \boldsymbol{g}|_p\,(\vec{u}, \vec{v}) = \boldsymbol{g}|_p\,(\vec{v}, \vec{u}),$$

3. non-degenerate :
$$\left(\exists \vec{u} \in T_p\mathcal{M} : \forall \vec{v} \in T_p\mathcal{M},\ \boldsymbol{g}|_p\left(\vec{u}, \vec{v}\right) = 0\right) \Rightarrow \vec{u} = \vec{0}.$$

For a given $p \in \mathcal{M}$ we will denote by (r, s) the signature[1.48] of $\boldsymbol{g}|_p$. The smoothness condition is naturally expressed by requiring that $\forall \vec{u}, \vec{v} \in \Gamma(T\mathcal{M})$, the function $\boldsymbol{g}(\vec{u}, \vec{v}) : \mathcal{M} \to \mathbb{R}$ defined by $[\boldsymbol{g}(\vec{u}, \vec{v})](p) \coloneqq \boldsymbol{g}|_p(\vec{u}|_p, \vec{v}|_p)$ is a smooth function.

Condition 3 in the above definition is very important. As we know from linear algebra, this condition implies that, for a given $p \in \mathcal{M}$, $\boldsymbol{g}|_p$ exhibits an isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$.[1.49] Indeed, from condition 1, given any $\vec{v} \in T_p\mathcal{M}$ we can define $\boldsymbol{g}|_p(\vec{v}, \cdot) \in T_p^*\mathcal{M}$ as the covector $\underline{\sigma} \in T_p^*\mathcal{M}$ such that $\underline{\sigma}(\vec{u}) = \boldsymbol{g}|_p(\vec{v}, \vec{u})$ for any $\vec{u} \in T_p\mathcal{M}$. Condition 3 ensures that we can always go the other way around; that is that given any $\underline{\sigma} \in T_p^*\mathcal{M}$ there exist a unique $\vec{v} \in T_p\mathcal{M}$ such that $\underline{\sigma} = \boldsymbol{g}|_p(\vec{v}, \cdot)$. Due to this isomorphism, $\vec{v}$ and $\boldsymbol{g}|_p(\vec{v}, \cdot)$ will naturally encode the same information.

The metric $\boldsymbol{g}$ is then a symmetric, smooth, $(0, 2)$-tensor field such that (due to condition 3) the matrix of its components in any basis is invertible.

Once introducing a basis of vector fields $\{\vec{e}_{(a)}\}$ and its dual basis $\{\underline{\theta}^{(a)}\}$, the metric will be given by

$$\boldsymbol{g} = g_{ab}\ \underline{\theta}^{(a)} \otimes \underline{\theta}^{(b)}, \tag{1.137}$$

where $g_{ab} = \boldsymbol{g}\left(\vec{e}_{(a)}, \vec{e}_{(b)}\right)$. In particular, introducing a coordinate basis $\{\partial_\mu\}$ and its dual basis $\{\mathrm{d}x^\mu\}$,

$$\boldsymbol{g} = g_{\mu\nu}\ \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \tag{1.138}$$

where $g_{\mu\nu} = \boldsymbol{g}(\partial_\mu, \partial_\nu)$.

The components of the inverse of the matrix of components g_{ab} are usually denoted g^{ab}. This clever placement of indices allows to naturally encode the isomorphism offered by the metric. Indeed, given a vector $\vec{v} = v^a\ \vec{e}_{(a)}|_p \in T_p\mathcal{M}$, the components of the covector $\underline{\sigma} = \boldsymbol{g}|_p(\vec{v}, \cdot)$ in the dual basis $\{\underline{\theta}^{(a)}|_p\}$ are $\sigma_a = v^b\ g_{ba}(p) =: v_a$. Conversely, given a covector $\underline{\sigma} = \sigma_a\ \underline{\theta}^{(a)}|_p \in T_p^*\mathcal{M}$, we can define a vector $\vec{v} \in T_p\mathcal{M}$ by imposing that its components in the basis $\{\vec{e}_{(a)}\}$ are $v^a \coloneqq \sigma_b g^{ba}(p) =: \sigma^a$. By construction, $\vec{v}$ will then correspond to the unique vector such that $\underline{\sigma} = \boldsymbol{g}|_p(\vec{v}, \cdot)$. It is in this sense that one can "lower and raise indices using the metric". This game of indices allows including calls to the isomorphism associated to $\boldsymbol{g}|_p$ as part of the notation; see also remark 1.10. Following the convention in the physics community, we will make frequent use of it in the following.

Remark 1.10. *The operation allowing to univoquely map a vector to a covector using the metric and the inverse operation allowing assigning a unique vector*

[1.48]The signature is defined by the number of positive and negative eigen values of the metric. This is generically denoted by (r, s), not to be mistaken with the rank of a tensor, where r denotes the number of negative and s the number of positive eigen values. Note that, the metric being non-degenerate, it cannot have 0 as an eigen value so $r + s = n$. Also, consequently, a metric will be positive-definite if and only if its signature is $(0, n)$.

[1.49]In terms of the concepts briefly sketched in remark 1.8, this would allow to define a soldering of the cotangent bundle.

to any covector are often referred to, in a rather poetic way, as the musical isomorphisms $\flat$ *and* $\sharp$*. For a given* $p \in \mathcal{M}$*, the application* $\flat : T_p\mathcal{M} \to T_p^*\mathcal{M}$ *is defined as the linear application which maps a vector* $\vec{v}$ *to the unique covector* $\vec{v}^\flat$ *such that for all vectors* $\vec{u}$*,* $\vec{v}^\flat(\vec{u}) := \boldsymbol{g}|_p(\vec{v}, \vec{u})$*. In other words, using our previous notation* $\vec{v}^\flat = \boldsymbol{g}|_p(\vec{v}, \cdot)$*. For this same point* $p \in \mathcal{M}$*, the inverse operation is the linear application* $\sharp : T_p^*\mathcal{M} \to T_p\mathcal{M}$ *that maps a given covector* $\underline{\sigma}$ *to the unique vector* $\underline{\sigma}^\sharp$ *for which* $\boldsymbol{g}|_p\left(\underline{\sigma}^\sharp, \vec{u}\right) := \underline{\sigma}(\vec{u})$ *for all vectors* $\vec{u}$*. In other words,* $\boldsymbol{g}|_p\left(\underline{\sigma}^\sharp, \cdot\right) := \underline{\sigma}$*. That is* $\left(\underline{\sigma}^\sharp\right)^\flat = \underline{\sigma}$*. Note that this operation is well defined only because the metric is non-degenerate. It is also straightforward, since the metric is non-degenerate, to get that* $\left(\vec{v}^\flat\right)^\sharp = \vec{v}$*. We then indeed have* $\sharp = \flat^{-1}$*.*

Now, if we introduce a basis $\{\, \vec{e}_{(a)}|_p \,\}$ *of* $T_p\mathcal{M}$ *and its dual basis* $\{\, \underline{\theta}^{(a)}|_p \,\}$*, we will have that* $\left(\vec{v}^\flat\right)_a = g_{ab}(p)v^b =: v_a$ *and* $\left(\underline{\sigma}^\sharp\right)^a = g^{ab}(p)\sigma_b =: \sigma^a$*. Thus, just as, in music, the* $\flat$ *allows lowering the key for a music note and the* $\sharp$ *allows going to a higher key, the musical isomorphisms are just another way to encode the idea of lowering and raising indices thanks to the metric. This notation just makes the use of the isomorphisms more explicit.*

As a final aside on this notation, note that, thanks to the metric, once given a point $p \in \mathcal{M}$*, we can always define the reciprocal basis* $\{\, \vec{e}^{(a)}|_p \,\}$ *of a given basis* $\{\, \vec{e}_{(a)}|_p \,\}$ *of* $T_p\mathcal{M}$ *as the unique basis of* $T_p\mathcal{M}$ *such that* $\boldsymbol{g}|_p\left(\vec{e}^{(a)}|_p, \vec{e}_{(b)}|_p\right) := \delta^a{}_b$*. From the musical isomorphisms, we thus see that*

$$\vec{e}^{(a)}|_p := \left(\underline{\theta}^{(a)}|_p\right)^\sharp,$$

were $\{\, \underline{\theta}^{(a)}|_p \,\}$ *is the dual basis of* $\{\, \vec{e}_{(a)}|_p \,\}$*.*

This notation then allows for a more basis independent way of expressing the isomorphism offered by the metric. That being said, in what follows, following to the ubiquitous convention in the physics community, we will refer to the musical isomorphisms only via the terminology "lowering and raising indices with the metric".

Tetrad field

Given a point $p \in \mathcal{M}$, we can then define the notion of an orthonormal basis of $T_p\mathcal{M}$. This will be a basis $\{\, \vec{e}_{(a)_p} \,\}$ such that $\boldsymbol{g}|_p\left(\vec{e}_{(a)_p}, \vec{e}_{(b)_p}\right) = \eta_{ab}$, where

$$(\eta_{ab}) = \mathrm{diag}\left(-1, \cdots, -1, 1, \cdots, 1\right). \tag{1.139}$$

In this expression, -1 will appear r times and 1 will appear s times if (r, s) denotes the signature of $\boldsymbol{g}|_p$.

Of course, we can extend this notion to bases of vector fields. We will say that a basis of vector fields $\{\, \vec{e}_{(a)} \,\}$ is orthonormal if, for all $p \in \mathcal{M}$ where this is defined, we have

$$g_{ab}(p) = \eta_{ab}. \tag{1.140}$$

Such bases are also referred to as tetrad fields, just tetrads, or vielbeins[1.50]. The dual basis of a tetrad is sometimes called a co-tetrad.

For later use, it is interesting to further develop (1.140) to compare the components of the metric in a tetrad to its components in a coordinate basis. More precisely, given a tetrad $\{ \vec{e}_{(a)} \}$ and a coordinate system $\{ x^\mu \}$ whose natural basis is $\{ \partial_\mu \}$, writing $\vec{e}_{(a)}|_p = e_a{}^\mu(p)\partial_\mu|_p$, we get that (1.140) is equivalent to the condition that for all $p \in \mathcal{M}$ where both the tetrad and coordinate system are defined

$$e_a{}^\mu(p)\; g_{\mu\nu}(p)\; e_b{}^\nu(p) = \eta_{ab}. \tag{1.141}$$

Contracting this equation with the inverse coefficients $e^a{}_\mu(p)$, this further rewrites as

$$g_{\mu\nu}(p) = e^a{}_\mu(p)\; \eta_{ab}\; e^b{}_\nu(p), \tag{1.142}$$

where we recall that η_{ab} is given by (1.139). This rather straightforward rewriting of the definition of a tetrad (1.140) yet allows emphasising an important point : (1.142) reveals that, locally, all the information encoded within the metric is actually equivalently encoded in its tetrads. Indeed, in this relation, η_{ab} are constant coefficients (fixed once and for all if we can know that the signature of the metric is always the same on the domain under study) so that the point dependence of $g_{\mu\nu}$ (and hence of $\boldsymbol{g}$) is really fully encoded in the coefficients $e^a{}_\mu$ (and hence in $\{ \vec{e}_{(a)} \}$).

Scandalmongers might be tempted to synthesise this property by saying that a tetrad is "the square root of the metric". We should not rely on this mischievous terminology in the following but it illustrates the fact that, when working in a coordinate system, one can transform every expression involving the components of the metric to make it depend only on the components of a tetrad. For example, if we want to compare locally two different metrics that one could define on the manifold, this could be done by using a coordinate system, a choice of tetrad for each metric and by studying the difference of the appropriate contractions, with η_{ab}, of the tetrad components in the coordinate system as a way to encode the difference in the components of the metrics.

Of course, this correspondence between the components of the metric and that of a tetrad is not one to one. Any tetrad can be used to obtain the same expression for the components of the metric and two tetrads are always related, on the overlap of their domain of definition, by a local generalised orthogonal transformation (see below).

Generalised orthogonal group and local Lorentz transformations

For definiteness, let us also recall that, once given a symmetric and non-degenerate bilinear map with a given signature (r, s) on a vector space, one can define the generalised orthogonal group, usually denoted $\mathrm{O}(r, s)$, as the group of linear transformations that preserves this map. By a slight misuse of terminology,

[1.50] In 4 dimensions, one usually uses the term vierbein. This is a little "pun" that comes from the german where "viel" stands for "many" and "vier" for "four".

we might consider it as the group of isometries of the space equipped with the aforementioned bilinear map.

More precisely, we can define $\mathrm{O}(r,s)$ as the group obtained by considering the set of $\mathbb{R}_{n\times n}$ matrices $\Lambda = \left(\Lambda_a{}^b\right)$ satisfying

$$\Lambda_a{}^k\ \eta_{kl}\ \Lambda_b{}^l = \eta_{ab} \tag{1.143}$$

equipped with the usual matrix multiplication, where η_{ab} is given by (1.139) for the r and s corresponding to the signature of the given bilinear map.

From the geometrical point of view, elements of $\mathrm{O}(r,s)$ will correspond to the matrix relating orthonormal basis. Indeed, if $\{\,\vec{b}_{(a)}\,\}$ and $\{\,\vec{b}'_{(a)}\,\}$ are two bases of our vector space such that $\{\,\vec{b}_{(a)}\,\}$ is orthonormal and $\vec{b}'_{(a)} = \Lambda_a{}^b\ \vec{b}_{(b)}$, $\{\,\vec{b}'_{(a)}\,\}$ will be orthonormal if and only if the coefficients $\Lambda_a{}^b$ satisfy (1.143).

Now, clearly, we can do such a construction as soon as we have a symmetric, non-degenerate bilinear map with a given signature (r,s) on a vector space. From the point of view of our manifold, with the metric $\boldsymbol{g}$ we do have such a map defined on $T_p\mathcal{M}$ for any $p\in\mathcal{M}$. We can then define local generalised orthogonal transformations. The idea is to consider two bases of vector fields $\{\,\vec{e}_{(a)}\,\}$ and $\{\,\vec{e}'_{(a)}\}$. In general, these will be related by a point dependent transformation $\vec{e}'_{(a)}|_p = \Lambda_a{}^b(p)\ \vec{e}_{(b)}|_p$. If both bases are tetrads and the metric signature is always (r,s) on the overlap of their domain of definition, this means that for any point p where the previous relation makes sense, we should have $\Lambda(p) \coloneqq \left(\Lambda_a{}^b(p)\right) \in \mathrm{O}(r,s)$.

This quick recap gives us the opportunity to point out a few things :

1. Up to now, when dealing with changes of basis, we generically noted the link between basis as $\vec{e}'_{(a)} = \Lambda_a{}^b\ \vec{e}_{(b)}$ and the link between the corresponding dual basis as $\underline{\theta}'^{(a)} = \Lambda^a{}_b\ \underline{\theta}^{(b)}$. This choice of notation is equivalent to say that we note $\Lambda = \left(\Lambda_a{}^b\right)$ the matrix whose coefficients relates the basis $\{\,\vec{e}'_{(a)}\}$ to the basis $\{\,\vec{e}_{(a)}\,\}$ and that, to simplify the notation, we pose $\Lambda^{-1} = \left(\left(\Lambda^{-1}\right)_a{}^b\right) \eqqcolon \left(\Lambda^b{}_a\right)$ so that the difference of notation between the matrix and its inverse is understood in the placement of the indices of the coefficients.

   In the previous sections, when we had not yet defined a metric, this was nothing more than a convenient notation. Now that we have a metric, we see that $\Lambda\in\mathrm{O}(r,s)$ if and only if

   $$\Lambda^{-1} = \eta\Lambda^T\eta^{-1}, \tag{1.144}$$

   where $\eta = (\eta_{ab})$, $\eta^{-1} = \left(\eta^{ab}\right) = \eta$ and $\Lambda^T = \left(\left(\Lambda^T\right)^a{}_b\right) \coloneqq \left(\Lambda_b{}^a\right)$. In this case, if we follow the usual practice and use the components of the metric to lower indices and the components of its inverse to raise indices, we would really have that $\left(\Lambda^{-1}\right)_a{}^b = \eta_{ak}\left(\Lambda^T\right)^k{}_l\,\eta^{lb} \eqqcolon \left(\Lambda^T\right)_a{}^b \eqqcolon \Lambda^b{}_a$ corresponds to (1.144) and should not be true for any change of basis.

Let us then emphasise that this choice we made previously to the introduction of a metric should not be harmful if we consider the following : In the previous sections, since there was no metric defined on the manifold, there was no risk to mislead the coefficients of Λ^{-1} with those of Λ^T as we had no way to "lower or raise indices". Now, in situations where we do have a metric, this notation should clearly be reserved to elements of $\mathrm{O}(r,s)$. What saves us, so to say, is that, when having a metric, we will almost only deal with orthonormal (or with coordinate) basis when writing expressions in components and this is equally true when considering changes of basis. This will happen naturally since, by construction, orthonormal bases are the ones that simplify drastically the form of the metric. They are then far more convenient for a practical purpose. Furthermore, the advantage of our potential misuse of the notation is that equations presented in the previous sections are then already in the good shape for when we should use them in context where we do have a metric and where the coefficients for the changes of basis should be understood as elements of $\mathrm{O}(r,s)$.

2. We used the term "metric" to name $\boldsymbol{g}$ and dubbed the elements of $\mathrm{O}(r,s)$ as the corresponding "isometries". This is, of course, a misuse of language in the sense that $\boldsymbol{g}$ will allow defining a norm and a distance in the proper mathematical sense – and thus deserve its name of a metric in the sense of something that allows measuring distances – if and only if its signature is $(0,n)$. In other words, if and only if $\boldsymbol{g}|_p$ defines a symmetric non-degenerate and positive-definite bilinear map for each $p \in \mathcal{M}$. It is only in this case that the elements of $\mathrm{O}(0,n) \simeq \mathrm{O}(n)$ can properly be understood as the linear isometries of the corresponding distance.

   Of course, now that this is clarified, we will follow the ubiquitous convention in physics and continue to misuse the terminology.

3. For our physical purpose, when we will assume to work on a 4 dimensional manifold equipped with a Lorentzian metric (a metric of signature $(1,3)$), the corresponding group of interest will consistently be the Lorentz group $\mathrm{O}(1,3)$. Following the present discussion, we will then be allowed to perform local Lorentz transformations on our manifold. We could even do this on n dimensions in general by considering a Lorentzian metric as a metric of signature $(1,n-1)$, in which case we would deal with the generalised Lorentz group $\mathrm{O}(1,n-1)$; see remark 1.11.

Remark 1.11. *Strictly speaking, a Lorentzian is a metric for which one of the eigenvalues has the opposite sign with respect to the* $n-1$ *others. We can then equivalently define a Lorentzian metric as a metric of signature* $(1,n-1)$ *or as a metric of signature* $(n-1,1)$.

In fact, in general, the structure given by a metric $\boldsymbol{g}_+$ *of signature* (r,s) *is the same as the one of a metric* $\boldsymbol{g}_-$ *of signature* $(n-r,n-s)$ *since metrics of signature* (r,s) *are in one-to-one correspondence with metrics of signature* $(n-r,n-s)$ *via the relation* $\boldsymbol{g}_+ \to -1 \cdot \boldsymbol{g}_+$.

In this text, we will almost always adopt the mostly plus convention, i.e. *we will concentrate on metrics of signature* (r, s) *where (without loss of generality)* $r \leq s$.

Following this convention, a Lorentzian metric will then be a metric of signature $(1, n-1)$.

1.3.10 Metric Compatible Connection

Of course, this metric structure can be defined on our manifold independently of the presence of any linear connection.

Covariant derivative of the metric

If we have also endowed our manifold with a linear connection, we can compute the covariant derivative of the metric tensor. (1.95) will naturally gives that for any $\vec{V} \in \Gamma(T\mathcal{M})$

$$\nabla_{\vec{V}} \boldsymbol{g} = V^c \left(\vec{e}_{(c)}(g_{ab}) - \omega^d_{ac}\, g_{db} - \omega^k_{bc}\, g_{ak} \right) \underline{\theta}^{(a)} \otimes \underline{\theta}^{(b)}. \tag{1.145}$$

When both a linear connection and a metric are defined, we can then study the compatibility of these two objects by means of the behaviour of $\nabla_{\vec{V}} \boldsymbol{g}$. This quantity controls the changes in the scalar product of vector fields since $\forall \vec{u}, \vec{v}, \vec{V} \in \Gamma(T\mathcal{M})$,

$$\nabla_{\vec{V}} \Big(\boldsymbol{g}(\vec{u}, \vec{v}) \Big) = \left(\nabla_{\vec{V}} \boldsymbol{g} \right) (\vec{u}, \vec{v}) + \boldsymbol{g} \left(\nabla_{\vec{V}} \vec{u}, \vec{v} \right) + \boldsymbol{g} \left(\vec{u}, \nabla_{\vec{V}} \vec{v} \right). \tag{1.146}$$

The left-hand side of the expression corresponds to the derivative in the direction of $\vec{V}$ of the real-valued function $\boldsymbol{g}(\vec{u}, \vec{v})$ encoding the scalar product of $\vec{u}$ and $\vec{v}$ at any point where they are both defined. If $\vec{u}$ and $\vec{v}$ are "constant in the direction of $\vec{V}$", *i.e.* if they are parallel transported along $\vec{V}$ so that $\nabla_{\vec{V}} \vec{u} = \vec{0} = \nabla_{\vec{V}} \vec{v}$, the last two terms in the right-hand side of (1.146) vanish. When then see that the scalar product of $\vec{u}$ and $\vec{v}$ can be constant in the direction of $\vec{V}$ if and only if $\left(\nabla_{\vec{V}} \boldsymbol{g} \right) (\vec{u}, \vec{v}) = 0$. So, in general, the scalar product of vector fields will not be constant in a given direction even if both vector fields are and the changes of this quantity are controlled by $\nabla_{\vec{V}} \boldsymbol{g}$.

Metric compatible connection

Intuitively, a connection and a metric will be compatible with each other if the parallel transport and the evaluation of the scalar product of vector fields commute with each other in the sense that, given two vector fields parallel transported along a curve, their scalar product is constant along the curve. As we saw, this will depend on the behaviour of the covariant derivative of the metric.

We will say that a linear connection is metric compatible if

$$\forall\, \vec{V} \in \Gamma(T\mathcal{M}),\ \nabla_{\vec{V}} \boldsymbol{g} \equiv 0. \tag{1.147}$$

In other words, this corresponds to the requirement that the metric tensor is parallel transported along any curve.

This condition can also be expressed by means of the so-called non-metricity tensor. This is the $(0,3)$-tensor field $\mathbf{Q}$ defined by the requirement that its components in the basis $\underline{\theta}^{(a)} \otimes \underline{\theta}^{(b)} \otimes \underline{\theta}^{(c)}$ are given by

$$Q_{abc} := \nabla_c g_{ab} = \vec{e}_{(c)}(g_{ab}) - \omega^{d}_{ac}\, g_{db} - \omega^{k}_{bc}\, g_{ak}. \tag{1.148}$$

In other words, $\mathbf{Q} := \nabla \boldsymbol{g}$. A metric compatible connection is then a connection for which

$$\mathbf{Q} \equiv 0. \tag{1.149}$$

Imposing this condition has a consequence on the form of the connection coefficients associated to a tetrad field. Indeed, if $\{\, \vec{e}_{(a)} \,\}$ is a tetrad, one obtains from (1.147) or (1.149) that

$$\omega_{abc} = -\omega_{bac}, \tag{1.150}$$

where we have set $\omega_{abc} := \omega^{k}_{bc}\, g_{ak}$. Note that the same will then obviously apply to the spin connection coefficients, *i.e.* we will have

$$\omega_{ab\mu} = -\omega_{ba\mu}, \tag{1.151}$$

where $\omega_{ab\mu} := \omega^{k}_{b\mu}\, g_{ak}$. Equations (1.150) and (1.151) then provide two other equivalent formulations of the metric compatibility condition valid only if the basis used is a tetrad.

The metric compatibility of a connection will also raise an extra property of the curvature tensor. Using (1.150) and (1.105), we will obtain that, once expressed using a tetrad and its co-tetrad, the components of $\mathbf{R}$ must satisfy

$$R_{cdab} = -R_{dcab}, \tag{1.152}$$

where, as usual, we defined $R_{cdab} := g_{ck}\, R^{k}{}_{dab}$. As always, this property was established using a particular basis (a tetrad) but, $\mathbf{R}$ being a tensor, it should be valid in any basis.

We can also remark that (1.152) implies

$$R^{c}{}_{cab} = 0. \tag{1.153}$$

1.3.11 Levi-Civita and Generic Linear Connections

Levi-Civita connection

Physicists are used to the fact that there is a unique torsion-free ($\mathbf{T} \equiv 0$) and metric compatible ($\mathbf{Q} \equiv 0$) connection associated to a given metric. This connection is known as the Levi-Civita connection and is entirely specified once given that metric.

This is conveniently proved by looking at the form of the Christoffel symbols under the assumption that the connection is torsion-free and metric compatible. The vanishing of torsion is easily implemented in terms of those coefficients by (1.122). The metric compatibility will then be considered in terms of the components of $\mathbf{Q} = \nabla \boldsymbol{g}$ in the same coordinate basis and the computation of $Q_{\rho\mu\nu} + Q_{\nu\rho\mu} - Q_{\mu\nu\rho}$ will give

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu} \right). \tag{1.154}$$

We can then see the Levi-Civita connection as the (unique) linear connection that one can define using only the choice of a metric, even though we have seen above that no metric is required to define a linear connection in general.

By definition, the only non-trivial tensor associated to the Levi-Civita connection is its curvature tensor. This tensor will simultaneously satisfy the constraints imposed by the vanishing of torsion (that is (1.123) and (1.124)) and non-metricity (that is (1.152)). In this context, it is also convenient to note that, using (1.123), (1.152), the natural antisymmetry for the last two indices and some contractions with the metric, we can derive an extra property of the curvature tensor of the Levi-Civita connection

$$R_{abcd} = R_{cdab}. \tag{1.155}$$

In other words, the curvature tensor of the Levi-Civita connection is symmetric under the simultaneous exchange of its first and third and second and fourth indices.

Finally, we will have that the Ricci tensor of the Levi-Civita connection is symmetric, *i.e.*

$$R_{ab} = R_{ba}. \tag{1.156}$$

This is implied by (1.155), using the appropriate contractions, but we can also obtain it directly by considering that both (1.125) and (1.153) should hold.

Before going further, we should also mention the following property of the covariant derivative associated to the Levi-Civita connection : given a vector field $\vec{v} \in \Gamma(T\mathcal{M})$ we have that, using a coordinate system $\{ x^\mu \}$,

$$\nabla_a v^a = \frac{1}{\sqrt{|g|}}\, \partial_\mu \left(\sqrt{|g|}\ v^\mu \right), \tag{1.157}$$

where $g \coloneqq \det(g_{\mu\nu})$ is the determinant of the matrix containing the components of the metric in the chosen coordinate system. Anticipating what we will do hereinafter, the interest of this relation comes from the fact that, when added to the Lagrangian density of a theory, such terms will only produce boundary terms; which will then not contribute to the field equations.

Generic linear connection

As we just saw, (1.154) establishes the existence and unicity of the Levi-Civita connection once given a metric. Interestingly, as soon as we have a metric, we can connect any linear connection to the Levi-Civita connection. Indeed, if we consider a generic linear connection, the computation of $Q_{\rho\mu\nu} + Q_{\nu\rho\mu} - Q_{\mu\nu\rho}$ will allow extracting its Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ as

$$\Gamma^{\rho}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu} - \frac{1}{2}\left(Q^{\rho}{}_{\mu\nu} + Q_{\nu}{}^{\rho}{}_{\mu} - Q_{\mu\nu}{}^{\rho}\right) - \frac{1}{2}\left(T^{\rho}{}_{\mu\nu} - T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu\nu}{}^{\rho}\right), \quad (1.158)$$

where $\overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}$ denote the Christoffel symbols of the Levi-Civita connection defined in (1.154) and where the indices of the torsion and non-metricity tensors have been lowered and raised using the metric.

This computation gives us access to several interesting results.

First of all, once given a metric, a linear connection is entirely specified once given its non-metricity and torsion tensors. This generalises the result of (1.154) and implies as corollaries that any metric-compatible connection is entirely specified once given its torsion, that any torsion-free connection is entirely specified once given its non-metricity and that there could indeed only be one torsion-free and metric-compatible connection.

This also gives us that the linear connection should be computed from specific combinations of the non-metricity and torsion, known respectively as the disformation and contorsion tensor fields. The disformation is the $(1,2)$-tensor field, denoted $\mathbf{D}$, whose components in a basis of vector fields $\{\,\vec{e}_{(a)}\,\}$ are defined as

$$D^{a}{}_{bc} \coloneqq -\frac{1}{2} g^{ak}\left(Q_{kbc} + Q_{ckb} - Q_{bck}\right). \quad (1.159)$$

The contorsion is the $(1,2)$-tensor field, denoted $\mathbf{K}$, whose components in a basis of vector fields $\{\,\vec{e}_{(a)}\,\}$ are defined as

$$K^{a}{}_{bc} \coloneqq -\frac{1}{2}\left(T^{a}{}_{bc} - g_{ck}\; g^{al}\; T^{k}{}_{lb} + g_{bk}\; g^{al}\; T^{k}{}_{cl}\right). \quad (1.160)$$

This allows writing (1.158) as

$$\Gamma^{\rho}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu} + D^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu}. \quad (1.161)$$

We can also see from (1.158) or (1.161) that the coefficients of a general linear connection can be decomposed into three distinct parts. The "first" part, the Levi-Civita connection coefficients, will transform according to the law of transformation of connection coefficients (in general, (1.83)) while the "second" and "third" parts, the disformation and contorsion, will transform according to the law of transformation of $(1,2)$-tensors. This is, of course, consistent with the fact that the coefficients of the general connection should, in total, transform according to the law of transformation of connection coefficients. This also allows us, using (1.87), to write the coefficients ω^{b}_{ac} describing the general connection in an arbitrary basis of vector fields $\{\,\vec{e}_{(a)}\,\}$ as

$$\omega^{b}_{ac} = \overset{\circ}{\omega}{}^{b}_{ac} + D^{b}{}_{ac} + K^{b}{}_{ac}, \quad (1.162)$$

where $\overset{\circ}{\omega}{}^b_{ac}$ denote the coefficients associated to the Levi-Civita connection in the corresponding basis. Of course, the same relation will hold for the spin connection coefficients $\omega^b_{a\mu}$.

Once again, let us briefly comment on the choice of notation. In the following, when we will have to compare a generic linear connection to the Levi-Civita one, we will denote the Christoffel symbols of the Levi-Civita connection by $\overset{\circ}{\Gamma}{}^\rho_{\mu\nu}$ and reserve the notation $\Gamma^\rho_{\mu\nu}$ for the generic linear connection – as we just did in (1.158). Nevertheless, when it will be clear that we are working with the Levi-Civita connection (*i.e.* when $\Gamma^\rho_{\mu\nu} = \overset{\circ}{\Gamma}{}^\rho_{\mu\nu}$), we will allow ourselves to omit the $\circ$, as we did in (1.154), in order to have a more compact notation.

1.3.12 Flat Connection in the Presence of a Metric

If a metric is defined on our manifold, a linear connection will be entirely specified if one fixes his torsion and non-metricity as established by (1.158). If we proceed in this way, the curvature tensor will then also be fixed via the connection coefficients. However, this does not mean that we cannot impose *a priori* constraints on the curvature tensor. We already saw in section 1.3.8 that we can construct linear connections with vanishing curvature ($\mathbf{R} \equiv 0$). This property allowed us to find bases of vector fields $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ such that relations (1.129) and (1.130) are satisfied.

If a metric is defined on our manifold, a natural question to ask is whether such a basis could be a tetrad. This property is indeed tempting as it would give us (at least) one basis for which the connection coefficients vanish (1.130) while the metric has its simplest form (1.140). Once again, this would drastically simplify lots of practical calculations.

According to (1.146), this should not be possible in general. Indeed, if (1.129) holds, we would have a basis of vector fields $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ for which the last two terms in the right-hand side of (1.146) systematically vanish. But for this basis to be a tetrad, *i.e.* for (1.140) to hold, we want the left-hand side of (1.146) to vanish for any $\vec{V} \in \Gamma\,(T\mathcal{M})$. We thus see that this would indeed be impossible unless $\mathbf{Q} \equiv 0$.

This then points to an interesting case that we did not discuss yet : a flat ($\mathbf{R} \equiv 0$) metric compatible ($\mathbf{Q} \equiv 0$) connection. In this case, since the connection is flat, all results from section 1.3.8 will hold. Furthermore, since the connection is metric compatible, nothing will prevent the basis $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ from being a tetrad. Indeed, if we make sure that the basis $\{\,\vec{e}_{(a)_p}\,\}$ in (1.127) is orthonormal, (1.129) and (1.146) will ensure that $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ is a tetrad.

In this case, we thus obtain a very interesting extra property of the connection coefficients :

We just argued that for a flat metric compatible connection, we could construct a tetrad $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ satisfying (1.129) but, obviously, this does not imply that (1.130) will hold for any tetrad. Instead, we will have that the connection coefficients ω^b_{ac} for an arbitrary tetrad $\{\,\vec{e}_{(a)}\,\}$ will satisfy (1.131) but, this time, for a matrix $\Lambda(p) = \left(\Lambda_a{}^b(p)\right) \in \mathrm{O}(r,s)$ for any point p where relating the two tetrads makes sense. We then see that the connection coefficients of a flat metric compatible connection are generated by local generalised orthogonal transformations. Furthermore, using arguments similar to those of section 1.3.8, we will have that two local generalised orthogonal transformations will generate the same connection coefficients if and only if they are related to each other by a global (point independent) generalised orthogonal transformation.

This then draws a one-to-one correspondence between a flat metric compatible connection and a pair composed of tetrads and equivalence classes[1.51] of local generalised orthogonal transformations carefully patched everywhere on the manifold. As always, we can thus use this property as a way to define flat metric compatible connections if we want to.

Of course, similar properties will hold in terms of the spin connection coefficients, according to (1.133).

A tetrad $\{\,\vec{e}^{\,\star}_{(a)}\,\}$ for which the connection coefficients vanish is usually called a Weitzenböck tetrad. In the same spirit, a manifold with a metric and a flat metric compatible connection is called a Weitzenböck, or a teleparallel, manifold.

In the following, when working on a Weitzenböck manifold, if it is clear from context that we are using a Weitzenböck tetrad, we might omit the $\star$ in the notations.

Without surprise, the notion of teleparallel manifold leads us to a situation very close to Euclidian (or Minkowski) space since teleparallel manifolds admit orthonormal basis of vector fields for which the connection coefficients vanish. Let us then recall, as pointed out in section 1.3.8, that the aforementioned basis can match with coordinate basis if and only if the connection – already flat and metric compatible – is also torsion-free.

1.3.13 Line Element and Geodesics

Length of a curve and line element

Once given a metric on a differential manifold (independently of the presence or not of a connection) we can define the notion of length of a curve. Depending on the signature of the metric, this will require to be a bit careful but the idea is the same as what we know from Euclidean space.

Given a metric $\boldsymbol{g}$, we can define the (pseudo-)norm of a vector $\vec{v} \in T_p\mathcal{M}$ as $\boldsymbol{g}|_p\,(\vec{v},\vec{v}) \in \mathbb{R}$. Depending on the signature of $\boldsymbol{g}|_p$, this quantity could be postivite, null or negative. [1.52] It is then interesting to consider curves $\mathscr{C}$ whose tangent vector $\vec{v}_\lambda$ satisfy either $\boldsymbol{g}|_{\mathscr{C}(\lambda)}\,(\vec{v}_\lambda,\vec{v}_\lambda) > 0,\ \forall\lambda$ or $\boldsymbol{g}|_{\mathscr{C}(\lambda)}\,(\vec{v}_\lambda,\vec{v}_\lambda) < 0,\ \forall\lambda$.

[1.51] The equivalence relation being that two local generalized orthogonal transformations are equivalent if they are related by a global generalized orthogonal transformation.

[1.52] So, again, this quantity will really define a norm only when $\boldsymbol{g}|_p$ is positive-definite. In this case, the norm will be defined as $||\vec{v}|| \coloneqq \sqrt{\boldsymbol{g}|_p\,(\vec{v},\vec{v})}$.

If the metric is a Lorentzian metric, specific names are given to the different possibilities. Given a Lorentzian metric $\boldsymbol{g}$, a curve $\mathscr{C}$ will be called

1. Timelike if $\forall\lambda,\ \boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)<0$,
2. null (or lightlike) if $\forall\lambda,\ \boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)=0$,
3. spacelike if $\forall\lambda,\ \boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)>0$.

For definiteness, let us here assume that we work with a curve $\mathscr{C}$ for which $\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)>0,\ \forall\lambda$ (the reasoning is identical if $\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)<0,\ \forall\lambda$ as long as we add a -1 factor in front of the metric). Considering that the curve is defined on $\mathrm{Dom}\left(\mathscr{C}\right)=[a,b]$ for some $a,b\in\mathbb{R}$, the length of the curve is defined as

$$L(\mathscr{C}):=\int_a^b\sqrt{\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)}\,\mathrm{d}\lambda. \tag{1.163}$$

If we introduce a coordinate system $\{\,x^\mu\,\}$ and write the coordinates of the points along $\mathscr{C}$ as $x^\mu(\lambda)$, using (1.69) and the bilinearity of the metric, we will find that

$$L(\mathscr{C})=\int_a^b\sqrt{g_{\mu\nu}\left(x^\alpha(\lambda)\right)\dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)}\,\mathrm{d}\lambda. \tag{1.164}$$

Considering curves for which $\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)$ as a definite sign, we can always, without loss of generality, restrict to the case $\left|\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda,\vec{v}_\lambda\right)\right|=1,\ \forall\lambda$. This will be done by considering reparametrizations of the curves.

Strictly speaking, this consists in considering curves not as $\mathscr{C}:\mathbb{R}\to\mathcal{M}$ but as equivalence classes $[\mathscr{C}]$ of such functions under the relation defined, given $\mathscr{C}_1:[a,b]\to\mathcal{M}$ and $\mathscr{C}_2:[c,d]\to\mathcal{M}$, as

$$\mathscr{C}_1\sim\mathscr{C}_2\Leftrightarrow\begin{pmatrix}\mathscr{C}_1=\mathscr{C}_2\circ r\ \text{with}\ r:[a,b]\to[c,d]\\ \text{a strictly increasing diffeomorphism}\end{pmatrix}. \tag{1.165}$$

In this context, choosing a parametrization of a curve $[\mathscr{C}]$ consist in choosing a representative $\mathscr{C}$ in the class and a reparametrisation consist in choosing another representative in the same class. Once given $\mathscr{C}:\mathbb{R}\to\mathcal{M}$, choosing a parametrization of $[\mathscr{C}]$ then corresponds to choose a $r:\mathbb{R}\to\mathbb{R}$ satisfying the necessary requirements and to work with the representative $\mathscr{C}\circ r$. In the following, depending on the context, we will call such a r a parametrization or a reparametrisation.

A reparametrisation will then simply rescale the speed along the curve. Indeed, given $\mathscr{C}_1$ and $\mathscr{C}_2$ such that $\mathscr{C}_1\sim\mathscr{C}_2$, if we denote the vector tangent to $\mathscr{C}_1$ at $\mathscr{C}_1(\lambda)$ as $\vec{v}_{1,\lambda}$ and the one tangent to $\mathscr{C}_2$ at $\mathscr{C}_2(\lambda)$ as $\vec{v}_{2,\lambda}$, we will have that

$$\forall\lambda\in\mathrm{Dom}(\mathscr{C}_1),\vec{v}_{1,\lambda}=r'(\lambda)\ \vec{v}_{2,r(\lambda)},\ \text{with}\ r'(\lambda)>0. \tag{1.166}$$

It is then interesting to note that the length defined via (1.163) is invariant under reperametrizations in the sense that given $\mathscr{C}_1:\mathbb{R}\to\mathcal{M}$ and $\mathscr{C}_2:\mathbb{R}\to\mathcal{M}$ such that $[\mathscr{C}_1]=[\mathscr{C}_2]$, $L\left(\mathscr{C}_1\right)=L\left(\mathscr{C}_2\right)$. We can then consistently define the length of $[\mathscr{C}]$, noted $L([\mathscr{C}])$, as the length of any representative of $[\mathscr{C}]$ as defined by (1.163). It is in this sense that we can restrict to curves with normalised tangent vector without loss of generality when dealing with the length of a curve.

Using these notions, we can also define the arc length of a curve. Given $\mathscr{C} : [a,b] \to \mathcal{M}$ such that $L(\mathscr{C}) = l$, the arc length of $\mathscr{C}$ corresponds to the parametrization $s : [a,b] \to [0,l]$ for which $\mathscr{C} \circ s^{-1}$ is a curve, equivalent to $\mathscr{C}$ (*i.e.* a representative of $[\mathscr{C}]$), with normalised tangent vector. $\mathscr{C} \circ s^{-1}$ will then correspond to the curve $[\mathscr{C}]$ parametrised in terms of its arc length. Using (1.166) the arc-length must be the function solving

$$s'(\lambda) = \sqrt{\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda, \vec{v}_\lambda\right)},\ \forall \lambda \in]a,b[\tag{1.167}$$

and such that $s(a) = 0$. In other words,

$$s(\lambda) = \int_a^\lambda \sqrt{\boldsymbol{g}|_{\mathscr{C}(\tilde{\lambda})}\left(\vec{v}_{\tilde{\lambda}}, \vec{v}_{\tilde{\lambda}}\right)}\mathrm{d}\tilde{\lambda}. \tag{1.168}$$

Accordingly, the length of $\mathscr{C}$ will then be given by

$$L(\mathscr{C}) = \int_a^b s'(\lambda)\mathrm{d}\lambda = \int_0^l \mathrm{d}s = s(b). \tag{1.169}$$

For a given $\mathscr{C} : [a,b] \to \mathcal{M}$, the differential of the arc length is called the line element and is then given by

$$\mathrm{d}s = \sqrt{\boldsymbol{g}|_{\mathscr{C}(\lambda)}\left(\vec{v}_\lambda, \vec{v}_\lambda\right)}\ \mathrm{d}\lambda. \tag{1.170}$$

Using a coordinate system $\{ x^\mu \}$, the line element is given by

$$\mathrm{d}s = \sqrt{g_{\mu\nu}\left(x^\alpha(\lambda)\right)\dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)}\ \mathrm{d}\lambda, \tag{1.171}$$

so that its square can be expressed as

$$(\mathrm{d}s)^2 = g_{\mu\nu}\left(x^\alpha(\lambda)\right)\mathrm{d}x^\mu(\lambda)\mathrm{d}x^\nu(\lambda). \tag{1.172}$$

The similarity between equations (1.138) and (1.172) is responsible for another widespread misuse of notation in the physics community : the metric is very often denoted as $\mathrm{d}s^2$ instead of $\boldsymbol{g}$, especially when it should be written using a coordinate system. This is, of course, really a misuse in the sense that $\mathrm{d}s$ should only make sense along a given curve. Having noted the improper aspect of this notation, we will allow ourselves to use it extensively in the following.

Talking of misuse of notation, in the following, we will usually assimilate a curve in the sense of a class $[\mathscr{C}]$ to any of its representatives $\mathscr{C}$ to lighten the notations. For example, we will talk about curves parametrised by arc length without using brackets in our notations.

Extrema of the length functional and geodesics

From (1.163), we see that the metric allows defining a functional L that can be consistently defined on curves for which the pseudo-norm of the tangent vector has a definite sign. One can then consider the problem of finding the extremal curves for L. This will usually be done by fixing a coordinate system $\{ x^\mu \}$ so that the curves $\mathscr{C}$ can be described in terms of their coordinates $x^\mu(\lambda)$ and by studying the variations from (1.164). This will then require to study the Euler-Lagrange equations for the Lagrangian $\mathscr{L}\left(x^\alpha, \dot{x}^\beta\right) = \sqrt{g_{\mu\nu}\left(x^\alpha(\lambda)\right) \dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)}$.

We have already pointed out that L is invariant under reparametrizations so that we could always restrict ourselves to curves parametrised by arc length. In this case, it is possible to prove that the Euler-Lagrange equations for $\mathscr{L}$ are equivalent to those for the simpler Lagrangian $\tilde{\mathscr{L}}\left(x^\alpha, \dot{x}^\beta\right) = g_{\mu\nu}\left(x^\alpha(\lambda)\right) \dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)$.

The usual computation will then show that the extremal curves for L must satisfy

$$\ddot{x}^\rho\left(\lambda\right) + \overset{\circ}{\Gamma}{}^\rho_{\mu\nu}\big(x^\alpha(\lambda)\big)\ \dot{x}^\mu\left(\lambda\right)\dot{x}^\nu\left(\lambda\right) = 0, \tag{1.173}$$

where $\overset{\circ}{\Gamma}{}^\rho_{\mu\nu}$ is defined in the right-hand side of (1.154). From (1.103), we then see that, given a metric $\boldsymbol{g}$, the curves that extremize the functional L defined in (1.163) correspond to the geodesics of the associated Levi-Civita connection. Let us nevertheless emphasise that this does not mean that the manifold is assumed to be equipped with this connection (or with any other) ! The system (1.173) appears as the Euler-Lagrange equations associated to the problem of finding the extremal curves for L without any connection presupposed on the manifold but it can be recognised that the curves satisfying this system will coincide with the geodesics of the Levi-Civita connection.

1.3.14 Einstein Tensor

With the notions of differential manifold, linear connection and metric, we finally have all the necessary tools to enter the precise formulation of our different classical theories of gravity. We saw that the structure of a differential manifold equipped with a linear connection and a metric is mathematically very rich and, in a sense, we only scratched the surface – even though the above material should be enough for our needs.

That being said, if we technically have all the necessary mathematical structures, we are still missing the main character of general relativity : the Einstein tensor !

We saw that, given any linear connection, we can define both the curvature and the Ricci tensor. Now that we also have a metric, we can define the trace[1.53]

[1.53] Of course, the notion of trace can be defined independently of a metric but this notion only makes sense for tensors with at least one covariant and one contravariant index. This is, for example, what led us to define the Ricci tensor using a trace of the curvature tensor or what can be used to make sense of the trace of a $(1,1)$-tensor $\mathcal{T}$ by defining $\mathrm{Tr}(\mathcal{T}) \coloneqq \mathcal{T}^a{}_a$.

Strictly speaking, what the presence of a metric allows us to do is to define from the Ricci tensor the $(1,1)$-tensor field, say $\tilde{\mathbf{R}}$, of components $\tilde{R}^a{}_b \coloneqq g^{ac}R_{cb}$. We can thus define the of the Ricci tensor, also known as the Ricci scalar

$$R \coloneqq g^{ab} R_{ab}. \tag{1.174}$$

Using this scalar quantity, the metric and the Ricci tensor, the Einstein tensor is then defined as the $(0,2)$-tensor field $\mathbf{G}$ whose components G_{ab} in a given basis are

$$G_{ab} \coloneqq R_{ab} - \frac{1}{2} R\, g_{ab}. \tag{1.175}$$

If the definition of the Einstein tensor makes sense in any situation where both a linear connection and a metric are defined, this tensor does not possess that many interesting properties in the most general case. For example, it will not be symmetric in general since the Ricci tensor will fail to be symmetric.

The situation is quite different in the case of the Levi-Civita connection. Indeed, in this case, since the Ricci tensor is symmetric due to (1.156), so will be the Einstein tensor. We thus have

$$G_{ab} = G_{ba} \tag{1.176}$$

in this case. Furthermore, using the symmetry properties applicable to the curvature tensor of the Levi-Civita connection, namely (1.155), and the vanishing of the non-metricity (1.149), one will obtain, using appropriate contractions, that the differential Bianchi identity (1.124) implies

$$\nabla_a G^a{}_b = 0. \tag{1.177}$$

In other words, the Einstein tensor is divergence free. This property, valid only for the Levi-Civita connection, plays a crucial role in general relativity.

1.4 General Relativity

In this section, we recall the main features of the theory of general relativity. The precise formulation of the physical ideas underlying this theory will follow naturally from the mathematical tools presented in section 1.3.

We start by identifying the geometrical setup of the theory ($\sim$ the kinematics of the theory) in paragraph 1.4.1. We then present the equations fixing the dynamical content of the theory in paragraphs 1.4.2 and 1.4.3.

Our recap of the theory is here for completeness and to allow for latter comparisons (see section 1.6). More in-depth discussions can be found in [Wald, 1984], [Carroll, 1997] or [Misner et al., 1973].

Ricci scalar as $R \coloneqq \mathrm{Tr}(\tilde{\mathbf{R}})$. Of course, the informations encoded in $\tilde{\mathbf{R}}$ and in the Ricci tensor are identical since the metric is non-degenerate. It is then always relevant to "assimilate" $\tilde{\mathbf{R}}$ to the Ricci tensor and to see the Ricci scalar as the trace of the Ricci tensor itself.

1.4.1 Geometrical Setup

The theory of general relativity as formulated by Albert Einstein in 1915 takes place on a 4 dimensional spacetime. This assumption does not need that much motivation from the point of view of classical physics. According to the definitions presented in section 1.3, we will see that all the construct of the theory can be formulated without additional effort on a n dimensional spacetime if needed.[1.54]

In order to define a causal structure on our spacetime, it is also clear that one should demand the presence of a non positive-definite metric. To allow for the distinction between one time and three space dimensions, the metric of general relativity is assumed to be a Lorentzian metric.

The first main physical idea of the theory, as we already discussed (see section 1.2), concerns the formulation of the equivalence principle and the fact that gravity should not be understood as an external force anymore but integrated in the geometrical structure of spacetime. By considering that, in classical mechanics, the universality of gravitation implies that the effects of a uniform gravitational field can always be removed by going to an accelerated (non-inertial) frame and that, in special relativity, the fact that we work in an accelerated frame manifests itself at the level of the components of the metric, again, one will expect the gravitational field to have an influence on the spacetime metric.

In GR, this idea is pushed to its maximum by considering that, as in Minkowski spacetime, the metric is the only input needed to specify the geometry of spacetime. The covariant derivation of tensors will then be ensured by means of the associated Levi-Civita connection.

To summarise this in one sentence, spacetime in general relativity is assumed to be a 4 dimensional differential manifold $\mathcal{M}$ endowed with a Lorentzian metric $\boldsymbol{g}$ and equipped with the associated Levi-Civita connection, defined by its coefficients $\overset{\circ}{\omega}{}^{b}_{ac}$.

In the rest of this section, since we will only work with the Levi-Civita connection, we will omit the $\circ$ in the notation. In general, one will further denote the spacetime as $(\mathcal{M}, \boldsymbol{g})$, simply specifying its metric as it is understood that the connection employed will be the Levi-Civita one that can be regarded as a "byproduct" of the metric. This notation mimics the one we used in Minkowski spacetime (see section 1.1).

This idea that the connection is derived from the metric also permits to interpret the curvature $\mathbf{R}$ as a property derived from the metric. In this sense, even though we saw that the notion of curvature is *per se* a characteristic of the connection, we can consider it as the curvature of the metric and then really as the curvature of the spacetime $(\mathcal{M}, \boldsymbol{g})$.

Without trying to diminish the interest and physical legitimacy of this choice for the spacetime structure, it is interesting to mention that historically, when Einstein first searched for, and eventually arrived at, a relativistic theory of

[1.54] We will take advantage of this in some of our papers where we will study compact objects on a 5 dimensional spacetime.

the gravitational interaction (*i.e.* GR), the Levi-Civita connection was the only one "already available". At that time, Riemannian geometry was the most advanced mathematical theory allowing for a non-Euclidian geometry in 4 dimensions.[1.55] This time again, to not be too apocryphal, we should acknowledge how conceptually advanced it was, in 1915, to assimilate a physical process with a geometrical property and to abandon, for physical reasons, the use of Euclidian geometry. This sole idea could very well be considered the main legacy of general relativity. Nevertheless, while acknowledging this, it is interesting to note that Einstein's choice of $(\mathcal{M}, \boldsymbol{g})$ to describe the physical spacetime was not made by favouring the Levi-Civita connection over all other possible choices as there was, back in the days, no such a choice to be made.

1.4.2 Motion of Pointwise Particles

The second physical idea of the theory clarify how test particles will experience the presence of the gravitational field.

This postulate relies on the idea that gravity cannot be measured as a force. Indeed, similarly to what is done in Newtonian mechanics or in special relativity (see section 1.1), to be able to measure an interaction as a force, we should be able to measure it in terms of deviations from a situation where this interaction is not present. In a situation where no such type of interaction play a role, particles will move along "straight lines", *i.e.* geodesics. On account of the universality of gravity (see section 1.2), general relativity then postulates the complete impossibility of such measurements in the case of the gravitational interaction. Under this framework, it is then somehow natural to assign the motion of particles moving freely in the gravitational field to geodesics of the curved geometry.

Also, although we did not elaborate on this in this text, we should mention one property valid in (pseudo)-Riemannian geometry that further argues in favour of the geodesics of the Levi-Civita connection as the curves that should encode the motion of freely falling particles. Given a manifold equipped with a metric and the associated Levi-Civita connection $(\mathcal{M}, \boldsymbol{g})$, for any $p \in \mathcal{M}$, one can always locally construct a coordinate system – known as Riemann normal coordinates – for which the Christoffel symbols associated with this coordinate system precisely vanish at p. In general, *i.e.* in the presence of curvature, the coordinate system will only be able to ensure that the Christoffel symbols will vanish at this specific point p. Nevertheless, this can be done around any point (using a different coordinate system each time). This property can be interpreted as a "manifestation" of the equivalence principle and more precisely of the idea that gravity can locally be removed by passing in an appropriated frame. Indeed, at any point, by passing in Riemann normal coordinates the geodesic equation (1.103) would locally (*i.e.* "instantly", at one point) reduce to

[1.55] Actually, cutting a long story short, one could even say that several subsequent developpments in differential geometry – especially those leading to the emancipation of the notion of connection from a preexisting metric – were in part motivated by 1915's advent of general relativity. In this respect, one could see chapter 2 of [Blagojevic and Hehl, 2012].

its special relativistic equivalent (1.27) in agreement with Einstein's equivalence principle.

On account of these ideas, general relativity then precisely postulates that test particles should move along the geodesics of spacetime. In other words, their trajectory should satisfy (1.101). Once given a coordinate system $\{ x^\mu \}$ and initial position and 4-velocity, the trajectory is then found by solving (1.103).

As in special relativity, massive particles move along timelike curves while (in the geometrical optics approximation) light rays travel along null geodesics. Furthermore, in the case of timelike motions (the type of motion of physical observers), the arc length (1.168) will still be interpreted as the proper time measured by an observer moving along the curve under study.

The fact that light rays move along the (curved) spacetime geodesics just as massive particles is one of the main innovative predictions of general relativity compared to previous theories since it implies that the presence of a gravitational field should have an influence on the propagation of light even though light is massless.[1.56]

It is worth noting that here, since the manifold is endowed with the Levi-Civita connection, according to (1.173), this postulate can be equivalently stated by saying that test particles should move along curves of extremal length.

Both versions of the postulate can be regarded as equally valid (yet different, in general) covariant generalisations of the postulate from classical mechanics stating that free particles move along straight lines when observed in inertial frames.[1.57] In general relativity, the chosen connection ensures that these two formulations remain indistinguishable. This equivalence of formulation will then be lost in theories formulated using other geometrical inputs.

1.4.3 Einstein Field Equations

The last (but not least !) physical idea of general relativity dictates how the gravitational field is fixed by its sources.

If a covariant theory of the gravitational interaction is needed to remove the special role of inertial frames of Newtonian mechanics and to cohere with the findings of special relativity, no one will disagree on the fact that the Newtonian theory of gravity provides a successful way to describe gravitation in non-relativistic regimes. It is then clear that this theory should serve as a guideline in the construction of its more fundamental successors. In particular, whatever should be the correct covariant description of gravity, it should somehow reduce

[1.56] Note that, prior to the advent of GR *per se*, along his reflections on the equivalence principle, Einstein did already prefigure that light path should be bent close to massive objects. Yet, his preliminar calculations predicted a deviation twice smaller than GR's value and it is this value (GR's value) that is in agreement with experiment.

[1.57] Here, again, the generalisation does also rely on the meaning of the word "free" that is taken for "free of any non-gravitational interaction". Gravity is indeed incoroporated in the spacetime geometry in such a way that it cannot be measured by comparision with the motion of free particles (in the old sense of the term) since there is no such particles "free from gravity".

to the Newtonian one in the appropriate limit. This assessment is at the core of the procedure followed by Einstein to obtain the field equations of general relativity.

The first take-home message from Newtonian gravity points toward the source of the gravitational field. The source of the gravitational potential associated to a given object in Newton's theory of gravity is its mass distribution. This is encoded in Poisson's equation that fixes the gravitational potential by relating its Laplacian to the mass density of the source. On top of that, as we saw in section 1.1.5, one of the main findings of the theory of special relativity was to show that (rest-)mass itself is just a manifestation of the energy of a body and that the relevant concept to characterise an object (the one associated to a conservation law) is its energy-momentum tensor. The combination of these ideas leads to expect gravity to be actually sourced by the energy-momentum tensor of matter.

This leaves to find the appropriate tensorial quantity that can relate gravity to the energy-momentum tensor of matter. Here again, Einstein found inspiration in Newtonian gravity.

First of all, one needs to identify the quantity that should correspond to the gravitational potential in the Newtonian limit. This role will naturally be played by the metric tensor. In a nutshell, this can be justified by the idea that if gravity should be encoded in the geometry of spacetime, this is the only candidate in the sense that the other non-canonical structures defined on top of our manifold derive from the metric. This is also intuitively reinforced by comparing (1.103) to the equation of motion of a test body subject to the electromagnetic interaction on Minkowski spacetime (1.36). This comparison consists in spotting that the term proportional to the Christoffel coefficients mimics the Lorentz force[1.58] so that the Christoffel symbols themselves parallels the Faraday tensor. Under this analogy, we see that the Christoffel symbols derive from the metric just as the gravitational force derives from the gravitational potential. We have already argued that the action of gravity on test particles cannot be understood as a force in general relativity. In addition, since the Christoffel symbols do not define a tensor, they are, in any case, not what we would use to encode a physical force if we had to. Nevertheless, this analogy is worth noting since, for general relativity to reduce to Newtonian gravity in the appropriate limit, the equation of motion of test particles of general relativity should somehow reduce to the Newtonian description in this limit. Following this idea, the Newtonian force and potential should somehow emerge from (1.103). We then see that the only term from which they can is the one proportional to the Christoffel symbols. Also, if this term should somehow resemble the Newtonian force in this supposed limit, it should reduce to an expression depending on the first derivatives of the gravitational potential. The form of the Christoffel symbols of the Levi-Civita connection (1.154) then further suggests relating the metric (already related to gravity in the fully covariant picture) to the gravitational potential.

[1.58] except that it is quadratic and not linear in the particle 4-velocity

Before going further, let us mention that Einstein did this part of the argumentation in a slightly more elaborated way by comparing the geodesic deviation equation to the Newtonian's tidal equation. This way to proceed basically leads to the same conclusions.

We are then looking for a tensor derived from the metric. Once again, a look at Newtonian gravity provides further indications. To avoid conflict with the Newtonian picture in its domain of validity, the field equations of general relativity should reduce to Poisson's equation in the Newtonian limit. The whished tensor should then depend on second derivatives of the metric. Of course, since our manifold is endowed with the Levi-Civita connection, which is metric compatible, this tensor should be constructed from the second partial derivatives of the metric. According to the geometrical material presented in the previous section, it is then natural to look in the direction of the curvature tensor. This tensor is indeed constructed from partial derivatives of a connection – see (1.107) – which is itself constructed from partial derivatives of the metric – see (1.154). Since the curvature is a $(1,3)$-tensor field while the energy-momentum tensor should be a $(0,2)$-tensor field, it is even more natural to consider the Ricci tensor.

As argued in many places, on account for the above reasoning, one could write an equation like

$$R_{ab} = \kappa\, T_{ab}, \tag{1.178}$$

where T_{ab} are the components of the energy-momentum tensor of the matter sourcing the gravitational field and κ is a dimensional coupling constant, but this is not such a good idea (even though it is the form proposed by Einstein at some point). Indeed, as we already mentioned, the main interest of the energy-momentum tensor is that it should be associated to a conservation law which, in the covariant formulation, should be

$$\nabla_a T^a{}_b = 0. \tag{1.179}$$

This relation is not a consequence of (1.178) since in general, $\nabla_a R^a{}_b \neq 0$. If we try to add (1.179) as a constraint in addition to (1.178), we will be facing a strange situation since, for the Levi-Civita connection, the differential Bianchi identity implies

$$2\nabla_a R^a{}_b \equiv \nabla_b R. \tag{1.180}$$

Combining this identity to (1.178) and (1.179) would then lead to the constraint that $\nabla_b T^a{}_a = 0$. This implies that the trace of the energy-momentum tensor should be constant over spacetime. This is not a desirable property as one would expect the behaviour of T_{ab} and its trace to differ inside and outside of the matter distribution sourcing our gravitational interaction.

But enough beat around the bush, the "incompatibility" between (1.178) and (1.179) also suggests its own resolution since (1.180) is simply a rewriting of (1.177). We can then solve the problem of the above proposition by replacing R_{ab} by the components of the Einstein tensor G_{ab} in the right-hand side of (1.178), leading to the (correct) Einstein equations of general relativity

$$G_{ab} = \kappa\, T_{ab}. \tag{1.181}$$

The identity (1.177) will then consistently imply (1.179).

To complete the discussion, one still needs to fix the constant κ. This is done, once again, by looking at the Newtonian limit of the equation. For equation (1.181) to reduce to the Poisson equation, with the metric playing the role of the gravitational potential and the matter density coming from the energy-momentum tensor, one should have

$$\kappa = \frac{8\pi\mathcal{G}}{c^4}, \tag{1.182}$$

where $\mathcal{G}$ is Newton's constant and c the speed of light in vacuum. Einstein equations can then be written as

$$G_{ab} = \frac{8\pi\mathcal{G}}{c^4}T_{ab}. \tag{1.183}$$

Once written in a coordinate system, these equations reduce to 10 (highly non-linear) partial differential equations whose unknown (assuming that the matter content is known) are the 10 components of the metric in that coordinate system.[1.59]

1.4.4 Cosmological Constant

Actually, following the above reasoning which gives a central role to equation (1.179), one can consider a form of the equations slightly more general than (1.183). Indeed, since we work with a metric compatible connection, (1.179) will still be true if one writes

$$G_{ab} + \Lambda\ g_{ab} = \frac{8\pi\mathcal{G}}{c^4}T_{ab}, \tag{1.184}$$

where Λ is a constant. This constant is the famous cosmological constant. As we mentioned in the introduction chapter, it was first introduced (and fined tuned) by Einstein as a way to modify its fields equations (1.183) to ensure the existence of a(n unstable) static cosmological solution.

Despite this erroneous historical motivation, the cosmological constant found a place in modern physical theories such as in the ΛCDM model (again, see the brief discussion in the introduction chapter). It is also theoretically interesting as one can prove that the left-hand side of (1.184) gives the most general expression satisfying all our requirements and, especially, ensuring (1.179).

1.4.5 Einstein-Hilbert Lagrangian

An important fact to mention is that (with or without cosmological constant) Einstein's equations in vacuum, *i.e.* with $T_{ab} = 0$, derive from a variational

[1.59]These are the number of equations and unknown in 4 dimensions. In n dimensions, due to the symmetry of the Einstein and metric tensors, one has *a priori* $\frac{n(n+1)}{2}$ equations and the same number of unknown.

principle. If we fix an arbitrary coordinate system $\{ x^\mu \}$, these equations are obtained by extremizing the action

$$S_{\text{EH}} = \frac{1}{2\kappa} \int_{\mathcal{V}} (R - 2\Lambda) \sqrt{-g} \, \mathrm{d}^4 x, \tag{1.185}$$

where $\mathcal{V}$ is an arbitrary integration volume, $g = \det(g_{\mu\nu})$ is the determinant of the matrix of components of the metric in the coordinate system and κ is given by (1.182).

To search for the condition ensuring to extremize (1.185), one can express its variation δS_{EH} with respect to the metric $g_{\mu\nu}$ (or equivalently with respect to the inverse metric $g^{\mu\nu}$). Remark that the consistency of this way of proceeding is related to the fact that we work on a manifold endowed with the Levi-Civita connection associated to the metric. Indeed, in (1.185) one finds the Ricci scalar which, in all generality, is constructed from the metric and the Ricci tensor which is a tensor constructed from the connection. Consequently, one can make sense of (1.185) as soon as we are given a manifold, a metric and a linear connection but this expression would then depend on two *a priori* independent pieces. These two pieces should then *a priori* be varied independently to compute δS_{EH}. That being said, here, previously to the introduction of the dynamics, we assumed to work with a manifold on which we have a metric and the associated Levi-Civita connection (see section 1.4.1). In this case, we know that the connection is entirely determined by the metric from (1.154) and we want this property to be preserved when computing the variations of S_{EH}. This could *a priori* be done by considering variations of the metric and the connection and by imposing restrictions on the allowed variations of the connection so that (1.154) is preserved. Nevertheless, since (1.154) allows us to "eliminate the connection in terms of the metric", this can also be done by expressing S_{EH} as depending only of the metric and by taking variations of this expression with respect to the metric only.

Following this procedure, one can show that the variation of this action is given by :

$$\begin{aligned} \delta S_{\text{EH}} =& \frac{1}{2\kappa} \int_{\mathcal{V}} \left(R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \sqrt{-g} \, \delta g^{\mu\nu} \, \mathrm{d}^4 x \\ &+ \frac{1}{2\kappa} \int_{\partial\mathcal{V}} \left(g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} g_{\rho\sigma} - \delta\Gamma^{\lambda}_{\sigma\lambda} \right) \mathrm{d}\Sigma^\sigma \end{aligned}$$

where $\partial\mathcal{V}$ denotes the boundary of $\mathcal{V}$ and $\mathrm{d}\Sigma^\sigma$ "the normal to $\partial\mathcal{V}$".

Therefore, if we neglect the boundary terms, by virtue of Du Bois-Reymond's lemma, we will have $\delta S_{\text{EH}} = 0, \forall \delta g^{\mu\nu}$ iff

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} + \Lambda g_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1.186}$$

The action (1.185) is commonly called Einstein-Hilbert action.

We can also obtain the Einstein equations with cosmological constant in the presence of a source (1.184) by adding a Lagrangian characterising the source to the Einstein-Hilbert Lagrangian. Let $\mathscr{L}_{\mathrm{M}}$ be this Lagrangian. In this case, the total action S will be given by

$$S = S_{\mathrm{EH}} + S_{\mathrm{M}} = \frac{1}{2\kappa}\int_{\mathcal{V}} (R - 2\Lambda)\sqrt{-g}\ \mathrm{d}^4x\ + \int_{\mathcal{V}} \mathscr{L}_{\mathrm{M}}\sqrt{-g}\ \mathrm{d}^4x. \tag{1.187}$$

Under the assumption that $\mathscr{L}_{\mathrm{M}}$ can depend (besides the matter fields and their derivatives) on the metric but not on its derivatives, the variation of S_{M} with respect to the inverse metric gives

$$\delta S_{\mathrm{M}} = \int_{\mathcal{V}} \left(\frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathscr{L}_{\mathrm{M}} g_{\mu\nu}\right)\sqrt{-g}\ \delta g^{\mu\nu}\ \mathrm{d}^4x.$$

Therefore, the total variation of the action (1.187) is given (up to boundary terms) by

$$\delta S = \int_{\mathcal{V}} \left(\frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}R\ g_{\mu\nu} + \Lambda g_{\mu\nu}\right) + \left(\frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathscr{L}_{\mathrm{M}} g_{\mu\nu}\right)\right)\sqrt{-g}\ \delta g^{\mu\nu}\ \mathrm{d}^4x.$$

The action (1.187) will then be extremized by the solutions of the equation

$$\frac{c^4}{16\pi\mathcal{G}}\left(G_{\mu\nu} + \Lambda\ g_{\mu\nu}\right) + \left(\frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathscr{L}_{\mathrm{M}} g_{\mu\nu}\right) = 0. \tag{1.188}$$

If we set

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{-g}\mathscr{L}_{\mathrm{M}}\right) = -2\left(\frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathscr{L}_{\mathrm{M}} g_{\mu\nu}\right), \tag{1.189}$$

we obtain that (1.188) is equivalent to (1.184) once written in the coordinate system $\{ x^\mu \}$.

As a final note on this calculation, remark that, using (1.142), one could have eliminated the metric components in terms of the tetrad components in (1.185) (or in (1.187)). Following this procedure, we could have obtained the field equations by considering variations with respect to these tetrad components. This would give equivalent results – but with expressions written in terms of the components of the chosen tetrad.

1.5 Black Holes in General Relativity

One of the most striking objects predicted by general relativity (and an important object of study for this thesis) are, of course, black holes. These are the topic of this section.

There are enormous amounts of things that could (and maybe should) be said regarding the properties of black hole spacetimes in general and the black hole solutions in general relativity in particular. In fact, each of these two aspects of the topic could deserve its own chapter. That being said, not all of these properties would be useful for the results of this thesis.

During this thesis, since our focus was on the construction of new black hole solutions in alternative theories of gravity, when dealing with black holes we were mainly focussing on the fact that their spacetime geometry (as encoded in their metric) could emerge as solution of the set of equations of the theory under study. In this context, we have been interested in a classification of the possible solutions of our models as a function of the free parameters of the theory and, in this respect, we also studied the main physical characteristics of these solutions – such as their mass or their electric charge. Other more formal aspects were not investigated.

For this reason, to fit with the main focus of this thesis, in this section we will review the most important exact black hole solutions known in the context of general relativity by focussing on the form of the solution in an appropriate coordinate system and on the link between these different solutions. More complete discussions on their physical properties can be found in [Wald, 1984], [Carroll, 1997] or [Townsend, 1997], to cite a few examples.

1.5.1 Schwarzschild Black Hole

Famously, the first exact solution to Einstein equations (1.183) was found in 1916, soon after the equations were proposed, by Karl Schwarzschild; whose name was hence given to the solution. He studied the equations in vacuum ($T_{ab} = 0$) in the case of a static spherically symmetric spacetime.

In this text, despite focussing on the mathematical structures at hand, we carefully avoided the question of how spacetime symmetries can be formulated and studied in the context of differential geometry. Such a study would have been necessary to formally deal with the condition of spherical symmetry of our spacetime. Instead of properly correcting this flaw here, for the sake of conciseness, we will rely on an intuitive description.

The idea of symmetry is always related with the idea of invariance of a mathematical structure under a certain group of transformations. Here, for the spherical symmetry, this group should be the group of "spatial rotations", *i.e.* the special orthogonal group $SO(3)$. To see this as a symmetry of our spacetime, this group of transformations should act in a way that does not change spacetime's structure. Since, in general relativity, the geometrical structure of our spacetime is entirely encoded, apart from its differential structure, in the metric, the group of rotations should manifest via diffeomorphisms whose action preserves the metric of spacetime. At a technical level, this invariance of the metric will be encoded by demanding the vanishing of its Lie derivative (see remark 1.7) along the vector fields that will generate these isomorphisms. The implementation of the condition that the metric is static – that it does not "change in time" – will be done similarly by means of a one-parameter group of diffeomorphisms that will be interpreted as the "time translations" and that should leave the metric invariant. In fact, the existence of this "time translation isometry" corresponds to the idea that the spacetime is stationary. To really implement the idea that it is static, we should impose an extra condition which is a bit technical : that there exist a spacelike hypersurface orthogonal to the orbit of the isometry. We will comment on the consequence of this last requirement below.

With such an implementation of the notions of staticity and spherical symmetry, it is possible to construct coordinate systems adapted to the spacetime symmetries. If we denote such a coordinate system by $\{ x^\mu \} = \{ ct, r, \theta, \varphi \}$ one can show that the metric of a static spherically symmetric spacetime can always be written in the form

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -f(r)\ c^2\mathrm{d}t^2 + \frac{1}{g(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right), \end{aligned} \tag{1.190}$$

for two differentiable functions f and g. The fact that this is the metric of a static (and not only stationary) spacetime is encoded here in the fact that $g_{t\mu} = 0$, for $\mu = r, \theta, \varphi$. In other words, the extra condition defining staticity allows us to avoid the presence of terms of the form $\mathrm{d}t\mathrm{d}x^\mu$ (with $\mu \neq t$) in (1.190) when writing the metric in an appropriate coordinate system. This corresponds to the intuitive idea that, since the spacetime is static, nothing changes in time so that the time and space directions in (1.190) are completely "decorrelated".

In addition to this property, Schwarzschild's solution also add one important condition : the spacetime should be asymptotically flat. This time again, we are facing a condition that would deserve its own in-depth discussion. Since we want to focus on the construction of the solution, let us again rely on your intuition. The idea behind the concept of asymptotic flatness is well captured by its name. The idea is that, sufficiently far away from the centre of our spherically symmetric solution, we expect spacetime to look like a flat (Minkowski) spacetime. To state it differently, this corresponds to the idea that physical effects should decrease as one moves away from the source. In the context of our spherically symmetric spacetime endowed with the above coordinates adapted to its symmetries, this condition will manifest as a boundary condition fixing the behaviour of $f(r)$ and $g(r)$ as $r \to \infty$. Typically, if the solution should tend to a Minkowski spacetime as $r \to \infty$, we should have

$$f(r) \underset{r\to\infty}{\longrightarrow} 1,\ \ g(r) \underset{r\to\infty}{\longrightarrow} 1. \tag{1.191}$$

With the metric written as (1.190) and the boundary condition expressed as (1.191), one can solve the vacuum Einstein equations to obtain $f(r)$ and $g(r)$. On then finds

$$f(r) = 1 - \frac{r_s}{r} = g(r), \tag{1.192}$$

with

$$r_s = \frac{2\mathcal{G}M}{c^2}, \tag{1.193}$$

$\mathcal{G}$ being Newton's constant of gravitation, c the speed of light in vacuum and M a (positive) constant. This ((1.190) with (1.192)) is the celebrated Schwarzschild solution. The quantity r_s is known as the Schwarzschild radius.

We promised to not give too many details about this spacetime so let us simply mention that what makes this solution a black hole is the presence of an event horizon (a region inside of which any particle or light ray would be trapped forever) located at $r = r_s$ and the existence of a singularity (a point in spacetime at which the geometrical description breaks down since geometrical invariants diverges) at $r = 0$. It is, by the way, well known that, contrary to what happens for $r \to 0$, the pathological behaviour of the $g_{rr} = 1/g(r)$ metric component for $r \to r_s$ is a coordinate artefact that does not correspond to any pathology of the geometry itself.

To complete this discussion, we should also comment on the physical interpretation of the parameter M. This parameter is unfixed by the equations. This means that when we talked about "the" Schwarzschild solution, we were in fact referring to a one-parameter family of solutions (the parameter being either M or r_s). According to its units, the parameter M will be physically interpreted as the mass of the Schwarzschild black hole. It thus corresponds to the amount of mass that agglutinated inside the region of space enclosed with the Schwarzshild radius r_s *i.e.* inside the event horizon.

Despite being the first solution ever found to Einstein equations, the Schwarzschild solution already presents an interesting level of generality, see section 1.5.6.

1.5.2 Kerr Black Hole

Another very important black hole solution to Einstein equations in vacuum is the Kerr solution. Its importance relies on the fact that it corresponds to the most general solution of Einstein equations in vacuum that is stationary and asymptotically flat (see section 1.5.6 for a discussion of this property).

In terms of its geometry, the Kerr solution describes a spacetime that is stationary, axisymmetric and asymptotically flat. Axial symmetry basically corresponds to the idea of one specific axis of symmetry "in space". To sate it intuitively, instead of being invariant under the full group of spatial rotations $SO(3)$, an axisymmetric spacetime will only be invariant under the action of a subgroup corresponding to the rotations around a fixed axis. As for spherically symmetric spacetimes, it is possible to construct coordinates adapted to the symmetries of a stationary axisymmetric spacetime. Let $\{ x^\mu \} = \{ ct, r, \theta, \varphi \}$ be such a coordinate system.

In this adapted coordinate system, the Kerr solution is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\left(1 - \frac{r_s r}{\Sigma}\right) c^2\ dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ &\quad + \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma} \sin(\theta)^2\right) \sin(\theta)^2 d\varphi^2 - \frac{2 r_s r a \sin(\theta)^2}{\Sigma} c\ dt d\varphi, \end{aligned} \tag{1.194}$$

where r_s is defined as in (1.193),

$$a = \frac{J}{Mc}, \tag{1.195}$$

$$\Sigma = r^2 + a^2 \cos(\theta)^2, \tag{1.196}$$

and

$$\Delta = r^2 - r_s r + a^2. \tag{1.197}$$

Here, to contrast with the Schwarzschild solution, we can recognise that Kerr spacetime is stationary but non-static from the presence of a $\mathrm{d}t\mathrm{d}\varphi$ term in $\mathrm{d}s^2$.

The physical interpretation of the Kerr solution is very important. Just as for the Schwarzschild solution, "the" Kerr solution actually defines a family of solutions. In this case it is a two-parameter family (the two parameters being M and J or equivalently M and a, or r_s and a, ...). On account for the units, M will still be interpreted as the mass of the solution, while J will be interpreted as its angular momentum.

The geometry of Kerr spacetime is more involved than for Schwarzshild spacetime. Without giving a detailed presentation here, let us comment on some of the main points regarding the interpretation of the solution. A more in-depth discussion can be found in [Wald, 1984]. From (1.194), we see that the metric components can present a singularity if $\Sigma = 0$ or $\Delta = 0$. Similarly to the Schwarzshild case, one of these conditions is associated to a true (curvature) singularity while the other is only related to a coordinate singularity but allows finding the event horizon.

The singularity associated to $\Sigma = 0$ (*i.e.* $r = 0$ and $\theta = \pi/2$) is a true singularity. The interpretation of this singularity forces to take the interpretation of r as being a "radial coordinate" with a grain of salt.[1.60] More precisely, the locus of equation $r = 0$ cannot just be seen as a "point" as in the Schwarzshild case. The singularity appearing for $r = 0$ and $\theta = \pi/2$ should instead be interpreted as a ring singularity. This complicates the discussion of the topology of the solution but the important point for us here is that this indicates that the spacetime associated to the Kerr solution always contains a true singularity.

The other possible problem occurs if $\Delta = 0$. This case can be shown to correspond to a coordinate singularity that would locate the event horizon of the black hole. Of course, for this condition to be satisfied, one should have that $r_s^2 \geq 4a^2$, *i.e.* $M^4\mathcal{G}^2 \geq J^2c^2$. If this condition is satisfied, one finds that $\Delta = 0$ for $r = r_\pm$ with

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4a^2}\right). \tag{1.198}$$

[1.60] In fact, if one tries to intuitivelly interpret the coordinates (ct, r, θ, φ) in terms of "cartesian coordinates" (ct, x, y, z) with the z-axis alligned with the axis of symmetry of the solutions, r would correspond to the distance from the orgin along the z-axis but it is $\sqrt{r^2 + a^2}$ which would correspond to the distance from the origin measured in the "$z = 0$ plane" (which would correspond to $\theta = \pi/2$). The surface corresponding to $r = 0 \wedge \theta = \pi/2$ would then correspond to a circle of radius a in the "$z = 0$ plane". See [Visser, 2007] for more details.

In this case, one always has that $r_\pm > 0$ and that $r_+ \geq r_-$, so $r = r_+$ defines an (outer) horizon that always "hides" the ring singularity at $r = 0 \wedge \theta = \pi/2$ and the (inner) horizon at $r = r_-$. The surface $r = r_+$ will then correspond to the event horizon in this case. Also, since $r_s \geq r_+$, the event horizon is not located at $r = r_s$ anymore – unless $a = 0$.

On the contrary, if $r_s^2 < 4a^2$, there is no solution to the equation $\Delta = 0$. In this case the singularity is "naked" and the solution cannot be interpreted as a black hole. In the following, we should thus only consider the situation $r_s^2 \geq 4a^2$.

Another distinguishing feature of the Kerr solution is that, unlike in the Schwarzshild case (according to (1.192)), the event horizon (the region where $1/g_{rr} \to 0$) does not match with the region at which the "time direction becomes spacelike" ($g_{tt} \geq 0$). To find the region at which g_{tt} will changes sign, one should solve $\Sigma - r_s r = 0$. This then gives us two surfaces of equation

$$r_\pm^E = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4a^2\cos(\theta)^2}\right), \tag{1.199}$$

With $r_\pm^E > 0$, $r_+^E \geq r_-^E$. Between these two surfaces, $g_{tt} \geq 0$. Note that, since $r_+^E \geq r_+ \geq r_-^E$, there is a region (called the ergosphere) that is outside the event horizon ($r \geq r_+$) and for which this phenomenon occurs ($r_+^E \geq r$). We should not comment much more on the properties of the ergosphere here as it will be of little interest for the rest of our discussion. Yet, this already reveals another important difference with the Schwarzshild solution.

On account of these properties, the Kerr solution with $M^4\mathcal{G}^2 \geq J^2c^2$ might be seen as describing a rotating black hole of mass M and angular momentum J with an event horizon at $r = r_+$. The stationary but non-static character of the spacetime further reinforces the idea that the rotation happens "at a steady rate".

As a final comment on this solution, we should mention that the solution smoothly reduces to the Schwarzschild solution in the case $a \to 0$. In this sense, the Schwarzshild solution can be seen as a non-rotating (hence recovering the static character) black hole of mass M.

1.5.3 Einstein-Maxwell Equations

The first two black hole solutions discussed in this section were vacuum solutions. The next two solutions that we should present correspond to solutions to Einstein-Maxwell equations. These correspond to the field equations obtained by coupling general relativity to Maxwell's electromagnetism.

Before entering the discussion of the solutions, we should say a word on how the model is constructed.

Einstein-Maxwell theory gives us an interesting example raising the question on how to convey physical interactions well described (in the absence of gravitation) on Minkowski spacetime to curved spacetime. According to Einstein equivalence principle, the laws describing electromagnetic processes on curved spacetime should locally reduce to their special relativistic form once expressed in an appropriate coordinate system around a point. It is well known that this constraint is a necessary but not a sufficient one to define a mathematically univoque procedure mapping the laws of physics from flat (Minkowski) to curved spacetime. Indeed, there are, in general, infinitely many non-equivalent equations on curved spacetime that reduces to a given equation on flat spacetime.

In lack of a clear mathematical obligation to favour one equation over the others, one should thus define a principle dictating the transition from flat to curved spacetime – and then test the viability of this principle in terms of its experimental successes. According to the idea that the passage from special to general relativity mainly account to a passage from Minkowski metric (and its associated, trivially defined, flat, metric compatible and torsion-free connection) to a generic Lorentzian metric and its associated Levi-Civita connection, a natural way to proceed consist in applying the following replacement rule to the laws of physics from special relativity

$$\begin{cases} \boldsymbol{\eta} & \rightarrow \boldsymbol{g} \\ \boldsymbol{\partial}_\mu & \rightarrow \nabla_\mu \end{cases} . \tag{1.200}$$

This procedure is known as the minimal coupling procedure. By construction, it will automatically lead to the correct description on flat spacetime. More than that, when observed at a point using Riemann normal coordinates, one will recover the fact that the equations reduce to their special relativistic form.

The last thing that we should point out in this respect is that, despite its seeming definiteness, (1.200) does not automatically lead to a univoque procedure in general. Typically, in the case of Maxwell's equations, on account for (1.41), one could wonder if the principle should be applied to the equations (1.38) in terms of the Faraday tensor or the vector potential. In this case (see [Wald, 1984], [Carroll, 1997] or [Misner et al., 1973]), contrary to what might seem intuitive, one will not arrive at the same equations in terms of the vector potential with both procedures. This ultimately rely on the fact that, when second (or higher order) derivatives are implied in the equations, the minimal coupling procedure cannot give a prescription on the order in which the derivatives should be taken. On flat space, indeed, when expressed in an inertial frame, the laws of physics involves partial derivatives which, as we know, commute with each other. On the contrary, on curved spacetime, even when considered in a coordinate system, covariant derivatives do not commute with each other on account of (1.104). What would result from the application of (1.200) will then depend in which order one writes the flat spacetime partial derivatives.

To resolve this ambiguity in the case of Maxwell's equations one will usually rely on two facts. First of all, in classical situations, the vector potential is not directly measurable in electromagnetism[1.61], what is measurable are the electric and magnetic fields, that is to say, the Faraday tensor. This then argues

[1.61] When adding quantum mechanics in the picture, this statement should be contrasted in the context of the Aharonov–Bohm effect. Yet, here, since we are discussing classical (not quantized) teories – and this because there is, to this date, no satisfactory quantum description of gravity – we can consider that no direct measure of the vector potential should be possible.

in favour of an application of the principle in (1.38) in terms of the Faraday tensor instead of the vector potential. Second, this idea is further reinforced by the fact that, more than just getting covariant equations, in the context of electromagnetism, one would also like to preserve the conservation law of the 4-vector current density (1.42) when written on curved spacetime. Thanks to the specificities of the Levi-Civita connection (namely (1.157)), this property is ensured if one applies (1.200) to (1.38) directly in terms of the Faraday tensor. This is not automatically the case if one does it in terms of the vector potential.

Finally, it is also worth noting that the form of (1.41) will be unchanged on curved spacetime. This may look like a consequence of (1.200) and the vanishing of torsion of the Levi-Civita connection but it is actually deeper than that. Fundamentally, the antisymmetry of the Faraday tensor motivates to encode it as a 2-form $\mathbf{F}$. In this case, (1.38b) encodes the fact that this differential form is exact. In this context, (1.41) corresponds to the writing in a coordinate system of the relation $\mathbf{F} = \mathrm{d}\mathbf{A}$, where $\mathbf{A} = A_\mu \mathrm{d}x^\mu$ is a 1-form that corresponds to the "vector potential". Since the geometry of differential forms is inherent to the structure of a differential manifold, it is thus independent from the chosen connection. The link between $F_{\mu\nu}$ and A_μ should then always assume the form (1.41).

These observations then lead to the following formulation of Einstein-Maxwell equations. The system is defined on a differential manifold $\mathcal{M}$ endowed with a Lorentzian metric $\boldsymbol{g}$ and the corresponding Levi-Civita connection. On top of that, one defines the vector potential (by means of a 1-form $\mathbf{A}$). Once written in a coordinate system $\{ x^\mu \}$, the field equations of the system are given by Einstein equations (1.183), where the components of the energy-momentum tensor $T_{\mu\nu}$ are given by (1.46), and Maxwell fields equations (1.38a) with $F_{\mu\nu}$ given by (1.41).[1.62] Of course, in each relation involving the electromagnetic variables (except for (1.41) as a matter of principle), the aforementioned special relativistic formulas should be adapted by means of (1.200).

In the next two paragraphs of this section, we should study two important solutions of the Einstein-Maxwell field equations.

1.5.4 Reissner-Nordström Black Hole

In the realm of exact black hole solutions in (4 dimensional) general relativity, the Reissner-Nordström solution can be seen as a complementary extension of the Schwarzschild solution compared to the Kerr solution. While Kerr spacetime generalises the Schwarzchild one by adding an angular momentum, the Reissner-Nordström solution does it by adding, instead, an electric charge. As the schwarzschild solution, Reissner-Nordström's spacetime is assumed to be static, spherically symmetric and asymptotically flat. In a coordinate system $\{ x^\mu \} = \{ ct, r, \theta, \varphi \}$ adapted to this situation, the metric will thus assume the

[1.62] In this case, (1.38b) becomes an identity following from (1.41).

form

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -f(r)\ c^2\mathrm{d}t^2 + \frac{1}{g(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right). \end{aligned} \tag{1.201}$$

Since the Reissner-Nordström solution concerns a situation for which our black hole is only electrically charged, and in order to have spherical symmetry also on the electromagnetic side, the vector potential is further assumed to be of the form

$$\mathbf{A} = A_\mu \mathrm{d}x^\mu = V(r)\mathrm{d}t. \tag{1.202}$$

Here, the condition of asymptotic flatness will manifest on $f(r)$ and $g(r)$ via (1.191). Since the function $V(r)$ will source the spacetime curvature by means of Einstein equations, to get asymptotic flatness, one should thus also demand that

$$V(r) \underset{r\to\infty}{\longrightarrow} V_\infty. \tag{1.203}$$

Also, since the field equation only involves derivatives of $\mathbf{A}$, this constant can always be fixed so that $V_\infty = 0$.

Solving Einstein-Maxwell equations by taking (1.191) and (1.203) into account, the Reissner-Nordström solution then corresponds to

$$f(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} = g(r) \tag{1.204}$$

and

$$V(r) = -\frac{Q}{4\pi\varepsilon_0 r}, \tag{1.205}$$

with r_s given by (1.193) and

$$r_Q^2 = \frac{\mathcal{G}\ Q^2}{4\pi\varepsilon_0 c^4}. \tag{1.206}$$

As for the former solutions, the Reissner-Nordström solution actually depicts a family of solutions. Here, it is a two-parameter family (the parameters being M and Q or, equivalently r_s and r_Q, ...). On account of the units, M will be interpreted as the mass of the solution and Q as its electric charge.

Qualitatively speaking, the Reissner-Nordström solution present some similarities with both the Schwarzshild and Kerr solutions. As in the Schwarzshild solution, the only conditions that could lead to a pathological behaviour of the metric components are $r = 0$ and $f(r) = 0$. Also, like the Schwarzshild solution, one can prove that the singularity at $r = 0$ is a true singularity while the singularity associated to $f(r) = 0$ is only a coordinate one. However, more like the Kerr solution, the condition $f(r) = 0$, *i.e.* $r^2 - r_s r + r_Q^2 = 0$, does not necessarily have solutions. If $r_s^2 \geq 4r_Q^2$ – that is $M^2\mathcal{G} \geq Q^2/(4\pi\varepsilon_0)$ – the equation admits two solutions at $r = r_\pm$ with

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right). \tag{1.207}$$

In this case, one will still have $r_s \geq r_+ \geq r_- > 0$, so that the locus $r = r_+$ will correspond to the event horizon of the solution.

If $r_s^2 < 4r_Q^2$, the equation admits no solution. There is thus no event horizon to "hide" the singularity and the solution fails to describe a black hole.

On account of these properties, we have that the Reissner-Nordström solution with $M^2\mathcal{G} \geq Q^2/(4\pi\varepsilon_0)$ might be seen as a static spherically symmetric and asymptotically flat black hole of mass M and electric charge Q, with an event horizon at $r = r_+$.

Similarly to the Kerr solution, we see that the Reissner-Nordström solution reduces to the Schwarzschild solution in the limit $Q \to 0$.

1.5.5 Kerr-Newman Black Hole

The last 4 dimensional exact black-hole solution that we would like to comment on is the Kerr-Newman black hole. It completes the above presentation in the sense that it corresponds to a solution of Einstein-Maxwell field equations that has the same geometrical symmetries as the Kerr black hole. According to the above interpretations, it should thus correspond to a charged, steadily rotating and asymptotically flat black hole. As always, one can construct coordinates $\{ x^\mu \} = \{ ct, r, \theta, \varphi \}$ that will be adapted to the whished symmetries.

In such a coordinate system, the Kerr-Newman solution is given by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(1 - \frac{r_s r - r_Q^2}{\Sigma}\right) c^2\ \mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 \\ &+ \left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin(\theta)^2}{\Sigma}\right)\sin(\theta)^2 \mathrm{d}\varphi^2 - \frac{2\left(r_s r - r_Q^2\right) a \sin(\theta)^2}{\Sigma} c\ \mathrm{d}t\mathrm{d}\varphi, \end{aligned} \tag{1.208}$$

and

$$\mathbf{A} = A_\mu \mathrm{d}x^\mu = -\frac{Q}{4\pi\varepsilon_0}\frac{r}{\Sigma}\left(\mathrm{d}t - a\sin(\theta)^2 \mathrm{d}\varphi\right), \tag{1.209}$$

where

$$\Delta = r^2 - r_s r + a^2 + r_Q^2, \tag{1.210}$$

and where r_s, a, Σ and r_Q are respectively defined by (1.193), (1.195), (1.196) and (1.206). Here, we then see that the Kerr-Newman solution describes a three-parameter family of solutions (the parameters being M, J and Q, or any relevant combination of related quantities).

The Kerr-Newman solution will qualitatively present all the same features as the Kerr solution. Indeed, the metric components can present a singularity if $\Sigma = 0$ or $\Delta = 0$.

The singularity associated to $\Sigma = 0$ (*i.e.* $r = 0$ and $\theta = \pi/2$) is a true singularity (with the same properties as for the Kerr solution) while the one associated to $\Delta = 0$ is not and will give access to the event horizon.

For the condition $\Delta = 0$ to be satisfied, one should have that $r_s^2 \geq 4\left(a^2 + r_Q^2\right)$, *i.e.* $M^2\mathcal{G}^2 \geq J^2c^2/M^2 + \mathcal{G}Q^2/(4\pi\varepsilon_0)$. If this condition is satisfied, one finds that $\Delta = 0$ for $r = r_\pm$ with

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4\left(a^2 + r_Q^2\right)}\right). \tag{1.211}$$

In this case, one will still have $r_s \geq r_+ \geq r_- > 0$, so that the locus $r = r_+$ will correspond to the event horizon of the solution.

If the condition $r_s^2 \geq 4\left(a^2 + r_Q^2\right)$ is violated, there is no solution to the equation $\Delta = 0$ and the solution cannot be interpreted as a black hole. So, here again, we will only consider the case where the condition is satisfied.

The solution will then also present an ergosphere since the regions at which g_{tt} will changes sign should be found from $\Sigma - r_s r + r_Q^2 = 0$. This then gives us two surfaces of equation

$$r_\pm^E = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4a^2\cos(\theta)^2 - r_Q^2}\right), \tag{1.212}$$

With $r_+^E \geq r_+ \geq r_-^E > 0$. This thus gives an ergosphere for $r_+^E \geq r \geq r_+$.

On account of these properties, if $M^2\mathcal{G}^2 \geq J^2c^2/M^2 + \mathcal{G}Q^2/(4\pi\varepsilon_0)$, the Kerr-Newman solution can be seen as a charged rotating black hole of mass M, electric charge Q and angular momentum J with an event horizon at $r = r_+$.

This solution is thus the most general one among the four that we presented here. It generalises the Schwarzschild solution by adding both a charge and an angular momentum. We see that, in the limit $a \to 0$, the solution reduces to the Reissner-Nordström solution, in the limit $r_Q \to 0$, the solution reduces to the Kerr solution and then, obviously, the solution reduces to the Schwarzschild solution if both $a \to 0$ and $r_Q \to 0$.

1.5.6 Israel's Theorem

To complete the interpretation of the solutions presented in this section, we should here comment on their level of generality. This part of the discussion mostly comes from the corresponding one in chapter 12 of [Wald, 1984] – One should also consider the "box 33.1" of [Misner et al., 1973].

The importance of this discussion relies on the fact that, to this date, the most general solution of Einstein equations is far from being known, even in vacuum. This observation may get one worried about the quite small number of exact solutions that are known. Especially considering that we mostly know stationary solutions of Einstein's equations in vacuum.

However, in this respect, a big ray of sunshine comes from a series of results obtained by Israel, Carter, Hawking, Ellis and Robinson between 1967 and 1975 that allowed to demonstrate that the Kerr metric describes the most general family of stationary, asymptotically flat black holes solving Einstein equations in vacuum. It is on these different arguments that we would like to say a few words.

Israel's theorem for Einstein equations

The first argument of this proof – which was historically one of the last to be formally established – is due to Hawking. He showed that the intersection of the event horizon of a stationary black hole and a Cauchy surface must be topologically identical to the two-sphere. In other words, "at any time" the event horizon of a stationary black hole has the same topology as a sphere.

The second step, also due to Hawking, is the proof that a stationary black hole must either be static or axisymmetric. Hawking actually showed that the event horizon of a stationary black hole is a lightlike hypersurface and that the generators of this hypersurface either coincide with the time-translation isometry or with the vector field generating another one-parameter family of isometries that commutes with the time isometry. In the latter case, by taking a linear combination of the two isometries, Hawking obtained an isometry whose orbits were closed *i.e.* an axial isometry. Note that the fact that the generators of the horizon are Killing vectors makes this horizon a Killing horizon, by definition.

Let us emphasise the fact that, up to now, the results obtained are rigorous results of differential geometry more than of general relativity in the sense that we have not yet imposed Einstein equations (or any other field equation fixing the behaviour of the spacetime geometry).

The case of a topologically spherical static black hole was studied by Israel who showed that the Schwarzschild solution covers the most general family of such black holes.

Finally, the case of a stationary but non-static topologically spherical black hole was studied by Carter and Robinson who established that the most general family of black holes satisfying these properties was a two-parameter family. As the Kerr metric covers such a two-parameter family (the parameters being M and J), the Kerr solution thus describes the most general family of axisymmetric, stationary and asymptotically flat black holes solving Einstein equations in vacuum.

On thus arrives to the fact that the Kerr solution describes the most general family of stationary and asymptotically flat black hole solutions from Einstein equations in vacuum. Indeed, according to the above discussion, the most general family of solutions stationary but non-static is described by (1.194) and the most general family of static solutions by (1.190). But (1.190) can be obtained from (1.194) in the limit $J \to 0$, hence allowing for the above formulation.

Israel's theorem for Einstein-Maxwell equations

It is interesting to note that, as stated here, these results do not necessarily apply as straightforwardly to solutions of Einstein-Maxwell equations. It is thus important to say that these results generalise to the case of Einstein-Maxwell equations, giving a proof of the fact that the Kerr-Newman solution indeed describes the most general family of solutions to Einstein-Maxwell equations that are stationary and asymptotically flat.

This family of results is usually collectively referred to as Israel's theorem. This term then refers to the fact that the Kerr-Newman solution describes the most general family of stationary and asymptotically flat black hole solutions of Einstein-Maxwell equations.

Israel's theorem is also referred to as the no-hair theorem for black holes *a.k.a.* the proof that "black holes have no hair". This name comes from the fact that what Israel's theorem establishes, physically, is that, should general relativity and the minimally coupled version of Maxwell's electromagnetism be correct, the final result of the gravitational collapse of a body, if it ends up to be a black hole, should be described by the Kerr-Newman solution. The link to the "final result of the gravitational collapse of a body" is that, again assuming that this collapse will result in the formation of a black hole, the final state is expected to be stationary (if it is the final sate, no dynamical processes remains, leading to this hypothesis) and asymptotically flat (to ensure that the effects decrease far away from the source).[1.63] Israel's theorem thus shows that this final state will only depend on three parameters, M, J and Q, respectively interpreted as the mass, angular momentum and electric charge of the solution. It is in this sense that "black holes have no hair", *i.e.* depend on no other parameter than these three.

Black Holes have No-Hair . . . Unless when they do

To conclude this discussion, it is thus important to emphasise that, if Israel's theorem relies on solid mathematical hypothesis, the very generic and popular statement that "black holes have no hair" is, on the contrary, very fuzzy. Indeed, what Israel's theorems says is "black holes obtained as stationary and asymptotically flat solutions of Einstein-Maxwell equations are characterised by the Kerr-Newman solution (1.208–1.210) and are thus fully specified once given the three parameters M, J and Q."

This precision might seem repetitive and redundant considering the above discussion but, due to the very subject of this thesis, it is important that we acknowledge that the catchy sentence "black holes have no hair" have anchored in the mind of a part of the physics community the idea that black holes (in the broader possible sense of the term) cannot, by any means, be endowed with non-trivial field profiles (except for an electric field as in (1.209)). This simplistic and extreme understanding of the sentence is, of course, completely wrong. Israel's theorem, as its name suggests, is a theorem that does thus rely on some specific hypothesis and cannot make any claim in situations that does not fulfil them. In chapter 2, we should see several no-scalar-hair theorems proving the impossibility, under some hypothesis, to endow a black hole with a

[1.63] Note that, to really be able to say that a Kerr-Newmann black hole is the result of the gravitational collapse of a body in any possible case, one should also, in addition to an electromagnetic field, consider what would happen to fields describing the strong and weak nuclear forces and conclude that charges associated to these interactions would indeed end "hidden" at the end of the process. This question is out of the scope of our discussion but we should point out that it has been addressed, for more details see [Wald, 1984] and box 33.1 of [Misner et al., 1973].

non-trivial scalar field but also ways to circumvent these results ! In particular, all the black hole solutions studied in this thesis – and any scalarised black hole solution across the literature – are explicit counter-examples of the extremist (and wrong) understanding of the sentence “black holes have no hair”. For a more detailed discussion in this respect, one could see chapter 2 and [Herdeiro and Radu, 2015].

To avoid closing this remark without any form of nuance, we should also acknowledge that, even though black hole solutions allowing circumventing Israel’s theorem (or other types of no-hair theorems) can be constructed in some circumstances, the question of the stability of these solutions remains an important issue to be able to really consider these solutions as physically consistent. Indeed, to really be able to interpret these solutions as the possible result of the collapse of some astrophysical body, they should be proven stable against perturbations to establish their physical viability. Once a solution is constructed, this question is thus very important to address. That being said, this is a bit beyond the scope of the present discussion.

1.6 Teleparallel Equivalent of General Relativity

In this section, we present in a nutshell the main characteristics of two theories of gravity formulated using different geometrical tools than those of general relativity but leading to equivalent physical predictions. We emphasise some of the main differences of interpretation and briefly discuss how these classically equivalent theories point toward different ways to extend the content of general relativity.

1.6.1 Geometrical Setup

As we saw in section 1.4, the theory of general relativity is formulated on a curved spacetime. More precisely, it assumes that spacetime is a differential manifold endowed with a Lorentzian metric and equipped with the associated Levi-Civita connection. We reviewed how the physical assumptions of general relativity combined with this geometrical structure to lead to predictions regarding the motion of test particles, via equation (1.173), and to the Einstein field equations relating this spacetime geometry to the matter content sourcing the gravitational field, via (1.183) or (1.184).

More importantly, in section 1.4.5, we saw that Einstein equations (with or without a cosmological constant) can be derived from a variational principle if we take the Lagrangian density to be the Ricci scalar of the Levi-Civita connection (1.185). This last observation is at the core of our presentation of the construction below.

First of all, it is interesting to further emphasise that the use of the Levi-Civita connection is a (very reasonable) postulate of general relativity but not a logical necessity.[1.64] That being said, it is also important to stress that, if the use of this specific connection is not mandatory, the presence of a metric on our spacetime manifold – in the sense of definition (1.136) – is arguably unavoidable as soon as we need to introduce a causal structure. So it is conceivable to construct a covariant theory of gravity based on a spacetime manifold defined as a differential manifold endowed with a (*a priori* Lorentzian) metric and an independent linear connection. But it is also clear that, even if the Levi-Civita connection is not the one that we physically interpret as THE connection of the theory, it will still be defined (or, say, definable) since, in any case, we need a metric.

To state it differently, unlike what is done in general relativity, the Levi-Civita connection does not need to be given any ontological status in the theory, yet, mathematically speaking, such a connection is automatically determined by the choice of a metric and this metric is always given an ontological status as it is used to encode the causal structure of spacetime. This fact is important to emphasise since, as we know from section 1.3.11, in such a situation, any linear connection can be related to the Levi-Civita connection according to (1.162).

Ricci scalar of a generic linear connection in the presence of a metric

The fact that, in the presence of a metric, any linear connection can be described in terms of its coefficients using (1.162) – or equivalently (1.161) – imply a relationship between the curvature tensor of a generic linear connection and the one of the Levi-Civita connection.

Using (1.161) in (1.107) – or (1.162) in (1.106) – one will get that, using a generic basis of vector fields $\{\, \vec{e}_{(a)} \,\}$,

$$\begin{aligned} R^a{}_{bcd} = \ & \overset{\circ}{R}{}^a{}_{bcd} + 2\left(D^a{}_{k[c|}K^k{}_{b|d]} + K^a{}_{k[c|}D^k{}_{b|d]}\right) \\ & + 2\left(D^a{}_{k[c|}D^k{}_{b|d]} + \overset{\circ}{\nabla}_{[c|}D^a{}_{b|d]}\right) \quad , \\ & + 2\left(K^a{}_{k[c|}K^k{}_{b|d]} + \overset{\circ}{\nabla}_{[c|}K^a{}_{b|d]}\right) \end{aligned} \tag{1.213}$$

where we used a $\circ$ to denote quantities computed using the Levi-Civita connection. From this relation, by contraction, one immediately obtains a similar relation for the Ricci tensor

$$\begin{aligned} R_{bd} = \ & \overset{\circ}{R}_{bd} + 2\left(D^a{}_{k[a|}K^k{}_{b|d]} + K^a{}_{k[a|}D^k{}_{b|d]}\right) \\ & + 2\left(D^a{}_{k[a|}D^k{}_{b|d]} + \overset{\circ}{\nabla}_{[a|}D^a{}_{b|d]}\right) \quad , \\ & + 2\left(K^a{}_{k[a|}K^k{}_{b|d]} + \overset{\circ}{\nabla}_{[a|}K^a{}_{b|d]}\right) \end{aligned} \tag{1.214}$$

[1.64] This choice is, from a modern point of view, strongly tied to the physical requirement that gravity cannot be interpreted as a force in general (*i.e.* unless some specific conditions are met). We shall come back on this important idea in section 1.6.2.

and, by contraction with the metric, for the Ricci scalar

$$\begin{aligned} R = \; & \overset{\circ}{R} + 2\left(D^{a}{}_{k[a|}K^{kb}{}_{|b]} + K^{a}{}_{k[a|}D^{kb}{}_{|b]}\right) \\ & + 2\left(D^{a}{}_{k[a|}D^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}D^{ab}{}_{|b]}\right) \\ & + 2\left(K^{a}{}_{k[a|}K^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}K^{ab}{}_{|b]}\right) \end{aligned} \quad . \tag{1.215}$$

This last relation can finally be rewritten as

$$\begin{aligned} \overset{\circ}{R} = \; & R - 2\left(D^{a}{}_{k[a|}K^{kb}{}_{|b]} + K^{a}{}_{k[a|}D^{kb}{}_{|b]}\right) \\ & - 2\left(D^{a}{}_{k[a|}D^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}D^{ab}{}_{|b]}\right) \\ & - 2\left(K^{a}{}_{k[a|}K^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}K^{ab}{}_{|b]}\right) \end{aligned} \quad . \tag{1.216}$$

Before going further, let us emphasise that in the right-hand side of (1.216) the curvature, disformation and contorsion tensors are computed from the generic linear connection but that the covariant derivatives are covariant derivatives with respect to the Levi-Civita connection. The importance of this point is related to (1.157).

The left-hand side of (1.216), *i.e.* the Ricci scalar of the Levi-Civita connection, is the Lagrangian density of the Einstein-Hilbert action (1.185) (without cosmological constant). Equation (1.216) then shows that, given any linear connection, one can construct a Lagrangian density equivalent to the one of general relativity. The Lagrangian density of this theory will be given by the right-hand side of (1.216) – in which we can possibly neglect the total divergencies on account of (1.157).

Of course, following the way we introduced it, one could question the naturalness of a theory involving such a Lagrangian density. In particular, one could wonder if such a theory can stand on its own from a physical perspective. One could indeed doubt that such a theory can emerge from "reasonable" physical postulates, without a systematic reference to general relativity to sustain it. We will come back on this important issue at the end of this section. Before that, let us be a bit more precise on the geometrical structures that we will be interested in during this section by discussing a bit more consequences of (1.216) in two important special cases.

Teleparallel equivalent of general relativity

The first example of a geometrical setup that differs from that of general relativity but can lead to equally successful predictions consists in a 4 dimensional differential manifold endowed with a Lorentzian metric and a flat metric-compatible linear connection. The properties of this type of connection have been discussed in section 1.3.12. In other words, the geometrical setup here consists in a 4 dimensional teleparallel manifold further equipped with a Lorentzian metric.

In this case, it is interesting to remark that, apart from the metric, the geometrical properties of our spacetime will come from the torsion of the connection. In this case, contrarily to general relativity, this property cannot be attributed to the metric and should really be seen as a property of the connection itself. As a consequence, if we define our physical spacetime simply as a differential manifold and see the metric and the connection as additional structures placed on top of it, one cannot say that torsion is a property of spacetime. One can circumvent this by defining the physical spacetime as the full structure (manifold+metric+connection) but, even in this case, part of the geometrical properties of this spacetime (typically, its torsion) will be independent of the spacetime metric.

Since the connection is assumed to be flat ($\mathbf{R} \equiv 0 \Rightarrow R \equiv 0$) and metric compatible ($\mathbf{Q} \equiv 0 \Rightarrow D^a{}_{bc} \equiv 0$), (1.216) simplify as

$$\mathring{R} = -2\ K^a{}_{k[a|}K^{kb}{}_{|b]} - 2\ \mathring{\nabla}_{[a|}K^{ab}{}_{|b]}. \tag{1.217}$$

This relation can be rewritten in a more compact form as

$$\mathring{R} = -T + B, \tag{1.218}$$

where

$$\begin{aligned} T &\coloneqq 2\ K^a{}_{k[a|}K^{kb}{}_{|b]} = K^a{}_{ka}K^{kb}{}_b - K^a{}_{kb}K^{kb}{}_a \\ &= \frac{1}{4}T_{abc}T^{abc} + \frac{1}{2}T_{abc}T^{cba} - T^a{}_{ak}T^b{}_b{}^k \\ &= \frac{1}{2}T_{abc}\left(K^{bca} + g^{ac}g^{kb}T^l{}_{lk} - g^{ab}g^{dc}T^l{}_{ld}\right) =: \frac{1}{2}T_{abc}\ S^{abc} \end{aligned} \tag{1.219}$$

and

$$\begin{aligned} B &\coloneqq -2\ \mathring{\nabla}_{[a|}K^{ab}{}_{|b]} = \mathring{\nabla}_b K^{ab}{}_a - \mathring{\nabla}_a K^{ab}{}_b \\ &= 2\ \mathring{\nabla}_a\left(T^b{}_{bc}\ g^{ca}\right) =: 2\ \mathring{\nabla}_a\left(T_{\text{vec}}\right)^a. \end{aligned} \tag{1.220}$$

The scalar field T defined by (1.219) is known as the torsion scalar. The $(3,0)$-tensor field of components S^{abc} defined by identification in (1.219) is known as the superpotential. The vector field $\vec{T}_{\text{vec}}$, constructed using a trace of the torsion tensor and a contraction with the metric, defined by identification in (1.220) is usually referred to as the vector torsion.

Equation (1.218) then suggests the existence of a theory of the gravitational interaction based on the torsion tensor of a Weitzenböck connection that would lead to field equations equivalent to Einstein equations of general relativity. This theory is known as the teleparallel equivalent of general relativity (abbreviated TEGR or $\mathrm{GR}_{||}$).

We will elaborate a bit more on the field equations of TEGR in section 1.6.3 but we first want to comment on the consequences of this new geometrical setup on the interpretation of the gravitational interaction by investigating the motion of test particles. This is the topic of section 1.6.2.

Before entering this discussion, we briefly mention below, for completeness, another geometrical setup that can be interesting when looking at alternatives to general relativity suggested by (1.216).

Symmetric teleparallel equivalent of general relativity

The second example of a geometrical setup that differs from that of general relativity but can lead to equally successful predictions consists in a 4 dimensional differential manifold endowed with a Lorentzian metric and a flat torsion-free linear connection. It is worth emphasising that, since we have a metric defined on our spacetime manifold, despite having vanishing curvature and torsion, the connection is not necessarily trivial since it can exhibit non-metricity.

In this case, apart from the metric, the properties of spacetime will be defined by the non-metricity tensor. The situation here is, a bit more nuanced than in the previous case. Indeed, if the non-metricity tensor cannot entirely be seen as a property of the metric (in the sense that it depends on the connection) it cannot either be seen as coming only from the connection (since it is defined from the covariant differential of the metric tensor) even though specifying the non-metricity tensor will fully define the connection.

Since the connection is assumed to be flat ($\mathbf{R} \equiv 0 \Rightarrow R \equiv 0$) and torsion-free ($\mathbf{T} \equiv 0 \Rightarrow K^a{}_{bc} \equiv 0$), (1.216) simplify as

$$\overset{\circ}{R} = -2\ D^a{}_{k[a|} D^{kb}{}_{|b]} - 2\ \overset{\circ}{\nabla}_{[a|} D^{ab}{}_{|b]}. \tag{1.221}$$

Similarly to what we discussed here above for TEGR, this relation establishes that, given a flat torsion-free linear connection, the Lagrangian density

$$Q := -2\ D^a{}_{k[a|} D^{kb}{}_{|b]} = D^a{}_{kb} D^{kb}{}_a - D^a{}_{ka} D^{kb}{}_b \tag{1.222}$$

will give rise to field equations equivalent to Einstein equations but where the tensor field encoding gravity is the non-metricity tensor of the connection.

The theory based on the Lagrangian density (1.222) is known as the symmetric teleparallel equivalent of general relativity (STEGR for short). The name can be understood from the two conditions imposed on the connection. Since the curvature vanishes, the manifold will present most of the properties of a teleparallel manifold as discussed in section 1.3.8 excepted that one cannot construct a tetrad for which the connection coefficients vanish as discussed in section 1.3.12. On the other hand, the vanishing of torsion implies that the Christoffel symbols of the connection are symmetric under the exchange of their two lower indices, according to (1.122). This last property did not hold in TEGR.

In the rest of this section, we will focus our discussion on TEGR. Most of the conclusions and interpretations should hold similarly in STEGR. This choice is motivated by two complementary reasons. The first one is related to the naturalness of the theory and is briefly addressed in section 1.6.4. The second reason is more pragmatic. An in-depth discussion of STEGR would not be as meaningful in the context of this thesis as for TEGR. Indeed, TEGR is the framework in which the most recent paper of this thesis takes place, see appendix F. It is then more important to present the features and consequences of this theory for the sake of our discussion.

1.6.2 Motion of Pointwise Particles

Let us then assume to work on a Weitzenböck manifold for the rest of this section. In the previous paragraph, we gave strong indications of the possibility to construct a theory where the gravitational interaction is encoded in the torsion of a Weitzenböck connection and leading to field equations equivalent to the Einstein equations of general relativity. Before digging into the details of these field equations, we should point out that the physical content of general relativity does not reduce to Einstein field equations. General relativity also specifies how the presence of a gravitational field influences the motion of pointwise particles. For a covariant theory of gravity to be completely equivalent to general relativity, it should also address this question.

As reviewed in section 1.4, in general relativity, one postulates that test bodies should move along the geodesics of the curved spacetime *i.e.* along the geodesics of the Levi-Civita connection. To obtain the exact same dynamical content as general relativity, our theory based on a Weitzenböck spacetime should then rely on an equivalent equation. One then faces again the question of the naturalness of this postulate. Can we make sense of the fact that pointwise particles will *de facto* move along the geodesics of the Levi-Civita connection associated to the spacetime metric without explicitly trying to cohere with general relativity?

The answer is yes. Indeed, we already pointed out that this postulate from general relativity could be equivalently stated as the fact that test bodies should follow trajectories that extremises the length functional (1.163). Stated in this way, the postulate does not directly rely to the presence or not of a Levi-Civita connection on our spacetime. It only requires the presence of a metric to be able to make sense of (1.163). And, once again, this postulate can naturally arise as a covariant generalisation of the postulate from classical mechanics stating that free particles move along straight lines when observed in inertial frames without any direct reference to the Levi-Civita connection. We thus have good reasons to use it for our theory independently of the fact that it will lead to general relativity's predictions.

Of course, the computation of the Euler-Lagrange equations associated to (1.163) will automatically lead to (1.173) which will *de facto* correspond to the geodesics of the Levi-Civita connection but the consequences of the postulate will be significantly different in terms of interpretation.

Since the Levi-Civita connection does not have any ontological status in the teleparallel theory, we aim to interpret (1.173) in terms of the Weitzenböck connection. This can be done using (1.161). This allows rewriting (1.173) as

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = K^\rho{}_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu, \tag{1.223}$$

where $\Gamma^{\rho}_{\mu\nu}$ and $K^{\rho}{}_{\mu\nu}$ are the Christoffel symbols and contorsion tensor of the Weitzenböck connection and where we have omitted the explicit dependencies in the descriptive parameter of the curve for conciseness. In terms of the components of the vector $\vec{v}_{\lambda}$ tangent to the curve, this can be written in an arbitrary basis as

$$\dot{v}^a + \omega^a_{bc}\ v^b v^c = K^a{}_{bc}\ v^b v^c. \tag{1.224}$$

We then obtain an equation fixing the motion of test particles that is "purged" from references to the Levi-Civita connection and that can be interpreted only in terms of the "true" connection of our Weitzenböck spacetime.

We thus see that particles will not move along the spacetime geodesics. The deviation from a geodesic motion is encoded in the right-hand side of the equation. This term is the one that shakes up the interpretation of how gravity interact with pointwise particles. The form of equation (1.223) reintroduces the interpretation of gravity as acting on test bodies by means of a (universal) force! This force is precisely given by the right-hand side of the relation; that is $K^{\rho}{}_{\mu\nu}\ \dot{x}^{\mu}\dot{x}^{\nu}$.

This important shift of interpretation needs further discussion. The idea of interpreting the quantity $K^{\rho}{}_{\mu\nu}\ \dot{x}^{\mu}\dot{x}^{\nu}$ as a force is clearly suggested by analogy with the Lorentz force but this interpretation rest on more than a mere analogy.

First, since the presence of a gravitational field is here attributed to the torsion of the Weitzenböck connection (fixed by means of field equations that we will discuss hereinafter), removing the gravitational interaction should correspond to the vanishing of torsion.[1.65] This would then immediately imply the vanishing of the right-hand side of (1.223) – or (1.224). We then consistently recover the idea that free particles[1.66] will move along "straight lines", a.k.a. geodesics, and that an interaction with a gravitational field causes a deviation from this idealised motion. This drives to the idea that gravity, while being universally present, can be understood as a force in the usual sense of the term inherited from Minkowski spacetime.

Second, remember that for this type of geometry, the interpretation of geodesics as straight lines in the usual sense of the term is reinforced by (1.134). This allows simplifying the left-hand side of (1.224) when working with a Weitzenböck tetrad. Note, however, that, due to its tensorial nature, the right-hand side – the force term – will still not vanish.

This observation motivates to interpret the Weitzenböck tetrads as orthonormal inertial frames, in the same sense as in Minkowski spacetime, and to see the connection coefficients as encoding inertial effects. Indeed, in this case, since the interaction of particles with gravity is already encoded in the tensorial term $K^b{}_{ac}\ v^a v^c$, the connection coefficients ω^b_{ac} loses all their dynamical content in the sense that the vanishing of these coefficients does not correspond to a physical effect; *e.g.* it has nothing to do with the vanishing of the gravitational field

[1.65] In this case, unless an incompatibility arises at the topological level, we would be back on Minkowski spacetime.

[1.66] Here, the term "free" has to be understood in the Newtonian sense of "free from any physical interaction, *including gravity*".

since, torsion being a tensor, it will remain non-vanishing in any basis.

This situation should be contrasted with what is done in general relativity. In general relativity, the equivalence principle is used to declare the impossibility to distinguish gravity from inertia. Analysing general relativity *a posteriori*, this is really this physical postulate that forces the choice of the Levi-Civita connection. We then arrived at a framework in which the notion of inertia loses its operational meaning in the sense that it cannot, not even in principle (and this is the crucial difference), manifest on its own. Admittedly, the idea that gravity can be locally compensated by inertial effects comes back through the back door when one considers the possibility to find frames in which the connection coefficients of the Levi-Civita connection vanish at a given point. But this is more of a "consistency check" than a way to define inertial effects. To state it differently, we are still unable to measure those effects quantitatively; we can only bring the idea that they exactly compensate the gravitational effects that still cannot be measured independently.

In a teleparallel theory of gravity, on the contrary, one really can disentangle inertia and gravity. This might also be seen by considering (1.162) written as

$$\overset{\circ}{\omega}{}^{b}_{ac} = \omega^{b}_{ac} - K^{b}{}_{ac}. \tag{1.225}$$

We already mentioned, from a mathematical perspective, that in this relation the non-tensorial nature of the coefficients ω^{b}_{ac} is entirely controlled by the non-tensorial behaviour of $\overset{\circ}{\omega}{}^{b}_{ac}$. For the sake of interpretation, we can here state it the other way around. We can thus interpret the Levi-Civita connection coefficients as involving both inertial and gravitational effects. Using this way of looking at the situation, (1.225) can really be seen as the splitting of the Levi-Civita connection into its inertial (non-tensorial) and gravitational (tensorial) parts. We then obtain a framework in which the notion of inertia remains meaningful in the usual sense.

The subtleties in this characterisation are that inertia keeps its meaning in principle but that gravity is, of course, always present in practice[1.67] and that inertial frames cease to be related to coordinate systems due to the presence of torsion.

This interpretation in which one recovers the ability to distinguish between inertial and gravitational effects is one of the very attractive features of a teleparallel theory of gravity.

To conclude this part of the discussion, note that this interpretation will hold in any theory of gravity based on a Weitzenböck spacetime, independently of the exact form of the field equations fixing the behaviour of the torsion.

1.6.3 Action and Field Equations

As promised, we now turn to the question of the formulation of the action and field equations of TEGR.

[1.67] Note that this feature is already present in Newtonian mechanics.

To start the discussion on this point, we will present some generic properties of a theory based on a Weitzenböck spacetime in which the field variables are taken to be a tetrad $\{\vec{e}_{(a)}\}$ (or, equivalently, its cotetrad $\{\underline{\theta}^{(a)}\}$) and the Weitzenböck connection (described in terms of the connection 1-forms $\omega^a{}_b$ related to the chosen tetrad). This part of the discussion is based on the material presented in [Bahamonde et al., 2021]. We will here present the conclusions and focus on their implications. In particular, we aim to emphasise the specific role of the tetrad and connection in this picture. For a more detailed derivation of the results, the reader should refer directly to [Bahamonde et al., 2021] where these are derived in great details.

Finally, to complete the loop, we will briefly discuss the implications of these properties in the specific case of TEGR.

Action and field equations for generic teleparallel theory

When writing a theory on a Weitzenböck spacetime that describes the dynamics of the interaction of some matter field with the Weitzenböck geometry, one will generally assume an action of the form

$$S = S_{\mathrm{g}}[\underline{\theta}^{(a)}, \omega^a{}_b] + S_{\mathrm{m}}[\underline{\theta}^{(a)}, \psi], \tag{1.226}$$

where ψ symbolically denotes the matter fields. The main if not only assumption at this stage is thus that the matter fields do not directly couple to the connection. To obtain the field equations of this theory, one should fix a coordinate system $\{x^\mu\}$ and study the variation of (1.226) with respect to the cotetrad $\underline{\theta}^{(a)} = e^a{}_\mu \mathrm{d}x^\mu$, the connection – encoded via $\omega^a{}_b = \omega^a_{b\mu}\mathrm{d}x^\mu$ – and the matter fields $\psi(x^\mu)$. Since the action will, in general, depend on these variables and on their derivatives up to a given order, as usual, the computation of δS will involve several integration by parts. As always, the boundary terms arising from these integration by parts will not contribute to the field equations. We can then neglect them when discussing the form of the field equations and write symbolically the variations of S_{g} and S_{m} (up to those boundary terms) as

$$\delta S_{\mathrm{g}} = -\int_{\mathcal{V}} \left(E_a{}^\mu\ \delta e^a{}_\mu + Y_a{}^{b\mu}\ \delta\omega^a_{b\mu}\right)\theta\ \mathrm{d}^4x \tag{1.227a}$$

$$\delta S_{\mathrm{m}} = \int_{\mathcal{V}} \left(\Theta_a{}^\mu\ \delta e^a{}_\mu + \Psi_{(K)}\ \delta\psi^{(K)}\right)\theta\ \mathrm{d}^4x\quad, \tag{1.227b}$$

where $\mathcal{V}$ is an arbitrary integration volume, $\theta \coloneqq \det\left(e^a{}_\mu\right) = \sqrt{-g}$ and where the capital index K numbers the matter fields. The field equations of the theory are then obtained from the condition $\delta S = 0$.

First of all, we can read the matter field equations from (1.227b). They are given by

$$\Psi_{(K)} = 0. \tag{1.228}$$

The variation with respect to the tetrad leads to

$$E_a{}^\mu = \Theta_a{}^\mu. \tag{1.229}$$

Here, the equation is written with mixed indices since we have considered the variation with respect to $e^a{}_\mu$. By further contracting this relation with the cotetrad and the metric, we can write it as $E_{\mu\nu} = \Theta_{\mu\nu}$. It is worth emphasising here that since this relation is just a rewriting of (1.229), there is no reason to expect some symmetry properties from either $E_{\mu\nu}$ or $\Theta_{\mu\nu}$ at this stage. So really, written in this way, (1.229) can be split into a symmetric and an antisymmetric part

$$\begin{cases} E_{(\mu\nu)} = \Theta_{(\mu\nu)} & (1.230\text{a}) \\ E_{[\mu\nu]} = \Theta_{[\mu\nu]} & (1.230\text{b}) \end{cases} .$$

The variation with respect to the connection is trickier. Indeed, since we assume to work on a Weitzenböck spacetime, we need to preserve the fact that the connection it is flat and metric compatible all along the computation. There are several equivalent ways to do this, none of which is completely trivial. One way to preserve the properties of the connection is to impose constraints on $\delta\omega^a_{b\mu}$ so that the curvature and non-metricity remain unchanged. From this we then see that the field equation associated to the connection cannot simply be $Y_a{}^{b\mu} = 0$. By carefully taking care of this crucial aspect, one will finally get that the field equations for the connection are of the form

$$W_{[\mu\nu]} = 0, \tag{1.231}$$

where $W_{[\mu\nu]}$ is constructed from $Y_a{}^{b\mu}$, its partial derivatives and some specific contractions of $Y_a{}^{b\mu}$ with the tetrad and connection coefficients. Pay attention to the antisymmetry here. This comes as a result of the restrictions imposed to $\delta\omega^a_{b\mu}$ to preserve the defining properties of the connection.

At this point, let us emphasise that, unless other properties are required on the action, the field equations for the tetrad (1.230) and the connection (1.231) are *a priori* independent from each other.

Of course, in the most general case, the setup described by (1.226) could lack some important physical properties. Typically, when building a physical theory, one cannot allow for any possible dependence in the tetrad and the spin connection. In a nutshell, one will usually obtain the Lagrangian densities $\mathscr{L}_{\rm g}$ and $\mathscr{L}_{\rm m}$, for the gravitational and matter part of the action respectively, by considering contractions of tensorial quantities built from these objects. The actions $S_{\rm g}$ and $S_{\rm m}$ will then, by construction, be invariant under general coordinate transformations (or, equivalently, under diffeomorphisms) and under local Lorentz transformations. None of these properties were used in the above computation. In the following we should discuss the outcome of these fundamental properties. As we will see, these will bring some simplifications in the field equations.

Consequences of the local Lorentz invariance

First of all, let us consider the invariance under local Lorentz transformations. To provide the explicit calculation, one should first express the form of the variations $\delta e^a{}_\mu$ and $\delta\omega^a_{b\mu}$ under this type of transformation. More precisely, one will be interested to these quantities at first order for infinitesimal Lorentz transformations. The variation will then be expressed in terms of the generators of the Lorentz group λ_{ab}. As we know, these will be such that $\lambda_{(ab)} = 0$. One will then insert the corresponding expressions for $\delta_\lambda e^a{}_\mu$ and $\delta_\lambda\omega^a_{b\mu}$ in (1.227) and impose the vanishing of the corresponding $\delta_\lambda S_{\rm g}$ and $\delta_\lambda S_{\rm m}$ (up to boundary terms) to obtain the consequences of this invariance.

The condition $\delta_\lambda S_{\rm g} = 0$ (up to boundary terms) will then imply

$$W_{[\mu\nu]} = E_{[\mu\nu]}. \tag{1.232}$$

Remark that this condition was derived without requiring the field equations to be satisfied. This condition is then a consequence of the invariance under local Lorentz transformations that holds both on shell and off shell.

In turn, the condition $\delta_\lambda S_{\rm m} = 0$ (up to boundary terms) will lead to

$$\Theta_{[\mu\nu]} = 0. \tag{1.233}$$

Here again, the condition is derived without requiring the field equations to be satisfied. It is then valid both on shell and off shell. However, the derivation of this equation makes explicit use of the fact that the matter action is independent of the connection.

The local Lorentz invariance of the theory will then lead to simplified field equations. Indeed, using (1.233), the tetrad field equations reduce to

$$\begin{cases} E_{(\mu\nu)} = \Theta_{\mu\nu} & (1.234\text{a}) \\ E_{[\mu\nu]} = 0 & (1.234\text{b}) \end{cases}.$$

From (1.232), (1.231) and (1.234b), we also see that the field equations for the connection become redundant with the antisymmetric part of the tetrad equations. This property makes sense from a physical point of view. It expresses the fact that the connection itself is empty of dynamical content in the sense that the connection coefficients encode only inertial effects that can always be removed by choosing an inertial frame (*i.e.* by choosing the tetrad to be a Weitzenböck tetrad). This then ensures that one can always consistently solve the dynamics by choosing a Weitzenböck tetrad. Let us nevertheless emphasise that, in this case, one should make sure that the Weitzenböch tetrad solves both the symmetric and antisymmetric part of the tetrad field equations.

In practice, since the antisymmetric part of the tetrad equations is independent of the matter content of the theory, one will usually start by classifying the tetrads satisfying (1.234b) in the situation of interest (or at least by finding some of them). One will then insert this form in (1.234a) and try to solve it. In this sense, (1.234b) can also be seen as the condition ensuring that a tetrad found by solving (1.234a) is really a Weitzenböck tetrad.

Before going further let us mention, following [Bahamonde et al., 2021], that imposing the local Lorentz invariance of the theory is equivalent to require that the matter part of the action does only depend on the tetrad through the metric and its derivatives and that the gravitational part does only depend on the metric, the torsion and its covariant derivatives. This is in agreement with the above idea that, in practice, one will always construct the Lagrangian density of the theory through tensorial quantities.

In this case, remark that, if the matter part of the action does depend on the metric but not on its derivatives, $\Theta_{\mu\nu}$ will be given by (1.189). In other words, provided one consistently fixes the constants in the action, it will correspond to the same energy-momentum tensor as the one encountered in general relativity. In teleparallel theories, one will prefer to denote this tensor $\Theta_{\mu\nu}$ as we did here rather than $T_{\mu\nu}$ as we did before to avoid any possible confusion with a quantity constructed from the torsion tensor.

Consequences of the diffeomorphism invariance

The invariance under local coordinate transformations – or under diffeomorphisms – does also bring some worth noticing properties. In this case, one will do the computation by considering that, infinitesimally, these transformations are generated by vector fields. The variation of any tensor field will then be given by its Lie derivative.

The invariance of the gravitational part of the action will then bring two identities

$$\overset{\circ}{\nabla}_\mu E^{(\mu\nu)} = 0, \quad E^{[\rho\nu]} K_{\rho\nu}{}^{\mu} - \overset{\circ}{\nabla}_\nu E^{[\mu\nu]} = 0, \tag{1.235}$$

both valid on shell and off shell. Nonetheless, the derivation will make use of the local Lorentz invariance via (1.232). It is interesting to emphasise that these are then geometrical identities satisfied by the tensor field $E_{\mu\nu}$ entering the field equations. Depending on how precisely the gravitational action is constructed from the torsion tensor, these will be related to (or even consequences of) the Bianchi identities. For this reason, one usually refers to (1.235) as the Bianchi identities of the theory.

The invariance of the matter part of the action, on its side, will lead to the on-shell conservation of the energy-momentum tensor of matter

$$\overset{\circ}{\nabla}_\mu \Theta^{\mu\nu} = 0. \tag{1.236}$$

Contrary to the other identities following from one invariance property discussed here, the derivation of this relation makes explicit use of the matter field equations (1.228). This is usual but should be stressed to further emphasise that it is not the case for the other identities derived above.

The case of TEGR

To complete the discussion, let us here discuss specifically the situation in TEGR. In this case, the Lagrangian density is given by

$$\mathscr{L}_{\text{TEGR}} = -\frac{1}{2\kappa} T, \tag{1.237}$$

where T is the torsion scalar defined in (1.219) and κ the constant given in (1.182). By construction, this will, of course, lead to an action that is invariant under local Lorentz transformations and coordinate changes. Consequently all the above results apply. Let us also mention one last time that we could have included the boundary term B defined in (1.220) in (1.237) without altering the field equations.

On account of (1.218), the tensor field $E^{\text{TEGR}}_{\mu\nu}$ entering the field equations will then be such that

$$E^{\text{TEGR}}_{\mu\nu} \equiv \frac{1}{\kappa}\overset{\circ}{G}_{\mu\nu}, \tag{1.238}$$

where, on the right-hand side, the Einstein tensor is expressed only in terms of the tetrad and cotetrad coefficients to match with the teleparallel field variable; the metric is thus replaced in terms of the tetrad and cotetrad via (1.142).

We then see that (1.176) implies $E^{\text{TEGR}}_{[\mu\nu]} \equiv 0$ so that the antisymmetric part of the field equations (1.234b) becomes trivial. This property makes sense if one recalls that (1.234b) is equivalent to the field equation for the connection. By construction, the TEGR action is equivalent to the Einstein-Hilbert action. As we already mentioned, this action can be written using only the tetrad. In the context of a teleparallel theory of gravity, we are thus working with a Lagrangian density independent of the Weitzenböck connection. It is then normal to have that the antisymmetric part of the field equation becomes trivial.

Finally, (1.238) will ensure the equivalence between (1.234a) and (1.183), as expected. Note that, following the conventions of this section, the energy-momentum tensor of matter, as defined in (1.189), is written $\Theta_{\mu\nu}$.

To conclude this discussion of the field equations of TEGR, we should emphasise an important point: the tetrad solving the field equations is not unique. This is, at the same time, an obvious and a crucial property of the teleparallel equivalent of general relativity. It is obvious since we know that the TEGR action should be equivalent to the Einstein-Hilbert action so that the field equations should be equivalent to Einstein equations (1.183). Since it is well known that, given appropriate boundary conditions, Einstein equations will allow determining a unique metric, the dynamical content can only determine a tetrad up to an arbitrary local Lorentz transformation, on account of (1.142). This is then also a crucial property since (1.142) also establishes that, given a tetrad solution of the field equation, all the other solutions are determined from this one by application of a local Lorentz transformation. It is thus sufficient to find one tetrad solution of the equations (by any clever mean) to be able to characterise all the set of solutions. Also, regarding the spacetime geometry, any tetrad is as valid as another to reconstruct the metric and the torsion tensor.

This property is also quite fundamental from a physical point of view. Indeed, a tetrad is nothing more than a frame. It is then important that our laws of physics do not point toward a unique (privileged) reference frame. Here, the field equations only determine a class of frames that, in terms of the Weitzenböck geometry, corresponds to the class of inertial frames. Yet the theory does not point these frames as privileged. These are the ones found by solving the equations and, since the action is actually independent of the Weitzenböck connection, we can always consistently consider that these tetrads are the Weitzenböck tetrads. This is this property that allows, in the case of TEGR, to get the inertial frames as naturally arising from the field equations without any specific restriction[1.68]. That being said, these are not intrinsically favoured by our physical laws nor central for their formulation. They are just, in practice, convenient to solve them.

As a final note, remark that this property that, given one tetrad solution of the field equations, any other tetrad also solving the equations can be obtained by an arbitrary local Lorentz transformation is a specific property of TEGR. It is not necessarily true for a generic teleparallel theory. Yet, the above concern on the fact that our laws of physics should not point to a privileged frame will still not be a problem. Indeed, as long as the theory is invariant under local Lorentz transformations applied on *both* the tetrad and the spin connection *at the same time*[1.69], we will be ensured that the field equations will only allow determining an equivalent class of this pair (tetrad and connection). This will thus only determine the geometry of spacetime without explicitly favouring a frame for their formulation. Which subset of these transformations (if there ever is any) will specifically allow mapping a Weitzenböck tetrad to another Weitzenböck tetrad is another – trickier – question whose answer is still unknown for a completely generic teleparallel theory.

1.6.4 Naturalness and Interpretation

Finally, to conclude this brief presentation of the teleparallel equivalents of general relativity (TEGR and STEGR), let us say a word on what we called the naturalness of these theories.

In the above presentation, we mainly focussed on how these theories will cohere with the predictions of general relativity. This was a presentation choice motivated by the many successes of general relativity at the observational level. Putting emphasis on the equivalence with general relativity aimed to reveal these alternative theories as viable models of the gravitational interaction from the phenomenological point of view. The drawback of this choice is that it can legitimately raise suspicions on the ability of these theories to stand on their own on the theoretical side. We then wanted to emphasise here that many authors have worked on the question and raised strong arguments suggesting that these theories (especially TEGR) can be built from scratch in such a way that the equivalence with general relativity can really be seen as a consequence of this independent construction.

To our knowledge, most of the work in this direction has been devoted to TEGR and theories of gravity based on torsion in general (in opposition to theories purely based on the non-metricity tensor). In this respect, we should mention that TEGR has been claimed to provide a description of gravity as a gauge

[1.68] This might be seen as a consequence from the fact that the antisymmetric part of the field equations is trivial in the specific case of TEGR.

[1.69] Remember that this should always be the case provided our action is built from tensorial quantities.

theory of the translational group. This is, of course, a very attractive feature of TEGR as it formulates gravity in the same language as the other fundamental interactions. The interested reader should refer to [Bahamonde et al., 2021] and [Aldrovandi and Pereira, 2014] for more in-depth discussions presenting this point of view. One could also refer to [Blagojevic and Hehl, 2012] where the different attempts to formulate the gravitational interaction as arising from a gauge theory are presented more systematically and with valuable references to the historical developments of the subject.

To the best of our knowledge, STEGR has attracted significantly less attention. This is most likely due to the lack of strong independent physical motivations as the ones found for TEGR. That being said, we would like to mention here that this "geometric trinity of gravity" (GR, TEGR and STEGR), as it has been dubbed, raises some important questions from the perspective of the philosophy of sciences. The idea that these three theories enjoy a full mathematical equivalence when considering their (classical) dynamical content while resting on distinct interpretations of both the mathematical objects and the physical concepts of the theory set up very puzzling questions regarding what should be given an ontological status (*i.e.* what should be looked as "physical") in this context. We will not enter this discussion here as it is outside the scope of this thesis to a great extent but we wanted to mention it as an interesting – if not important – question that one should address when trying to be realist regarding our descriptions of Nature.

Chapter

On Scalar Fields and their Couplings to Gravity

2

In this chapter, we introduce some elements of one of the most common (classical) modification to general relativity (GR) : scalar-tensor gravity, *i.e.* a modification of GR where a scalar field is present.

For this chapter, since it is the framework that we used for almost all the work of this thesis, we will focus on modified theories of gravity based on the usual formulation of GR on a differential manifold endowed with a metric and the associated Levi-Civita connection. More information on the framework used for our work based on the teleparallel equivalent of general relativity can be found in appendix F and references therein.

We will start in section 2.1 by giving a few motivations for scalar-tensor theories of gravity in general. Then, in sections 2.2 and 2.3, we will review the quantum mechanical origins of the notion of scalar field in Minkowski spacetime and discuss how such a field can be studied on curved spacetime. This would define the idea of a minimal coupling of the scalar field to gravity. Finally, in section 2.4, we will present some classes of alternative couplings to gravity, focussing on the ones that will be emphasised in this thesis.

This chapter then aims to serve as a (obviously incomplete) state of the art of the scalar-tensor theories of gravity relevant for this thesis. On account of the subject of this thesis, we will look at the topic through the lens of the study of compact objects. In particular, we will consider the implications of the different couplings in the context of black holes physics as a guide line for the discussions of this chapter.

2.1 Motivation(s)

Motivating the introduction of any alternative theory of gravity requires to emphasise the loopholes in the very successful description of gravitational phenomenons given by GR. We already presented some of these "flaws" in the introduction chapter. Let us here have a brief second glance at these to emphasise how one can treat these questions, more specifically, in the context of modifications of GR involving a scalar field.

On an experimental level, GR has proven its accuracy to describe phenomenons from the solar system scale up to the cosmological scale. One of its first historical success was to provide a consistent resolution of the Mercury perihelion problem. But the adequation of GR with experiment does not reduce to this explanation. One of the main predictions of GR was the possibility for the existence of gravitational waves and, with the evolution of experimental technics, GR has recently proved its ability to predict not only the existence but the shape and properties of gravitational waves (the first direct detection being the GW150914 event in 2016). Other experimental successes of GR also includes the phenomenon of gravitational lensing (as emphasise by the Event Horizon telescope in 2019) and, at cosmological scale, the accurate description of the Cosmic Microwave Background when considered in the context of the ΛCDM model. All these experimental checks already make GR an incredible theory. Also on a purely theoretical ground, as discussed in chapter 1, the simplicity and elegance of the theory argue in favour of a geometrical description of gravity, as first offered by GR.

That being said, despite those undeniable successes, the necessity to go beyond the description of gravity offered by GR is also well admitted since the theory suffers from some problems on both the experimental and theoretical levels.

On one side, there is the question of the origin and composition of dark matter and dark energy since these seem to be unavoidable pieces of our current understanding of the universe (that account for around 95% of the matter-energy content of the observable universe according to the best fit of the ΛCDM model) whose existence precisely evades the descriptions offered by GR and the standard model of particle physics. On the other side, one can mention the elusiveness of a UV completion of GR providing a quantum theory of gravity as a necessity to question the validity of the theory on a purely theoretical basis.

In both of these contexts, studying alternative theories to GR at a classical level can be seen as a way to make a step in the right direction and to obtain a better understanding of what makes GR so special; even if these models do not necessarily provide a definitive solution to the aforementioned problems.

To complete this tricky task, one then needs to find its way in the narrow path between the experimental successes of GR and the theoretical puzzles left unsolved. In this respect, compact objects provide very interesting laboratories because they allow testing theories in extreme conditions which can potentially reveal their limits.

On account for their "simplicity", black holes are privileged gravitational systems for theoretical investigations. They allow testing the behaviour of matter under the most extreme conditions, to challenge the cosmic censorship and the possibility for the existence of spacetime singularities. They can also be used to provide constraints on the theories to consider as we shall see through this chapter. In addition, as mentioned above, the recent developments of experimental tests of black hole properties via gravitational lensing raises the need for the study of black holes in alternative models of gravity in order to compare them to the GR's predictions.

Obviously, all these motivations in themselves do not indicate the theoretical framework to favour. One out of many, the one chosen in this thesis, consists in the introduction of new degrees of freedom in the theory as a possible (classical) explanation for the unrated phenomenons. These new degrees of freedom can be seen as potential new particles or as new components in the description of the gravitational interaction itself. They can also be considered as effective descriptions of more fundamental processes yet to be discovered.[2.1]

In this respect, the simplest candidate to encode new degrees of freedom is indisputably a scalar field. Scalar fields represent the simplest type of covariant objects. For this reason, they are found in many places in physics. A few examples include :

- In low energy effective limits of some string theories,
- In the Kaluza-Klein model where a scalar degree of freedom manifests after dimensional reduction of a $5D$ gravitational theory initially aiming to unify gravitation with electromagnetism,
- In cosmology, where those fields might be used to model dark matter and dark energy
- In the standard model of particle physics with the Brout-Englert-Higgs boson
- ...

Scalar fields are also frequently used as effective descriptions for more complex phenomenons, such as in solid-state physics, with the so-called phonons, to mention just one example.

We should also emphasise that the experimental discovery of the BEH boson at CERN in 2012 constitutes in itself a motivation to study scalar fields as fundamental pieces of our model. Especially (even though this is an intuitive claim) for theories of gravity if one considers that the role of this boson in the standard model of particle physics is to give mass to the other fields.

In the rest of this chapter, we will recall some of the main attributes of scalar fields on flat and curved spacetime as a way to put our research work in context.

[2.1] In this thesis, whenever possible, we will remain agnostic on the interpretation of the new degrees of freedom introduced within the theory. We will nevertheless always keep in mind that these classical modifications of GR would ultimately need to be justified by a quantum descritpion of gravity and that our models are then most likely to describe a low energy effective field theory for this putative theory of quantum gravity.

2.2 Scalar Field on Minkowski Spacetime

In this section, we present the basic features of scalar fields on a (flat) Minkowski spacetime, focussing on the field equations for the scalar field and the Lagrangian formulation of the problem.

2.2.1 Klein-Gordon Equation

If one applies the so-called correspondence principle

$$\begin{cases} E \to i\hbar \dfrac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases} \tag{2.1}$$

to the definition of mechanical energy from Newtonian mechanics

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}) \tag{2.2}$$

one gets the wave equation of non-relativistic quantum mechanics, a.k.a. Schrödinger's equation,

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{x})\right)\psi(\vec{x},t), \tag{2.3}$$

where $\Delta = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian operator. This equation is extremely successful in describing non-relativistic quantum processes but fails to be compatible with special relativity in that it is not Lorentz invariant.

In an attempt to derive a relativistic quantum mechanical wave equation, one can follow the same philosophy and apply (2.1) to the formula for the energy from special relativity

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4 \tag{2.4}$$

or to its manifestly covariant version

$$p^\mu \eta_{\mu\nu} p^\nu + m^2c^2 = 0, \tag{2.5}$$

where the quantities were expressed on Minkokwsi spacetime using an inertial frame (see section 1.1). We then obtain the Klein-Gordon equation

$$\left(-\hbar^2\Box + m^2c^2\right)\phi(x^\mu) = 0, \tag{2.6}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2/c^2 + \Delta$ is the Dalembertian operator. Note that (2.6) is often more conveniently written as

$$\Box\phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{2.7}$$

This equation describes the behaviour of a massive scalar field (spin 0 particle of mass m) on Minkowski spacetime. Note that, unlike (2.3), (2.7) does not have any reference to complex numbers. Consequently, ϕ can be a real field.

In fact, ϕ can equally be taken complex but, due to the form of (2.7), both its real and imaginary parts (say ϕ_r and ϕ_i, respectively) would independently solve the equation. This model of a complex massive scalar field ϕ can then be seen as a model with two independent real ones, ϕ_r and ϕ_i, with the same mass m. The complex notation then simply becomes a convenient shortcut to write the system.

In both cases, since the Dalembertian operator is manifestly Lorentz invariant, (2.7) will be Lorentz invariant, as desired, provided $\phi\left(x^{\mu}\right)=\phi'\left(x'^{\mu}\right)$ under a Lorentz transformation $x^{\mu}\rightarrow x'^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}$.

Self-interaction for real scalar field

Considering a real scalar field, equation (2.7) can easily be generalised to include self-interactions of the scalar field. The idea would be to introduce a self-interaction potential $V(\phi)$ and to replace the right-hand side with the corresponding "force", $V'(\phi)$, where the $'$ here denotes a derivative of V with respect to its argument, leading to

$$\Box\phi=V'(\phi). \tag{2.8}$$

In the following, we will continue to call this equation the Klein-Gordon equation – and will in fact mostly refer to (2.8) when using this denomination for real scalar fields – even though it is technically a generalisation of *the* Klein-Gordon equation (2.7), which corresponds to the choice $V(\phi)=\frac{m^2c^2}{2\hbar^2}\phi^2$ in (2.8).

Self-interaction for complex scalar field

For a complex scalar field, the same trick can be applied but with a bit of extra-care. As we already mentioned, a complex massive scalar field $\phi=\phi_r+i\phi_i$ can be seen as a convenient shortcut for the study of two independent massive real scalar fields of equal mass, ϕ_r and ϕ_i. Considering the system from the point of view of ϕ_r and ϕ_i, if one wants to add mutual and/or self-interactions between those two fields, one would introduce an interaction potential

$$\tilde{V}:\begin{cases}\mathbb{R}^2\rightarrow\mathbb{R}\\ (\phi_r,\phi_i)\mapsto\tilde{V}(\phi_r,\phi_i)\end{cases}$$

and, similarly to what we did in (2.8), write the system as

$$\begin{cases}\Box\phi_r=\frac{1}{2}\partial_{\phi_r}\tilde{V}(\phi_r,\phi_i)\\ \Box\phi_i=\frac{1}{2}\partial_{\phi_i}\tilde{V}(\phi_r,\phi_i)\end{cases}, \tag{2.9}$$

where the $1/2$ factors were simply introduced for notational convenience. Of course, this system can still be synthesised in the form of a single complex equation :

$$\Box\left(\phi_r + i\phi_i\right) = \left(\frac{1}{2}\partial_{\phi_r} + \frac{i}{2}\partial_{\phi_i}\right)\tilde{V}(\phi_r, \phi_i). \tag{2.10}$$

The power of the complex notation can then be appreciated thanks to the following trick : starting from the problem in terms of the variables (ϕ_r, ϕ_i), perform the invertible "change of variables"

$$\begin{pmatrix}\phi_r \\ \phi_i\end{pmatrix} \mapsto \begin{pmatrix}1 & i \\ 1 & -i\end{pmatrix}\begin{pmatrix}\phi_r \\ \phi_i\end{pmatrix} =: \begin{pmatrix}\phi \\ \phi^*\end{pmatrix}. \tag{2.11}$$

Note that, despite being formally the complex conjugate of ϕ, ϕ^* is here seen as an independent variable[2.2]. Now, inverting (2.11), one can easily obtain that

$$\begin{cases}\partial_\phi = \frac{1}{2}\partial_{\phi_r} - \frac{i}{2}\partial_{\phi_i} \\ \partial_{\phi^*} = \frac{1}{2}\partial_{\phi_r} + \frac{i}{2}\partial_{\phi_i}\end{cases}. \tag{2.12}$$

Consequently, introducing

$$V\left(\phi, \phi^*\right) \coloneqq \tilde{V}\left(\frac{1}{2}(\phi + \phi^*), \frac{i}{2}(\phi^* - \phi)\right) = \tilde{V}(\phi_r, \phi_i),$$

one can rewrite (2.10) as

$$\Box\phi = \frac{\partial}{\partial\phi^*}V\left(\phi, \phi^*\right). \tag{2.13}$$

In the following, we will continue to call this equation the Klein-Gordon equation – and will in fact mostly refer to (2.13) when using this denomination for complex scalar fields – even though it is technically a generalisation of *the* Klein-Gordon equation (2.7), which corresponds to the choice $V\left(\phi, \phi^*\right) = \frac{m^2c^2}{\hbar^2}\phi\phi^*$ in (2.13).

It is worth noting that, playing the same game, we can obtain one equation for the complex conjugate

$$\Box\phi^* = \frac{\partial}{\partial\phi}V\left(\phi, \phi^*\right), \tag{2.14}$$

which is equivalent to (2.13)[2.3], ensuring that, even though we have seen ϕ and ϕ^* as independent variables, we still consistently have only one (two) equation(s) for one complex (two reals) variable(s). In other words, the model of one self-interacting complex scalar field corresponds to a very elegant way to consider a model for two mutually interacting real scalar fields.

Before concluding this paragraph, let us nevertheless stress that, in practice, when dealing with a complex scalar field, we will never refer to the interpretation of the system as composed of two mutually interacting real ones. We used the

[2.2] ϕ and ϕ^* are seen as the names of the new variables, $\phi_r + i\phi_i$ and $\phi_r - i\phi_i$ respectively.

[2.3] It is so since $\partial_{\phi^*} = \left(\partial_\phi\right)^*$ and $V\left(\phi, \phi^*\right) \in \mathbb{R}$.

above discussion as a (hopefully) natural way to introduce the notion of self-interaction for one complex scalar field[2.4] but the interest of considering complex scalar fields goes way beyond a simple shortcut in the notation[2.5]. It then makes complete sense to treat these complex fields as entities in themselves – rather than just a quick way to talk about two real fields – when discussing the physics of such systems.

Conserved current for complex scalar field

One important scenario for a complex self-interacting scalar field is the one where the self-interaction potential $V(\phi, \phi^*)$ depends only on the combination $|\phi|^2 = \phi\phi^*$. In this case, since

$$\frac{\partial}{\partial \phi^*} V(\phi\phi^*) = \phi\, V'(\phi\phi^*),$$

where the $'$ denotes a derivative of V with respect to its argument, (2.13) takes the form

$$\Box\phi = \phi\, V'(\phi\phi^*). \tag{2.15}$$

Consequently,

$$J^\mu = i\eta^{\mu\nu}\left(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi\right) \tag{2.16}$$

defines a conserved current associated to the field equation (2.15). Indeed, considering the combination $i\phi \times (2.15)^* - i\phi^* \times (2.15)$ and using the Leibnitz rule, one immediately gets

$$\partial_\mu\Big(i\eta^{\mu\nu}\left(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi\right)\Big) = 0.$$

This is a good example of a situation where it is more natural to discuss the problem in terms of the complex field ϕ directly.

This conserved current originates from the fact that, for potentials such that $V(\phi, \phi^*) = V(|\phi|)$, the model acquires a global $U(1)$ invariance $\phi \to e^{i\alpha}\phi,\ \alpha \in \mathbb{R}$. In other words, if ϕ solves (2.15), so will $e^{i\alpha}\phi$, $\forall\alpha \in \mathbb{R}$ (fixed). This invariance can be seen directly from (2.15) but is, as usual, more clearly connected to the conservation of (2.16) when looking at the Lagrangian formulation of the problem.

2.2.2 Klein-Gordon Lagrangian

The Klein-Gordon equation, for both real and complex scalar fields, can be seen as deriving from an extremal action principle.

Equation (2.8) will correspond to the condition of extremisation of the action

[2.4] This, and the peculiar physicist's habit of seeing ϕ^* as independent of ϕ despite being its complex conjugate.

[2.5] This is especially true when one tries to identify internal symmerties, as we should briefly exemplify hereinbelow.

$$S_{\mathrm{KG}\mathbb{R};\mathbb{M}_4} = -\int_{\mathcal{V}} \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)\sqrt{-\eta}\;\mathrm{d}^4x, \tag{2.17}$$

where $\mathcal{V} \subseteq \mathbb{M}_4$ indicates an arbitrary integration volume on the (fixed) Minkowski background, $\eta_{\mu\nu}$ the components of the Minkowski metric written on an inertial frame with a priori arbitrary coordinates[2.6] and $\eta = \det(\eta_{\mu\nu})$.

In turn, equation (2.13) will correspond to the condition of extremisation of the action

$$S_{\mathrm{KG}\mathbb{C};\mathbb{M}_4} = -\int_{\mathcal{V}} \left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* + V(\phi,\phi^*)\right)\sqrt{-\eta}\;\mathrm{d}^4x, \tag{2.18}$$

where here again ϕ and ϕ^* are seen as independent variables. Actually, (2.13) will be obtained considering the Euler-Lagrange equation for ϕ^* while (2.14) corresponds to the one for ϕ. In this context, the action will possess a global $U(1)$ invariance if and only if $V(\phi,\phi^*) = V(|\phi|)$. According to Nœther's theorem, the presence of this invariance will be responsible for the existence of a conserved current, whose expression is precisely given by (2.16).

Note that, in practice, when it will be clear from the context whether ϕ is real or complex, we will simply note the action $S_{\mathrm{KG};\mathbb{M}_4}$ and denote $\mathscr{L}_{\mathrm{KG};\mathbb{M}_4}$ the corresponding Lagrangian density – the expression inside the parenthesis in (2.17) and (2.18).

2.3 Minimally Coupled Scalar Field on Curved Spacetime

We will now turn to the question of studying scalar fields as a possible source of the gravitational field on curved spacetime. We start by presenting the field equation and the Lagrangian formulation of the problem. We also present a no-(scalar-)hair theorem for black holes due to Bekenstein motivating the material of the next section, *i.e.* the introduction of some additional couplings between the scalar and metric degrees of freedom.

2.3.1 Klein-Gordon Equation on Curved Spacetime

As already pointed out in section 1.5.3, even in the quite simple framework of general relativity where spacetime is seen as a differential manifold $\mathcal{M}$ endowed with a generic Lorentzian metric $\boldsymbol{g} = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ and the corresponding Levi-Civita connection, it is *a priori* highly non-trivial to know how one could generalise the laws of physics known on flat spacetime to curved spacetime. This also applies to the Klein-Gordon equation. From the mathematical point of view, there are indeed lots of possible terms that could be included in the equation, involving typically the spacetime curvature, which would identically

[2.6] $\eta_{\mu\nu}$ will then a priori be function of the spacetime coordinates.

vanish and lead to the correct description on Minkoswki spacetime. From a physical point of view, as we did in the case of the electromagnetic field, the simplest (and arguably the most natural) way to proceed would be to simply ensure the general covariance of the equation by means of the following rule :

$$\begin{cases} \eta_{\mu\nu} \to g_{\mu\nu} \\ \partial_\mu \ \ \to \nabla_\mu \end{cases} . \tag{2.19}$$

One would then simply replace the Minkowski metric by the curved spacetime metric and the ordinary derivatives by covariant ones. In the context of general relativity, once again, this defines the notion of minimal coupling to the gravitational field. This covariantization procedure leads to a covariant field equation that automatically gives the correct description on Minkowski spacetime.[2.7]

The Klein-Gordon equation for a real (resp. complex) scalar field will then be identical to (2.8) (resp. (2.13)) but with the Dalembertian operator taken on curved spacetime $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

The rule (2.19) can also be applied to generalise most of the results known for scalar fields in flat spacetime to minimally coupled scalar fields on curved spacetime. For example, for a complex scalar field, if the potential satisfies $V(\phi, \phi^*) = V(|\phi|)$, one can obtain the existence of a covariantly conserved current J^μ whose expression is obtained by applying (2.19) to (2.16), giving

$$J^\mu = i g^{\mu\nu} \left(\phi \nabla_\nu \phi^* - \phi^* \nabla_\nu \phi\right). \tag{2.20}$$

In other words, in this case, J^μ will satisfy $\nabla_\mu J^\mu = 0$, provided ϕ solves the complex Klein-Gordon equation on curved spacetime.

2.3.2 Klein-Gordon Lagrangian on Curved Spacetime

It is also possible to apply the covariantization procedure at the level of the action. Applying (2.19) to (2.17) and (2.18) leads respectively to

$$S_{\mathrm{KG}\mathbb{R}} = -\int_{\mathcal{V}} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi)\right) \sqrt{-g}\ \mathrm{d}^4 x, \tag{2.21}$$

where $\mathcal{V} \subseteq \mathcal{M}$ here corresponds to an arbitrary integration volume on the curved spacetime manifold $\mathcal{M}$, for a real scalar field and

$$S_{\mathrm{KG}\mathbb{C}} = -\int_{\mathcal{V}} \Big(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi^* + V(\phi, \phi^*)\Big) \sqrt{-g}\ \mathrm{d}^4 x, \tag{2.22}$$

for a complex one. Here again, the model will possess a global internal $U(1)$ invariance iff $V(\phi, \phi^*) = V(|\phi|)$ and this invariance, if present, will be responsible for the conservation of the Nœther current (2.20).

[2.7] Starting from the curved spacetime description, this is obtained by an inversion of the arrows in (2.19).

One can easily check that the Euler-Lagrange equations associated with these actions are precisely those discussed in the previous paragraph. The relation (2.21) (resp. (2.22)) defines, in terms of the action, the notion of a real (resp. complex) scalar field minimally coupled to gravity.

2.3.3 Energy-Momentum Tensor of a Minimally Coupled Scalar Field

As discussed in section 1.4.5, to use a matter field, described by an action S_{M}, as source of the gravitational field, it is important to characterise its backreaction on the metric. In other words, it is important to know the quantity it will induce in the metric field equation : its energy-momentum tensor (1.189).

For a real scalar field minimally coupled to gravity, this tensor is

$$T_{\mu\nu}^{(\phi)\mathbb{R}} = \nabla_\mu \phi \nabla_\nu \phi - \left(\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + V(\phi) \right) g_{\mu\nu}. \tag{2.23}$$

For a complex scalar field minimally coupled to gravity, it is

$$T_{\mu\nu}^{(\phi)\mathbb{C}} = \nabla_\mu \phi \nabla_\nu \phi^* + \nabla_\nu \phi \nabla_\mu \phi^* - \Big(g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi^* + V\left(\phi, \phi^*\right) \Big) g_{\mu\nu}. \tag{2.24}$$

Note that, in practice, when it will be clear from the context whether ϕ is real or complex, we will drop the $\mathbb{R}$ and $\mathbb{C}$ symbols in the notation. We will simply use $T_{\mu\nu}^{(\phi)}$ for the energy-momentum tensor, S_{KG} for the action and $\mathscr{L}_{\mathrm{KG}}$ for the corresponding Lagrangian density – the expression inside the parenthesis in (2.21) and (2.22).

2.3.4 No-Hair Theorem

To pursue our goal of discussing the relevance of scalar fields for the study of compact objects, we will now present an important result due to Bekenstein which can be seen as a motivation for many of the work of this thesis. It is a no-scalar-hair theorem, a result proving the impossibility to endow a black hole with a non-trivial[2.8] scalar profile. As we will discuss below, trying to evade this result can be seen as a guideline to construct interesting/fruitful alternative scalar-tensor theories of gravity. Indeed, as for any mathematical theorem, this result rely on several hypotheses which are essential to its validity but whose physical necessity can be questioned.

In order to point out how the different hypothesis act on the validity of the result, we emphasise them in the statement of the theorem and indicate as clearly as we can where they are used in the steps of the proof.

[2.8] *i.e.* a scalar field profile that would be non-constant and hence backreact on the metric via its energy momentum tensor, leading to different black hole solutions than those known in the theory formulated without this scalar field.

The content of this section is essentially inspired by [Herdeiro and Radu, 2015].

No Scalar-Hair Theorem 1 (Bekenstein)**.** *Consider an asymptotically flat black hole spacetime.*

Hypothesis 1 : (**Symmetries of spacetime**)
Consider a stationary (and asymptotically flat) black hole spacetime

Hypothesis 2 : (**Symmetries of the scalar field**)
The scalar field shares the space-time symmetries.

Hypothesis 3 : (**Coupling condition**)
Consider a minimally coupled real scalar field :

$$S = \int_{\mathcal{M}} \left[F(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \dots) - \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] \sqrt{-g}\ \mathrm{d}^4 x.$$

Hypothesis 4 : (**Energetic condition**)
Assume a self-interaction potential V such that $\phi V'(\phi) \geq 0,\ \forall \phi$, with $V'(\phi) = \mathrm{d}V/\mathrm{d}\phi$, and such that $\phi V'(\phi) = 0$ for some discrete values of ϕ, say ϕ_i.

Then*, the scalar field must be trivial :* $\phi(x^\mu) = \phi_i, \forall x^\mu$ *on the black hole exterior region.*

Proof.
Under the first hypothesis[2.9], Hawking's *rigidity theorem* establishes that the spacetime must be *either static or axisymmetric*. For definiteness, let us focus here on the axisymmetric case.

If the spacetime is axisymmetric, it possesses one timelike Killing vector $\vec{m}$ and one space-like Killing vector with closed orbits $\vec{k}$.

Using a coordinate system (t, r, θ, φ) adapted to these isometries, one simply has $\vec{m} = \partial_t$ and $\vec{k} = \partial_\varphi$. The metric will then be such that $\partial_t g_{\mu\nu} = 0 = \partial_\varphi g_{\mu\nu}$ in this coordinate system.

From hypothesis 2, the scalar field will then also be such that $\partial_t \phi = 0 = \partial_\varphi \phi$. **Hypothesis 3** fixes that the scalar field must satisfy the Klein-Gordon equation (where $\approx$ denotes an on-shell equality)

$$\nabla_\mu \nabla^\mu \phi - V'(\phi) \approx 0.$$

Multiplying the above equation by ϕ and integrating over the black-hole exterior spacetime region $\mathfrak{E}$, one gets

$$\int_{\mathfrak{E}} \left(\phi \nabla^\mu \nabla_\mu \phi - \phi V'(\phi)\right) \sqrt{-g}\ \mathrm{d}^4 x \approx 0.$$

[2.9] and assuming the null energy condition

The core of the proof will then be to work with this (on shell) equality to obtain an integrand that will be positive definite (both on shell *and* off shell) so that the above condition will constrain the scalar field to be trivial.

Performing an integration by parts over the first term gives

$$\begin{aligned}0 &\approx \int_{\mathfrak{E}} \left(\phi\nabla^\mu\nabla_\mu\phi - \phi V'(\phi)\right)\sqrt{-g}\ \mathrm{d}^4x \\ &= -\int_{\mathfrak{E}} \left(\nabla^\mu\phi\nabla_\mu\phi + \phi V'(\phi)\right)\sqrt{-g}\ \mathrm{d}^4x + \int_{\mathfrak{E}} \nabla^\mu\left(\phi\nabla_\mu\phi\right)\sqrt{-g}\ \mathrm{d}^4x \\ &= -\int_{\mathfrak{E}} \left(\nabla_\mu\phi\nabla^\mu\phi + \phi V'(\phi)\right)\sqrt{-g}\ \mathrm{d}^4x + \int_{\partial\mathfrak{E}} \left(\phi\nabla_\mu\phi\right)\sqrt{-g}\ n^\mu\ \mathrm{d}^3\sigma.\end{aligned}$$

We will now, first, argue that the boundary term actually vanishes.

Since $\mathfrak{E}$ corresponds to the black hole exterior spacetime, $\partial\mathfrak{E}$ consists in the event horizon $\mathcal{H}$ and the spacetime asymptotic region $\mathcal{H}_\infty$, so

$$\int_{\partial\mathfrak{E}} \phi\nabla_\mu\phi\sqrt{-g}\ n^\mu\ \mathrm{d}^3\sigma = \int_{\mathcal{H}} \phi\nabla_\mu\phi\sqrt{-g}\ n^\mu\ \mathrm{d}^3\sigma + \int_{\mathcal{H}_\infty} \phi\nabla_\mu\phi\sqrt{-g}\ n^\mu\ \mathrm{d}^3\sigma.$$

The term computed on $\mathcal{H}_\infty$ vanishes since the spacetime is asymptotically flat (and we should then have $\nabla_\mu\phi \to 0$ "sufficiently fast" when approaching $\mathcal{H}_\infty$ to ensure that the energy-momentum tensor (2.23) will vanish and allow the asymptotic flatness).

The term computed on $\mathcal{H}$ also vanishes : Since the event horizon of a stationary, asymptotically flat black hole is a *Killing horizon*, n^μ, the normal to $\mathcal{H}$, will be a linear combination of the Killing fields ∂_t and ∂_φ.

Now, since $\partial_t\phi = 0 = \partial_\varphi\phi$ (**hyp. 2**), we get that $\left(n^\mu\nabla_\mu\phi\right)|_{\mathcal{H}} = 0$, which ensures that the term vanishes[2.10].

With this at hand, our previous equation reduces to

$$\int_{\mathfrak{E}} \left(\nabla_\mu\phi\nabla^\mu\phi + \phi V'(\phi)\right)\sqrt{-g}\ \mathrm{d}^4x \approx 0.$$

We now want to justify that each term in the integrand is a positive quantity over $\mathfrak{E}$.

First, $\nabla_\mu\phi\nabla^\mu\phi \geq 0$: Indeed, since ϕ shares the spacetime symmetries (**hyp. 2**), it must be invariant under both Killing fields ∂_t and ∂_φ. As a consequence, its gradient must be orthogonal to these vectors and will then be either spacelike or null; that is $\nabla_\mu\phi\nabla^\mu\phi \geq 0$.

To be able to conclude now, we will rely on **hypothesis 4** which will constrain the form of the potential to ensure that one always (*i.e.* both on shell and off shell) have :

$$\int_{\mathfrak{E}} \left(\nabla_\mu\phi\nabla^\mu\phi + \phi V'(\phi)\right)\sqrt{-g}\ \mathrm{d}^4x \geq 0.$$

[2.10] Provided ϕ and $\mathrm{d}^3\sigma$ are both finite at the horizon.

The only way to saturate the lower bound of this inequality is then to have $\phi(x^\mu) = \phi_i$ everywhere on $\mathfrak{E}$, with ϕ_i such that $\phi_i V'(\phi_i) = 0$.

$\square$

This result calls for several comments.

First of all, we have chosen, following [Herdeiro and Radu, 2015], to present the proof in the case of an axisymmetric spacetime. This choice helped us to identify the Killing fields and to explicitly write down an appropriated coordinate system. We should nevertheless emphasise that the proof works the same if the spacetime is static; one loses the Killing vector field $\vec{k}$ but it is entirely spectator in the argument. In fact, one can even drop the use of the null energy condition. In this case, we cannot use Hawking's rigidity theorem, but this does not necessarily affect the proof. The only important thing is the stationarity of spacetime, which ensures the existence of the timelike Killing $\vec{m}$ that reduces to ∂_t in an appropriated coordinate system, and the fact the event horizon is a Killing horizon. As long as both these properties are preserved, the proof will still hold.

One important characteristic of the proof is that it makes no explicit use of the metric field equation. This allows the result to be applied in quite a generic class of gravitational theories and demonstrates that the important condition is the minimal coupling. We should nevertheless stress that the metric field equation is indirectly used once when we use the form of the scalar field's energy-momentum tensor $T^{(\phi)}_{\mu\nu}$ to constrain the behaviour of $\nabla_\mu \phi$ in the asymptotic region $\mathcal{H}_\infty$. The fact that $T^{(\phi)}_{\mu\nu}$ should vanish on $\mathcal{H}_\infty$ to allow for the asymptotic flatness of spacetime[2.11] can be seen as a consistency condition on the metric equation rather than a use of it. While discussing this consistency condition, let us also quote that the vanishing of (2.23) will also impose that $V(\phi) \to 0$ when approaching $\mathcal{H}_\infty$. Since the theorem establishes that the scalar field must assume a constant value ϕ_i on $\mathfrak{E}$, this value – in addition to satisfy $\phi_i = 0 \vee V'(\phi_i) = 0$ – must be such that $V(\phi_i) = 0$ to ensure the consistency of the metric equation. The existence of such a value ϕ_i might then be seen as a condition on the self-interaction potential ensuring the consistency of the problem.

It is also important to realise that the above result is actually representative of a all class of results. The skeleton of this pioneer theorem from Bekenstein can actually be used to develop no-hair theorems in different situations. It is, for example, relatively easy to adapt the proof in the case of a minimally coupled self-interacting complex scalar field, making a few assumptions on the self-interaction potential and changing the energetic condition.

We could also mention that, apart from the energetic condition used at the very end, the core of the proof, aiming to obtain of a positive definite integrand, uses only the assumptions on the symmetries of the solution.

[2.11] unless some very specific, but unlikely, cancelation occurs in the metric equation, the vanishing of $T^{(\phi)}_{\mu\nu}$ on $\mathcal{H}_\infty$ will be necessary for the consistency of the system.

Understanding the structure of this result and the role of each hypothesis can give deep insight on the way to construct black holes endowed with scalar hair. We will thus now comment on the different hypothesis concerning the scalar field (by decreasing order).

Hypothesis 4 : This assumption might appear as the most artificial one. It is used only at the very last step of the proof and seems to be "fine-tuned" to reach the conclusion. This is true to some extent but the importance of hypothesis 4 does not rely on its explicit form. It is indeed easy to tweak a bit the above proof to arrive at a different integral which would require another condition on the potential to have a definite sign[2.12]. That being said, in any case, one would need such a condition. Hypothesis 4 then simply reveals that one cannot obtain a no-go result for an arbitrary potential. We should also mention that the condition imposed on the potential here is satisfied in the case of a massive scalar field since, in this case, the potential, in natural units, is $V(\phi) = \frac{m^2}{2}\phi^2$.

Hypothesis 3 : As we have already discussed, the minimal coupling is a simplicity hypothesis regarding the behaviour of the scalar field. It is thus very important for our purpose to realise that it is a hypothesis and not a necessary condition to study scalar fields on curved spacetime.

Hypothesis 2 : This is probably the most natural hypothesis. If one aims a system with certain symmetries, it is almost implicit that all the components of the system will be assumed to present these symmetries. The situation is nonetheless a bit more subtle here. If we just want to construct a black hole with certain properties (here asymptotic flatness and stationarity) endowed "by any means" with a non-trivial scalar profile, it is not mandatory to have a scalar field presenting exactly the same properties. For consistency of the metric equation, it is necessary that all the terms in this equation present the whished symmetries, but nothing more. In other words, it is $T^{(\phi)}_{\mu\nu}$, and not directly ϕ, who should necessarily share the spacetime symmetries. Of course, if ϕ violates some of the spacetime symmetries, the total system (composed by the black hole spacetime and the scalar field) will have a smaller symmetry group than that of spacetime itself, but it is *a priori* possible to have a scalar field that consistently violates some of the spacetime symmetries[2.13], making hypothesis 2 truly a hypothesis. This point is even more important to stress if one realises that this is a key hypothesis in the construction of the proof. It is indeed the

[2.12] As an example, instead of multiplying the Klein-Gordon equation by ϕ before doing the integration over $\mathfrak{E}$, we could have multiply the equation by $V'(\phi)$, giving

$$\int_{\mathfrak{E}} \left(V''(\phi)\, \nabla_\mu \phi \nabla^\mu \phi + \left[V'(\phi) \right]^2 \right) \sqrt{-|g|}\, \mathrm{d}^4 x \approx 0$$

after the integration by parts. The second term beying obviously non-negative, one would have imposed convexity of the potential, *i.e.* $V''(\phi) \geq 0,\ \forall \phi$, to reach the conclusion.

[2.13] For example, assuming the potential V vanishes, $T^{(\phi)}_{\mu\nu}$ will only depend on the derivatives of ϕ, so one could try to have a real scalar field with a linear time dependence. This would lead to no direct contradiction even if we keep assuming that spacetime is stationary.

most used in the above proof.

Before concluding the discussions about this result, let us comment on the research possibilities suggested by an abandonment of the different hypothesis.

To abandon hypothesis 4 : This simply suggests considering more exotic self-interaction potentials. This is indeed always a possibility but one does not need the above theorem to motivate it.

To abandon hypothesis 3 : Questioning the validity of this hypothesis account for a challenge of the minimal coupling principle for scalar fields and for an opening to the numerous coupling possibilities available on curved spacetime. Intuitively stated, this corresponds to the idea that the simplest principle might not be the most interesting one. This is the research axis in which this thesis takes place. It is also the topic of the next section.

To abandon hypothesis 2 : As we have already discussed, this is a more subtle task to accomplish. It might nevertheless be fruitful. This was first demonstrated in [Herdeiro and Radu, 2014] where, enabling a harmonic time dependence for the scalar field, the authors were able to construct stationary asymptotically flat black holes endowed with a non-trivial $U(1)$-invariant complex scalar field minimally coupled to Einstein gravity.

2.4 Non-Minimal Couplings

In this section, we present the framework of Horndeski gravity. This corresponds to a general class of scalar-tensor theories of gravity presenting appealing properties. We will start by a more formal definition of what we mean by "appealing properties" and explain how Horndeski gravity satisfy these requirements. We will then turn to a summary of the rediscovery of Horndeski gravity in the context of the so-called Galileon theory. This should provide a more intuitive way to apprehend the theory. Finally, following the same approach as in the previous section, we will present a no-scalar-hair theorem for Horndeski gravity, due to Hui and Nicolis [Hui and Nicolis, 2013]. This will allow us to emphasise how evading this result naturally leads to some of the theories considered in this thesis.

2.4.1 Desired Properties and First Form for the Horndeski Lagrangian

The main interest of the notion of minimal coupling was its simplicity. If one aims to find other ways to couple scalar fields to gravity, it is necessary to select a set of guiding principles constraining the possible Lagrangian for the theory. Otherwise, the possibilities are virtually endless, but many of them certainly lack of physical interest.

Apart from this arguable simplicity, another advantage of the minimal coupling is how it naturally connects GR's formalism to the notion of scalar field known on Minkowski spacetime. More precisely,

1. the gravitational setup is unchanged with respect to GR : spacetime is a differential manifold equipped with a Lorentzian metric and the associated Levi-Civita connection,

2. the Lagrangian formulation of the theory leads to second order partial differential equations, as it is the case for both GR and scalar fields in Minkowski spacetime.

If we want to see our scalar-tensor theory as a generalisation of GR (which is a reasonable idea for a theory of gravitation considering the numerous successes of GR) the first property appears as a natural request – at least for a first attempt and considering that we want to place the generalisation in the scalar sector of the theory. The second property is more technical and its physical importance might then be more subtle. Yet, considering that theories allowing for higher-order equations of motion are usually plagued by unwished ghosts[2.14], it also appears *a priori* as a desirable property for a generalisation of GR – again, especially for a first attempt.

One very interesting question to ask is then :

"What is *the most general* theory (arising from a variational principle) including a single real scalar field, a single metric tensor equipped with its Levi-Civita connection and giving second order Euler-Lagrange equations ?"

This question has been first addressed by the mathematician Gregory Walter Horndeski in the early 1970s [Horndeski, 1974]. He derived the most general form for the Lagrangian of a 4 dimensional scalar-tensor theory presenting these properties.

Using Horndeski's original notation, adopting the convention that the action is written as

$$S = \int_{\mathcal{M}} \mathscr{L}_{\mathrm{H}}^{1974} \sqrt{-g}\, \mathrm{d}^4 x,$$

the Lagrangian density $\mathscr{L}_{\mathrm{H}}^{1974}$ is given by

$$\begin{aligned}
\mathscr{L}_{\mathrm{H}}^{1974} = \delta^{ijk}_{\mu\nu\sigma} \Big[& \kappa_1 \nabla^\mu \nabla_i \phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho} \nabla^\mu \nabla_i \phi \nabla^\nu \nabla_j \phi \nabla^\sigma \nabla_k \phi + \kappa_3 \nabla_i \phi \nabla^\mu \phi R_{jk}{}^{\nu\sigma} \\
& - 4\kappa_{3,\rho} \nabla_i \phi \nabla^\mu \phi \nabla^\nu \nabla_j \phi \nabla^\sigma \nabla_k \phi \Big] \\
& + \delta^{ij}_{\mu\nu} \left[\left(F + 2W\right) R_{ij}{}^{\mu\nu} + \left(2\kappa_8 \nabla_i \phi \nabla^\mu \phi - 4F_{,\rho} \nabla^\mu \nabla_i \phi\right) \nabla^\nu \nabla_j \phi \right] \\
& - 3 \Big[2\left(F + 2W\right)_{,\phi} + \rho\kappa_8 \Big] \nabla^\mu \nabla_\mu \phi + \kappa_9,
\end{aligned} \tag{2.25}$$

where $\rho = \nabla_\mu \phi \nabla^\mu \phi$, $R^\alpha{}_{\beta\mu\nu}$ is the Riemann tensor,

$$\delta^{i_1 i_2 \dots i_n}_{\mu_1 \mu_2 \dots \mu_n} = n! \delta^{[i_1}_{\mu_1} \delta^{i_2}_{\mu_2} \dots \delta^{i_n]}_{\mu_n},$$

[2.14] This is the so called Ostrogradski's instability [Ostrogradsky, 1850].

the κ_a ($a \in \{1,3,8,9\}$) are **arbitrary** functions $\kappa_a(\phi,\rho)$,

$$\kappa_{a,\rho} = \partial\kappa_a/\partial\rho,$$

$F = F(\phi,\rho)$ must satisfy

$$F_{,\rho} = \kappa_{1,\phi} - \kappa_3 - 2\rho\kappa_{3,\rho}$$

and $W = W(\phi)$ is also an **arbitrary** function.

This expression necessitates a few comments. First of all, let us note that this Lagrangian depends explicitly on the second-order derivatives of ϕ and $g_{\mu\nu}$ (via the Riemann tensor) even though the field equations are ensured to be at most of second-order. Secondly, the fact that it is the most general option presenting the whished properties is entirely contained in the few arbitrary functions present in its expression : the 4 functions $\kappa_a(\phi,\rho)$ and the function $W(\phi)$. The function $F(\phi,\rho)$ does not introduce any new freedom in the expression. Indeed $F_{,\rho}$ is fixed in terms of κ_1 and κ_3 so that F is determined up to an arbitrary function of ϕ which is basically represented by W since these two functions always appear in (2.25) via the combination $F + 2W$.

Being the most general Lagrangian density leading to second order field equations, (2.25) covers a larger class of well-known theories.

Example 2.1. *If $\kappa_a = 0$ for $a = 1,3,8$, $\kappa_9 = -2\kappa\Lambda$ and $F(\phi) + 2W(\phi) = \frac{\kappa}{2}$ with $\kappa = c^4/(16\pi\mathcal{G})$,* (2.25) *reduces to*

$$\begin{aligned}\mathscr{L}_{\mathrm{H}}^{1974} &= \frac{\kappa}{2}\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 2\kappa\Lambda = \frac{\kappa}{2}2!\delta^i_\mu\delta^j_\nu R_{ij}{}^{\mu\nu} - 2\kappa\Lambda \\ &= \kappa\left(R - 2\Lambda\right),\end{aligned}$$

i.e. *it reduces to the Einstein-Hilbert Lagrangian density with a cosmological constant Λ* (1.185)*.*

Example 2.2. *If $\kappa_a = 0$ for $a = 1,3,8$, and $F(\phi) + 2W(\phi) = 0$,* (2.25) *reduces to*

$$\mathscr{L}_{\mathrm{H}}^{1974} = \kappa_9(\phi,\rho).$$

So any function (at least $\mathcal{C}^2$) of ϕ and its kinetic term will lead to second order field equations. In particular, with $\kappa_9(\phi,\rho) = -\frac{1}{2}\rho - V(\phi)$, one recovers the Klein-Gordon Lagrangian (2.21)*.*

Example 2.3. *If $\kappa_a = 0$ for $a = 1,3,8$, $\kappa_9(\phi,\rho) = -\kappa\left(\frac{\omega(\phi)}{\phi}\rho + V(\phi)\right)$ and $F(\phi) + 2W(\phi) = \frac{\kappa}{2}\phi$, with $\kappa = c^4/(16\pi\mathcal{G})$, we get the Lagrangian density*

$$\begin{aligned}\mathscr{L}_{\mathrm{H}}^{1974} &= \frac{\kappa}{2}\phi\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 3\kappa\nabla^\mu\nabla_\mu\phi - \kappa\left(\frac{\omega(\phi)}{\phi}\rho + V(\phi)\right) \\ &= \frac{\kappa}{2}\phi 2!\delta^i_\mu\delta^j_\nu R_{ij}{}^{\mu\nu} - \kappa\left(\frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right) - 3\kappa\nabla^\mu\nabla_\mu\phi \\ &= \kappa\left(\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right) + \nabla_\mu\left(-3\kappa g^{\mu\nu}\nabla_\nu\phi\right).\end{aligned}$$

This corresponds to the pioneer scalar-tensor theory from Brans and Dicke (up to the divergency $\nabla_\mu \left(-3\kappa g^{\mu\nu}\nabla_\nu\phi\right)$ *which does not affect the field equations).*

Despite the fact that it provides the general framework to generalise minimally coupled scalar fields on curved spacetime, Horndeski's work sank into oblivion for a few decades before being rediscovered "accidentally" in the context of the Galileon theory. This is the topic of the next paragraph. It is worth noting that Horndeski cracked the problem from a purely mathematical point of view, with no aim for an application to physics whatsoever. This might partially explain why his result slipped through physicist's fingers for a while.

2.4.2 From Galileon Theory to Horndeski Gravity

Overlooked for a while, Horndeski theory was finally rediscovered in the early 2000s as a generalisation of the so-called Galileon theory. In this paragraph, we schematically present the steps in the construction of the Galileon theory and its generalisations, culminating with the rediscovery of the Horndeski Lagrangian. This discussion will allow us to shine a light on Horndeski theory from a different (and more recent) perspective. This will also lead to an equivalent but more user-friendly form of the Lagrangian density (2.25).

Galileon Theory

The model for the Galileon was first proposed in [Nicolis et al., 2009]. In this paper, the authors aim at finding criteria ensuring a consistent generalisation of GR. Here, by "consistent" we mean an alternative theory of gravity that can produce sizable deviations from GR at the cosmological scale but not at the solar system scale (*i.e.* locally and in the low field regime) where the theory is well established. This then corresponds to a theory with a built-in mechanism allowing to "screen" the modification at the solar system scale. Such a mechanism is then known as a screening mechanism.

Inspired by the DGP model (see remark 2.1), the authors focus their interest on the cases where the modification to GR can be encoded in a single real scalar degree of freedom, denoted by π, and where the screening mechanism is of Vainshtein type (see remark 2.2). They then aim to implement the screening mechanism by means of non-linearities in the field equation of π that would remain important even in situations where π can be treated as a small perturbation to spacetime geometry and appear only at linear level in the metric equations. This is their first assumption.

Remark 2.1. *The Dvali-Gabadadze-Porrati (DGP) model was a model aiming to modify GR via the study of the* 4 *dimensional gravitational theory emerging from the dynamics of a* 3*-brane on* 5 *dimensional Minkowski spacetime, see [Dvali et al., 2000]. One of the aims of the paper was to describe a mechanism allowing explaining the difference of behaviour of the gravitational potential at short and long distances.*

It turns out, see [Nicolis and Rattazzi, 2004], that the physics of the DGP model can be accurately captured at short distances[2.15] *by a* 4 *dimensional effective Lagrangian. This Lagrangian involves the usual* 4 *dimensional GR theory and a single real scalar field with some specific non-minimal derivative couplings – without any explicit reference to the* 5 *dimensional origin of the model.*

This property and the ability of the original DGP model to describe a theory of gravity with different behaviours at different scales are the main motivations of [Nicolis et al., 2009] to consider a single real scalar field subject to important non-minimal derivative couplings on top of a 4 *dimensional curved spacetime.*

Remark 2.2. *In a 1972 paper, see [Vainshtein, 1972], A.I. Vainshtein studied the problem of the finite discrepancies between GR and the perturbative approach to Fierz-Pauli massive gravity in the case of a very small (yet non-vanishing) graviton mass. This was indeed puzzling since one would sensibly expect the results of massive gravity to reduce to the GR ones in the limit of a vanishing graviton mass.*

*Vainshtein's paper is devoted to show, in a spherically symmetric case, how the perturbative approach (*i.e. *linearised in the metric functions) is inapplicable. This is due to higher order terms in the expansion that would be divergent – and then not negligible – in the zero mass limit, ruining the validity of the approximation.*

He also shows how a consistent zero mass limit can be recovered by going beyond the perturbative approach. The key to this point is to keep track of some specific non-linear terms in the extra-equation present in the massive case that remains important even for small graviton mass.

The prevalence of these non-linear terms in the dynamics is what screens the presence of the extra degree of freedom for very small mass and allows recovering an agreement with GR under a certain graviton mass.

This is, simplifying a little bit, the take-home message from Vainshtein's discussion : the concealment of the "unwanted" degree of freedom rely on the specific form of the non-linear terms in its equation.

Following this line, peoples have started to use the denomination "Vainshtein screening" *to refer to situations where the dynamical dissimulation of some degrees of freedom is due to specific non-linearities in the field equations.*

Their objective being to constrain the behaviour of π in contexts where it can be treated as a small perturbation to the geometry and in the low field regime, this motivates to discuss π's dynamics on Minkowski spacetime.

To achieve their goal, relying on cosmologically inspired arguments, the authors argue that the field equation for π will have to be invariant under the transformation

$$\pi \to \pi + b_\mu x^\mu + c, \tag{2.26}$$

for generic constants b_μ and c. This is their second assumption. The existence of this symmetry is what motivates the authors to call this scalar field "galileon"

[2.15] "short" meaning distances shorter (in geometrical units) that H^{-1}, where H is the Hubble parameter.

(see remark 2.3). Imposing this symmetry is equivalent to require that any π in the field equation is acted upon by *at least* two derivatives. But, since the avoidance of ghosts motivates to study theories with field equations of order two or less *i.e.* theories for which any π in the field equations is acted upon by *at most* two derivatives, they add this third assumption to arrive at the conclusion that the galileon theory must be such that the field equation for π is of the form

$$\frac{\delta S_\pi}{\delta \pi} = \mathcal{F}\left(\partial_\mu \partial_\nu \pi\right) \approx 0, \tag{2.27}$$

where $\mathcal{F}$ is a non-linear (according to the first assumption) Lorentz-invariant function of the tensor $\partial_\mu \partial_\nu \pi$.

Remark 2.3. *The name Galileon was chosen because of the similarities between* (2.26) *and the Galilean transformations from classical mechanics (a "*$0+1$ *field theory"). In particular* (2.26) *implies the invariance of the theory under a constant shift of* π*'s gradient* $\partial_\mu \pi \to \partial_\mu \pi + b_\mu$ *which can be seen as a generalisation of the Galilean symmetry* $\dot{x} \to \dot{x} + v_0$*. This is what motivates the name.*

The authors then show that, in 4 dimensions, the most general Lagrangian density satisfying (2.27) is a linear combination of 5 different terms

$$\mathscr{L}_{\text{Gal}} = \sum_{i=1}^{5} c_i \mathscr{L}_i, \tag{2.28}$$

where c_i are real constants and the $\mathscr{L}_i$'s can be found in appendix A. Let us nevertheless stress here that some of the terms $\mathscr{L}_i$ do explicitly depend on $\partial_\mu \partial_\nu \pi$ and that the equations are second order (and not fourth order as it would be the case in general) due to some specific cancellations of the higher order terms in the field equation. This is the key property in the construction of (2.28) (see appendix A for more details).

To summarise this discussion, Galileon theory describes the most general dynamics for a scalar field π on Minkowski spacetime such that the field equation is of the form (2.27). The Lagrangian for this theory has the relatively simple structure (2.28).

Generalised Galileon

The prevalence of non-linearities being the most important requirement for Vainshtein mechanism to have a chance to work, one natural way to generalise the Galileon construction is to relax a bit the form of the equation to allow for dependencies in the lower derivatives of π as well. That is

$$\frac{\delta S_\pi}{\delta \pi} = \mathcal{F}\left(\pi, \partial_\mu \pi, \partial_\mu \partial_\nu \pi\right) \approx 0, \tag{2.29}$$

where $\mathcal{F}$ is a non-linear Lorentz-invariant function. In [Deffayet et al., 2011] the authors prove that to obtain the most general theory satisfying this property, one just needs to replace the constants c_i in (2.28) by arbitrary functions of π

and its usual kinetic term $\partial_\mu \pi \partial^\mu \pi$. They then get the Lagrangian density for the generalised Galileon in the form

$$\mathscr{L}_{\mathrm{GG}} = \sum_{i=2}^{5} f_i \left(\pi, \partial_\mu \pi \partial^\mu \pi\right) \mathscr{L}_i, \tag{2.30}$$

where the $\mathscr{L}_i$'s are the ones found in (2.28).[2.16]

The generalised Galileon then provides, by construction, the most general scalar field action leading (at most) to a second order field equation on flat spacetime. In this respect, note that the usual Klein-Gordon Lagrangian density from (2.17) is trivialy recovered as a special case of the first term in (2.30).

Covariant generalised Galileon

In [Deffayet et al., 2011] again, the authors discussed the generalisation of (2.30) on curved spacetime. First, they show that the minimal covariantisation procedure (2.19) would lead to higher-order equations in terms of the metric function.[2.17] Then, they establish that these higher order terms in the field equations can be removed by adding (unique) appropriate counter-terms in the action. One then gets what we could call the covariant generalised Galileon theory, whose Lagrangian density is given by

$$\begin{aligned}\mathscr{L}_{\mathrm{CGG}} =& K(\phi,\rho) - G_3(\phi,\rho)\Box\phi + G_4(\phi,\rho)R + G_{4,\rho}(\phi,\rho)\left[(\Box\phi)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2\right] \\ & + G_5(\phi,\rho)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ & - \frac{1}{6}G_{5,\rho}(\phi,\rho)\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right],\end{aligned} \tag{2.31}$$

where we renamed the scalar field ϕ to distinguish it from the above constructions on flat spacetime,

$$\begin{aligned}\left(\nabla_\mu\nabla_\nu\phi\right)^2 &= \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi, \\ \left(\nabla_\mu\nabla_\nu\phi\right)^3 &= \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi, \\ \rho &= \nabla_\mu\phi\nabla^\mu\phi,\end{aligned}$$

and where the functions $G_i(\phi,\rho)$ ($i \in \{3,4,5\}$) and $K(\phi,\rho)$ are **arbitrary** functions. The functions K, G_3, $G_{4,\rho}$ and $G_{5,\rho}$ can be seen as the curved

[2.16] Note that the sum starts at $i = 2$ here since the first two terms can be reabsorded in the same aribtrary funtion of π and $\partial_\mu\pi\partial^\mu\pi$, called K in (2.31); see the form of $\mathscr{L}_1$ and $\mathscr{L}_2$ in appendix A.

[2.17] This happens because the cancelation of the higher order terms in the field equation on flat spacetime occurs thanks to the automatic commutation of partial derivatives ∂_μ. On curved spacetime, for a scalar field, a group of more than two covariant derivatives ∇_μ does not commute anymore, leading to terms involving derivatives of the Riemann tensor (*i.e.* third order derivatives of the metric) in the field equations.

spacetime extension of the terms[2.18] $f_i\left(\pi, \partial_\mu\pi\partial^\mu\pi\right)$ appearing in (2.30) for $i = 2, 3, 4, 5$ respectively.

In (2.31), the counter terms are the terms proportional to G_4 and G_5. They counter the higher order terms induced in the field equations by the variation of the terms proportional to $G_{4,\rho}$ and $G_{5,\rho}$ respectively. This is why in each of these two pairs of terms the multiplicative factors are not independent from each other. It is then clear that the above expression, just like its flat spacetime precursor, does still depend exactly on 4 arbitrary functions of the scalar field and its kinetic term (written as K, G_3, G_4 and G_5 for convenience).

Equivalence to Horndeski Lagrangian

Now, as spoiled as a motivation at the beginning of this paragraph, the covariant generalised Galileon theory has been proven to be equivalent to Horndeski gravity. Using the replacement rule :

$$
\begin{aligned}
K(\phi,\rho) &= \kappa_9(\phi,\rho) + \rho\int^{\rho}\left(\kappa_{8,\phi}(\phi,\rho') - 2\kappa_{3,\phi,\phi}(\phi,\rho')\right)\mathrm{d}\rho', \\
G_3(\phi,\rho) &= 6F_{,\phi}(\phi,\rho) + \rho\kappa_8(\phi,\rho) + 4\rho\kappa_{3,\phi}(\phi,\rho) - \int^{\rho}\left(\kappa_8(\phi,\rho') - 2\kappa_{3,\phi}(\phi,\rho')\right)\mathrm{d}\rho', \\
G_4(\phi,\rho) &= 2\left(F(\phi,\rho) + \rho\kappa_3(\phi,\rho)\right), \\
G_5(\phi,\rho) &= -4\kappa_1(\phi,\rho),
\end{aligned}
$$

(2.31) can be recast in the form (2.25), see appendix A of [Kobayashi et al., 2011].

This result nevertheless was *a priori* highly non-trivial ! Firstly, even though the generalised Galileon provided the most general Lagrangian density for a scalar field with second order field equation on flat spacetime there was, again, *a priori*, no reasons why its covariant extension should still be the most general possibility on curved spacetime; there might have been other curvature dependent terms missing. Secondly, the fact that the restriction of the Horndeski Lagrangian to Minkowski spacetime would give the most general Lagrangian density with second order field equation for the scalar field on flat spacetime was not obvious either. Horndeski's construction contains a condition on the metric field equation which is inapplicable to flat spacetime and could *a priori* restrict the obtained theory. The proof of the equivalence of these two theories was then non-trivial in both directions. That being said, the two theories are dynamically equivalent and then, to any classical purpose, we can (and will) take the Lagrangian density obtained from the covariant generalised Galileon for the one of Horndeski theory. In other words, in the following, when referring

[2.18] More precisely, these would correspond to the curved spacetime equivalents of $f_i\left(\pi, \partial_\mu\pi\partial^\mu\pi\right)\partial_\alpha\pi\partial^\alpha\pi$; see appendix A.

to the Horndeski Lagrangian density, we will refer to the Lagrangian density

$$\begin{aligned}\mathscr{L}_{\mathrm{H}} =& K(\phi,\rho) - G_3(\phi,\rho)\Box\phi + G_4(\phi,\rho)R + G_{4,\rho}(\phi,\rho)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]\\ &+ G_5(\phi,\rho)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\\ &- \frac{1}{6}G_{5,\rho}(\phi,\rho)\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla_\mu\nabla_\nu\phi)^2 + 2\,(\nabla_\mu\nabla_\nu\phi)^3\right],\end{aligned} \tag{2.32}$$

using the same conventions as in (2.31).

This new form does, indeed, present several advantages with respect to the original $\mathscr{L}_{\mathrm{H}}^{1974}$. First of all, from a physical perspective, its link to the Galileon theory gives more motivations to the study of this theory since it clearly indicates its filiation with Vainshtein screening. This makes it a good potential candidate for a consistent modification of GR.[2.19] Obviously, one should add to that the fact that its link to Horndeski's pioneer work makes it *the* good framework for a systematic study scalar-tensor theories of gravity satisfying the requirements of section 2.4.1. Indeed, there cannot be any (potentially interesting) missing term to consider. Also, from a more practical point of view, it is much easier to identify the different sectors of the theory from $\mathscr{L}_{\mathrm{H}}$ and it displays in a clearer way how this construction generalises both GR and a minimally coupled scalar field. We illustrate this last statement below with a few examples.

Example 2.4. *Considering the case where $G_3 = 0 = G_5$, $G_4 = \kappa$ and $K = -2\kappa\Lambda$, with $\kappa = c^4/(16\pi\mathcal{G})$ and Λ a real constant, one recovers the Einstein-Hilbert Lagrangian from example 2.1.*

Example 2.5. *The case where $G_i = 0$, $\forall i \in \{3,4,5\}$, corresponds to example 2.2.*

Example 2.6. *The case $G_3 = 0 = G_5$, $G_4 = \kappa\phi$ and $K = -\kappa\left(\frac{\omega(\phi)}{\phi}\rho + V(\phi)\right)$ corresponds to Brans-Dicke model from example 2.3.*

Example 2.7. *One can also show, see [Kobayashi et al., 2011], that, given a function $\mathcal{F}(\phi)$, the choice*

$$\begin{aligned} K &= 2\mathcal{F}^{(4)}\rho^2\,(3 - \ln|\rho/2|)\,,\\ G_3 &= -2\mathcal{F}^{(3)}\rho\,(7 - 3\ln|\rho/2|)\,,\\ G_4 &= -2\mathcal{F}^{(2)}\rho\,(2 - \ln|\rho/2|)\,,\\ G_5 &= -4\mathcal{F}^{(1)}\ln|\rho/2|,\end{aligned} \tag{2.33}$$

[2.19] Let us note here that Vainshtein screening motivates the study of a theory since it makes it a plausible candidate for a modification of GR that would be visible at the cosmological scale while being "hidden" at solar system scale (again, locally *and* in the low field regime). This, in itself, does not inform on the possibility for the theory to produce interesting compat objects, especially hairy black holes. As a consequence, despite being the original motivation to its introduction in the context of the Galileon theory, an eventual fail of (2.32) to produce a satisfactory Vainshtein screening or cosmological model would not necessarily spoil its interest for the study of hairy black holes.

where $\mathcal{F}^{(n)} = \mathrm{d}^n\mathcal{F}/\mathrm{d}\phi^n$*, will lead, after several integration by parts, to a non-minimal coupling of the form*

$$\mathcal{F}(\phi)\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\right),$$

where the term [2]

$$\mathscr{L}_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \tag{2.34}$$

is the Gauss-Bonnet invariant.

Example 2.8. *Finally, let us point out that the only explicit dependence in* ϕ *in* (2.32) *is in the arbitrary functions. This means that if we assume* $K_{,\phi} = 0 = G_{i,\phi}$, $\forall i \in \{3,4,5\}$, $\mathscr{L}_{\mathrm{H}}$ *will only depend on* ϕ*'s derivatives and the theory will be invariant under a shift of the scalar field* $\phi \to \phi + c$ *for any real constant* c*.*

For this sector of the theory, known as the shift-symmetric sector, according to Nœther's theorem, the scalar field equation reduces to a conservation law

$$\nabla_\mu J^\mu = 0, \tag{2.35}$$

with

$$J^\mu = \frac{\partial \mathscr{L}_{\mathrm{H}}}{\partial\left(\nabla_\mu \phi\right)} - \nabla_\nu \left(\frac{\partial \mathscr{L}_{\mathrm{H}}}{\partial\left(\nabla_\mu \nabla_\nu \phi\right)}\right). \tag{2.36}$$

2.4.3 No-hair Theorem for Horndeski Gravity

In this paragraph, we will review a no-hair theorem developed for Horndeski theory in [Hui and Nicolis, 2013]. Similarly to what we did in section 2.3.4, we will detail both the hypothesis and the proof of the result as a way to emphasise how it constrains the study of black holes in Horndeski gravity and what are the possible ways to evade it. This last part will be done in the next paragraph.

No Scalar-Hair Theorem 2 (Hui & Nicolis)**.** *Consider an asymptotically flat black-hole spacetime.*

Hypothesis 1 : (**Symmetries of spacetime**)
The spacetime is spherically symmetric.

Hypothesis 2 : (**Symmetries of the scalar field**)
The scalar field shares the spacetime symmetries.

Hypothesis 3 : (**Coupling condition**)
The scalar field is real and satisfy the field equations of the shift-symmetric sector of (2.32).

Hypothesis 4 : (**Regularity condition**)
The norm of the conserved current J^μ associated to the shift symmetry of the scalar field is finite at the black hole event horizon.

Then*, the scalar field must be trivial:* $\phi\left(x^\mu\right) = 0, \forall x^\mu$*.*

Proof.

Under the first hypothesis, it is well known that one can choose, without loss of generality, a coordinate system adapted to the spherical symmetry $\{x^\mu\} = \{t, r, \theta, \varphi\}$ such that the metric reads

$$\boldsymbol{g} = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + \rho^2(r)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right),$$

where f and ρ are generic functions of r. From **hypothesis 2**, one will then get that $\phi(x^\mu) = \phi(r)$.

As we already mentioned in example 2.8, **hypothesis 3** will ensure the existence of a conserved current J^μ of the form (2.36) associated to the scalar field equation. The core of the proof will then concern the form of J^μ. It can be decomposed into four steps.

Step 1: The only non-vanishing component of J^μ is J^r.

This follows **from hypotheses 1 and 2** (*i.e.* from the symmetries). Indeed, J^μ will be a covariant combination of ϕ, $g_{\mu\nu}$ and their derivatives. Assuming spherical symmetry, it can only depend on r.

J^θ and J^φ will then have to vanish: for any given r, J^θ and J^φ will define a constant vector over the two-sphere of radius $\rho(r)$. Since any regular vector on the two-sphere must vanish at least at one point, J^θ and J^φ must vanish identically.

J^t also vanishes: this comes from the fact that a non-vanishing J^t would select a preferred time direction. This would be in contradiction with the space-time symmetry. More precisely, J^t would have to change his sign under a time reversal $t \to -t$ to ensure the covariance of J^μ. Nevertheless, from the spherical symmetry, J^t can only depend on r and would then remain unchanged under this transformation. The only way to reconcile both facts is that $J^t = 0$.

Step 2: J^r vanishes at the event horizon.

This is a direct consequence of **hypothesis 4**. Indeed, since we now know that J^r is the only non-vanishing component of the current, its (pseudo-)norm is given by

$$J^\mu g_{\mu\nu} J^\nu = \frac{(J^r)^2}{f}.$$

Due to the spherical symmetry, the event horizon $\mathcal{H}$ of the black hole coincide with the locus where the time killing vector ∂_t becomes a null vector. This thus leads to

$$g_{\mu\nu}\left(\partial_t\right)^\mu\left(\partial_t\right)^\nu|_{\mathcal{H}} = g_{tt}|_{\mathcal{H}} = -f|_{\mathcal{H}} = 0.$$

In other words, the function f must vanish at the event horizon. The only way to maintain the finiteness of $J^\mu g_{\mu\nu} J^\nu$ on $\mathcal{H}$ is then to have $J^r|_{\mathcal{H}} = 0$.

Step 3: J^r vanishes everywhere.

We know that J^μ is a conserved current. From what we got at step 1, the conservation law can be straightforwardly integrated to obtain the form of J^r:

$$\frac{1}{\sqrt{-|g|}}\partial_r\left(\sqrt{-|g|}J^r\right) = 0 \Leftrightarrow \partial_r\left(\rho^2 J^r\right) = 0 \Leftrightarrow J^r(r) = \frac{c_0}{\rho^2(r)},$$

where c_0 is an integration constant. Note that $\rho|_{\mathcal{H}} \neq 0$ since it gives the radius of the constant-r spheres. The condition $J^r|_{\mathcal{H}} = 0$ from step 2 then gives $c_0 = 0$, so that

$$J^r(r) = 0, \forall r.$$

Step 4: ϕ vanishes everywhere.

The aim of this last step of the proof is to specify the form of J^μ and to use it to constrain the behaviour of ϕ itself. The central argument is that J^μ assumes the form

$$J^\mu = \Xi^{\mu\nu}\nabla_\nu\phi, \tag{2.37}$$

for a certain tensor $\Xi^{\mu\nu}\left(\partial_\alpha\phi, \partial_\alpha\partial_\beta\phi; g_{\rho\sigma}, \partial_\alpha g_{\rho\sigma}, \partial_\alpha\partial_\beta g_{\rho\sigma}\right)$ which will depend on polynomial combinations of its different variables and on the derivatives of the arbitrary functions present in (2.32) (see remark 2.4). In this case, for our spherically symmetric spacetime, this would reduce to

$$J^r = \Xi^{rr}\left(\phi'; f, \rho, f', \rho', f'', \rho''\right)\phi',$$

where the $'$ denotes a derivative with respect to r.[2.20]

The authors then discuss the behaviour of Ξ^{rr} in the asymptotic region. Given that spacetime is asymptotically flat, we must have $f \to 1$ and $\phi' \to 0$ for $r \to \infty$. They then claim that this will lead Ξ^{rr} to approach a *non-zero* constant provided that, in the weak field regime, the theory present a standard kinetic term[2.21] for ϕ. In other words, one would have

$$\Xi^{rr} \underset{r\to\infty}{\longrightarrow} C \neq 0.$$

This would come from the fact that, if we assume that, in the weak field regime, the dominant contribution of ϕ in the action comes from its standard kinetic term, the dominant term of J^r would be of the form $C\phi'$ for some non-zero constant C. Note that this would have to hold due to the ϕ' dependence of Ξ^{rr}, independently of the exact form of the metric. This means that what we really have is

$$\Xi^{rr} \underset{\phi'\to 0}{\longrightarrow} C \neq 0.$$

As a consequence, ϕ' must vanish everywhere. Indeed, we got from step 3 that J^r must identically vanish. Using the fact that Ξ^{rr} must be non-vanishing

[2.20]Note that in this case, the field equation for ϕ (2.35) being of second order, Ξ^{rr} cannot depend on ϕ''. This is a consequence of the spherical symmetry which turns the field equation into an ordinary differential equation (instead of a partial differential equation). In the general case, $\Xi^{\mu\nu}$ will explicitely depend on $\partial_\mu\partial_\nu\phi$ with no contradiction with the porperty that (2.35) remains second order. This is due to the specific form of the coupling terms in (2.32).

[2.21]This will be the case if the arbitrary function $K(\rho)$ in (2.32) is such that $K_{,\rho}(0) \neq 0$. This might be seen an the extra assumption related to the idea that ϕ is a scalar field in the usual sense (that is as in (2.21)) with several extra non-minimal couplings.

at spatial infinity (where $\phi' \to 0$), we obtain that ϕ' should remain equal to zero if one goes to a smaller radius. Otherwise, as soon as ϕ' deviates from 0, by continuity, Ξ^{rr} would remain different from zero and multiply a non-vanishing value of ϕ' to give $J^r \neq 0$, in contradiction with our previous conclusions. We must then have

$$\phi'(r) = 0, \forall r,$$

so that $\phi(r) = \phi_0, \forall r$. Of course, using the shift symmetry, this is equivalent to

$$\phi(x^\mu) = 0, \forall x^\mu.$$

□

Remark 2.4. *The explicit form of the conserved current J^μ can be obtained by a direct but tedious calculation from* (2.32) *and* (2.36)*. This explicit expression is not very enlightening for the above proof. Let us nonetheless comment on the most important fact asserted about J^μ: that it can always be written in the form* (2.37)*. Examining* (2.36)*, we see that J^μ is constructed from two distinct pieces.*

The first piece is $\dfrac{\partial \mathscr{L}_{\mathrm{H}}}{\partial(\nabla_\mu \phi)}$. In (2.32)*, the only dependence on $\nabla_\mu \phi$ is in the arbitrary functions and occurs only through the combination $g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The computation of this term will then require to compute the derivative of the arbitrary functions with respect to their argument times the quantity*

$$\frac{\partial}{\partial(\nabla_\mu \phi)}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) = 2g^{\mu\nu}\nabla_\nu\phi$$

and to multiply this by the corresponding factors from (2.32)*. This is then in agreement with* (2.37)*.*

The second piece, $\nabla_\nu\left(\dfrac{\partial \mathscr{L}_{\mathrm{H}}}{\partial(\nabla_\mu\nabla_\nu\phi)}\right)$, is the most involved one. The computation of $\dfrac{\partial \mathscr{L}_{\mathrm{H}}}{\partial(\nabla_\mu\nabla_\nu\phi)}$ will produce terms that will, in general, still explicitly depend on $\nabla_\alpha\nabla_\beta\phi$. The application of the ∇_ν operator will then produce two types of terms. Firstly, the ones coming from the derivatives of the arbitrary functions that will produce terms similar to what we described above. Secondly, terms coming from the application of ∇_ν on second-order (covariant) derivatives of ϕ that will then (seemingly) produce terms involving third-order derivatives and no $\nabla_\alpha\phi$ to factorise. Nevertheless, by construction of (2.32)*, we know that these terms can be removed. This operation will involve the permutation of two covariant derivatives acting on $\nabla_\alpha\phi$. This will therefore finally produce terms involving contractions of the Riemann tensor with first-order derivatives of ϕ, confirming the validity of* (2.37)*.*

It is interesting to note that the first two hypotheses of this result are similar to the first ones of Bekenstein's result. This observation reinforces the idea that it seems to be in favour of no-hair results to assume that the scalar field must satisfy the spacetime symmetries.

2.4.4 Evading the No-Hair Theorem

Soon after the above result was proposed in [Hui and Nicolis, 2013], some papers have pointed out ways to evade it. These constructions are based on a questioning of the hypothesis and on a meticulous examination of the last step of the proof.

In [Sotiriou and Zhou, 2014a, Sotiriou and Zhou, 2014b], the authors pointed out a potential loophole in step 4 of the proof, related to the asserted form of the conserved current J^μ. We already discussed the validity of (2.37) for a generic spacetime (see remark 2.4) but, in that paper, the authors remarked that, on a spherically symmetric spacetime, J^r can present a term independent of ϕ' provided some of the functions G_i in (2.32) present a pole for $\phi' = 0$ in such a way that their derivatives contain the right negative powers of ϕ' to cancel the corresponding ϕ' dependence in J^r. In that same paper, they explicitly showed that it was possible to do so by considering that these functions contain a piece of the form (2.33) with $\mathcal{F}(\phi) = \alpha\phi$, α being a real constant. In other words, they showed that a real scalar field with a non-minimal coupling to the Gauss-Bonnet invariant of the form

$$\mathscr{L} = R - \frac{1}{2}\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \alpha\phi\mathscr{L}_{\text{GB}}, \tag{2.38}$$

could evade the no-hair theorem from Hui and Nicolis.[2.22] This paper motivated further investigations of polynomial non-minimal couplings to the Gauss-Bonnet invariant (typically $\mathcal{F}(\phi) = \beta\phi^2$) and revealed the existence of a rich variety of black hole solutions, see for example [Silva et al., 2018].

In [Babichev and Charmousis, 2014], the authors found exact black hole solutions for a subclass of the shift-symmetric sector of (2.32)

$$\mathscr{L} = \kappa\, R - \beta\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \eta\, G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi, \tag{2.39}$$

where κ, η and β are real constants. They did it by questioning the second hypothesis and allowing a linearily time dependent scalar field of the form $\phi(t,r) = Qt + F(r)$, with Q a real constant. Their construction was also based on the idea of imposing $\Xi^{rr} = 0$ to ensure the regularity of the current norm at the horizon without inducing additional constraints on ϕ.

This brief list of papers studying hairy black holes in Horndeski gravity is obviously far from being exhaustive but we wanted to enlight these early contributions since these have been inspiring for the work of this thesis. We have indeed been mainly interested in the study of generic compact objects in scalar tensor theories of gravity presenting either a coupling to the Gauss-Bonnet invariant or a non-minimal derivative coupling to the Einstein tensor. A summary of our results is proposed in the next section.

[2.22] Note that this paper was not the first one to consider compact objects in the context of a non-minimal coupling with the Gauss-Bonnet invariant, see for example [Kanti et al., 1996]. We thank Manos Saridakis for pointing this out.

2.5 Original Results

To conclude this chapter with a clear connection to our work, we devote this section to a panorama of the original results obtained through this thesis. We present a small synopsis of the motivations and main results of each of the papers. The papers themselves have been placed in the appendixes B, C, D, E and F. As emphasised below, each paper led to a publication in a peer-reviewed journal.

Let us already emphasise that, for our study, unless explicitly stated otherwise, we focussed our attention to spherically symmetric systems. In every case that we considered, this allowed the field equations to reduce to ordinary differential equations instead of partial differential ones.

2.5.1 Previous Work

Prior to the work of this thesis, we had the opportunity to take our first step in the study of alternative theories of gravity during our master's thesis. This endeavour gave us the chance to publish two papers: one related to Einstein-Gauss-Bonnet gravity and one related to the non-minimal derivative coupling sector of Horndeski gravity.

We will not provide a detailed review of this work here but let us just say a few words on the context in which each paper takes place. This might be useful for the present discussion since some of the work of this thesis happened in the direct continuation of this preliminary research.

Black holes with scalar hairs in Einstein-Gauss-Bonnet gravity

In [Brihaye and Ducobu, 2016] we studied the behaviour of a complex scalar field minimally coupled to Einstein-Gauss-Bonnet gravity in 5 dimensions in the context of rotating spacetimes with two equal angular momenta. The paper highlights hairy black holes, boson stars and Q-balls solutions. It is worth noting that, since the spacetime is 5 dimensional, contrary to 4 dimensional situations, the Gauss-Bonnet invariant is not a total divergency. Consequently, when added to the Lagrangian of the theory, it will produce dynamical effects even without non-minimal couplings to the scalar field.

This paper gave us valuable experience for later work performed in [Brihaye and Ducobu, 2018] where, as emphasised below, despite not considering couplings to the Gauss-Bonnet invariant, we have been interested in the behaviour of scalar fields minimally coupled to Einstein gravity in 5 dimensions.

Slowly rotating neutron stars in the nonminimal derivative coupling sector of Horndeski gravity

In [Cisterna et al., 2016] we studied the behaviour of neutron stars in rotation in the same sector of horndeski gravity (and with a similar ansatz) that the one studied in [Babichev and Charmousis, 2014]; that is for a Lagrangian density of the form (2.39). The novelty of this paper with respect to previous ones relied on the inclusion of (small) rotational effects and on the use of more realistic equations of state.

As emphasised below, this paper was an inspiration of the work later performed in [Brihaye et al., 2020] in which the behaviour of neutron stars in this theory is compared to that of boson stars.

2.5.2 Nutty Black Holes in Galileon Scalar-Tensor Gravity

In this paper, we have extended the results of [Sotiriou and Zhou, 2014b] by considering solutions of the model described by the Lagrangian density (2.38) but on a spacetime presenting a NUT charge. This allowed for a non-trivial extension of these results by considering a spacetime that was stationary but explicitly non-static. We should point out that the interpretation of the solution as a rotating black hole is, nevertheless, delicate. Among other things, we should note that, despite being stationary but non-static, the NUT spacetime present, in addition to the time translation Killing vector, three Killing vectors that do still reproduce the algebra of $SO(3)$. Despite these intriguing properties, our interest was in the influence of the NUT charge on the structure of the singularities present inside the event horizon in the presence of a non-minimal coupling to the Gauss-Bonnet term. To illustrate the subtle influence of the NUT charge on the spacetime geometry, we also investigated the influence of this parameter and of the non-minimal coupling on the geodesic motions.

The behaviour of the solution in both the exterior and interior region and the motion of lightlike and timelike geodesics are thus discussed in detail.

Regarding the behaviour of the solution, our results shows that, while the solution always present two singularities inside the event horizon (one at the centre and another one between the centre and the event horizon) in the absence of the NUT charge, a non-vanishing NUT charge can change this pattern and remove the singularity that is not at the centre for small enough values of the non-minimal coupling parameter. Our results also demonstrate that the behaviour of this second singularity was crucial to understand the limitations in the domain of existence of solutions in terms of the NUT charge and non-minimal coupling parameter.

Regarding the influence on the geodesics, our results show that, independently of the value of the non-minimal coupling parameter, mimicking a rotation, a non-vanishing NUT charge avoid the presence of bounded trajectories confined within the equatorial plane. More precisely, the non-minimal coupling to the Gauss-Bonnet invariant causes quantitative changes but no qualitative changes.

The detailed presentation of these results can be found in appendix B. These results were published in International Journal of Modern Physics A, see [Brandelet et al., 2018].

2.5.3 Spinning-Charged-Hairy Black Holes in 5-d Einstein Gravity

In this paper, we have investigated the behaviour of scalar fields minimally coupled to Einstein gravity in 5 dimensions. Our work was motivated by the idea of extending the results of [Brihaye et al., 2014] and [Brihaye et al., 2016]. In [Brihaye et al., 2014], the authors have demonstrated the existence of a family of 5 dimensional rotating hairy black holes in the presence of a doublet of complex massive scalar fields minimally coupled to the 5 dimensional Einstein-Hilbert Lagrangian. The idea of our paper was to study the modifications of this pattern in the presence of an electromagnetic field if the doublet of scalar fields was allowed to carry an electric charge.

In the uncharged case, an important condition for the existence of regular solutions was the so-called synchronisation condition that states that the frequency characterising the harmonic time dependence of the scalar doublet should be equal to the rotation velocity of the black hole at the event horizon. Our results first show how this condition gets modified in the presence of an electric charge for the scalar field. We then studied the pattern of the hairy solutions as a function of the scalar electric charge. Without surprise, we found that there exist a maximal value for the electric charge above which, due to the importance of the electric repulsion in the scalar cloud, localised solutions cease to exist. What happened as a surprise, nevertheless, is the fact that, according to our numerical analysis, the critical value for the electric charge of the scalar field seems to be independent of the other parameters characterising the family of solutions.

The detailed presentation of these results can be found in appendix C. These results were published in Physical Review D, see [Brihaye and Ducobu, 2018].

2.5.4 Hairy Black Holes, Boson Stars and Non-minimal Coupling to Curvature Invariants

In this paper, we came back to 4 dimensional gravity and to the behaviour of solutions in theories presenting a non-minimal coupling (say $\mathcal{F}(\phi)$) to the Gauss-Bonnet invariant. The solutions presented in this paper can be divided into two groups: hairy black holes and boson stars.

Hairy black holes

Previously to our study, hairy black holes had notably been constructed for theories exhibiting a non-minimal coupling to the Gauss-Bonnet invariant in [Sotiriou and Zhou, 2014b] assuming a purely linear coupling ($\mathcal{F}(\phi) = \gamma_1\phi$) and, abandoning the shift symmetry of the scalar sector, in [Silva et al., 2018] under the assumption of a purely quadratic coupling ($\mathcal{F}(\phi) = \gamma_2\phi^2$). The pattern of solutions in these two cases are significantly different.

For the linear coupling, one can construct scalarised solutions through a continuous deformation of the Schwarzchild solution (obtained for $\gamma_1 = 0$) for sufficiently small values of γ_1. More precisely, hairy solutions could be found for $\gamma_1 \in [0, \gamma_{1,\max}]$. The reason for the existence of this value $\gamma_{1,\max}$ was related to the condition of regularity of the scalar field's first derivative at the black hole event horizon. Also, for this model, one could only construct unexcited solutions (*i.e.* solutions for which the scalar profile has no node). Due to the shift symmetry of the scalar sector, these solutions were dubbed "shift symmetric" scalarised black holes.

For the purely quadratic coupling, on the contrary, excited solutions did exist and, for a fixed number of nodes of the scalar field, hairy solutions usually existed for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,\max}]$, with $\gamma_{2,c} > 0$. This means that hairy black hole solutions do exist but appears spontaneously above a given critical value of the non-minimal coupling constant. In this case the existence of $\gamma_{2,c}$ and $\gamma_{2,\max}$ was also related to the condition of regularity of the scalar field's first derivative at the event horizon. This spontaneous appearance of solutions led to name hairy black holes in this model "spontaneously scalarised" black holes.

This important difference of behaviour led us to wonder what would happen if both types of couplings were present. In other words, we investigated the pattern of solutions in theories presenting a generic quadratic coupling to the Gauss-Bonnet invariant ($\mathcal{F}(\phi) = \gamma_1 \phi + \gamma_2 \phi^2$). Our results then show the existence of a family of solutions continuously extrapolating between the shift-symmetric and spontaneously scalarised black holes.

To complement this analysis, we also studied the influence of a mass for the scalar field on spontaneously scalarised solutions, showing that the inclusion of this term mostly shifted the pattern of solutions to higher values of γ_2.

Finally, in an appendix of this paper, we briefly commented on the case of a coupling to the Chern-Simons invariant showing that, for a spacetime presenting a NUT charge, the pattern of hairy black hole solutions was qualitatively similar to what we obtained in the case of a coupling to the Gauss-Bonnet invariant.

Boson stars

In this same paper, we also commented on the behaviour of boson stars in the case of a purely quadratic coupling ($\gamma_1 = 0$, $\gamma_2 \neq 0$). In this case, if the scalar field is taken to be massive, complex and to present a harmonic time dependence, it is possible to construct boson stars solutions without any non-minimal coupling. Solutions can also be constructed in this case if the scalar field is supplemented by a suitable potential. Our investigation can thus be seen as the study of the deformation of these solutions, their domain of existence and their stability as a function of the non-minimal coupling parameter γ_2.

In this respect, our main result concerns the classical stability of the solutions. We saw that, turning on γ_2, solutions can exist in a broader region of the parameter space and that the presence of the non-minimal coupling tend to enhance the classical stability of the solutions – as seen by comparing the mass and the particle number of the solution; this latter quantity being defined by means of the Nœther charge Q associated to the global $U(1)$ symmetry of the scalar part of the action.

The detailed presentation of the results of this paper can be found in appendix D. These results were published in Physics Letters B, see [Brihaye and Ducobu, 2019].

2.5.5 Boson and Neutron Stars with Increased Density

In this paper, inspired by the results of [Babichev and Charmousis, 2014], we investigated the behaviour of compacts objects other than black holes in the case of the Lagrangian (2.39), with $\beta = 0$, for a spherically symmetric spacetime supplemented by a real scalar field which, in addition to the radial coordinate, is also linearly time dependent. Neutron stars in this theory had already been studied previously in [Cisterna et al., 2015] and, with our collaboration, in [Cisterna et al., 2016].

Here, we provided the discussion of the behaviour of boson stars in this theory and we also revised and extended the discussion of [Cisterna et al., 2015] regarding the behaviour of neutron stars. We were mainly interested in the domain of existence of solutions and the influence of the non-minimal coupling parameter on the mass-radius relation for these objects.

Boson stars

For clarity of the discussion, let us emphasise that, when discussing boson stars, a second (complex) scalar field – distinct from the non-minimally coupled, say gravitational, one (which is real) – was considered as the matter content of the model and it is this scalar field that forms the boson star.

In this case, the part of the action related to the complex scalar field present a global $U(1)$ symmetry. The associated Nœther charge Q could then be used to discuss the classical stability of the solution. Also, to define the radius of the boson star, let us note that we used the definition of a mean radius $\langle R \rangle$ based on the integration of the quantity rj^t, where j^μ denotes the Nœther current associated with the global $U(1)$ symmetry.

We found that the non-minimal coupling had small influence on the mass of the solution – even though it could slightly increase or decrease the value of the maximal mass – but a significant influence on the radius and the Nœther charge. The presence of the non-minimal coupling consequently had significant importance on the classical stability of the solutions. The qualitative behaviour of the effect was also dependent on the sign of the non-minimal coupling constant η. The classical stability of the solutions was shown to be favoured in the case $\eta < 0$.

Neutron stars

In the case of neutron stars, the matter sector was modelled by means of a perfect fluid for which we studied different equations of state (EOS). First, to check the consistency of our results, we looked at the same EOS as the one used in [Cisterna et al., 2015]. Thanks to this analysis, we found some small inconsistencies in the results of this paper (see discussion of EOSI in our paper).

We then investigated the influence of the non-minimal coupling on the solutions for two other equations of state (see EOSII and EOSIII) and for $\eta < 0$. We discovered, in addition to the solutions obtained from a smooth deformation of the neutron star solutions known in GR (*i.e.* with $\eta = 0$), a second branch of solutions for which the mass-radius ratio tends to that of a (Schwarzshild) black hole.

Using a definition of the particle number density for these neutron stars, we were also able to discuss the classical stability of the solutions like we did for the boson stars. Similarly to the boson stars, we found that the presence of the non-minimal coupling with $\eta < 0$ increased the classical stability of the solutions. In particular, the "black hole like" solutions found on the new branch were shown to have an important binding energy (hence implying classical stability).

This similarity of behaviour between boson stars and neutron stars then suggests that, for $\eta < 0$, the real, time-dependent, gravitational scalar prevents these compact objects from collapsing to a black hole at the values known in GR. Indeed an increased central pressure seems to be allowed inside the stars. Our results also indicate that the compact objects studied here are stable with respect to a decay of the scalar cloud into its individual constituents (as a result of the discussion on the classical stability).

The detailed presentation of the results of this paper can be found in appendix E. These results were published in Physics Letters B, see [Brihaye et al., 2020].

2.5.6 Scalarized Black Holes in Teleparallel Gravity

This paper is the most recent project that we developed during this thesis. For this project, we made a transition from scalar-tensor theories of gravity of Horndeski type (based on a geometric setup similar to general relativity) to theories with a scalar field non-minimally coupled in the framework of teleparallel gravity (*i.e.* with a Weitzenböck spacetime as geometrical setup as discussed in chapter 1; see section 1.6). Hereinafter, we will call this type of model "scalar-torsion" theories to distinguish them from the Horndeski-like scalar-tensor theories.

In a series of papers, see [Hohmann, 2018a, Hohmann and Pfeifer, 2018, Hohmann, 2018b], Manuel Hohmann and Christian Pfeifer studied a general class of scalar-torsion gravity theories and identified a Lagrangian whose structure mimic what is known for scalar-tensor gravity. The question of the existence of compact objects for this type of theory then naturally arose.

In our paper, we provide, to the best of our knowledge, the first construction of hairy black holes in this type of scalar-torsion theories of gravity. We concentrated our study on spherically symmetric black holes endowed with a real non-minimally coupled spherically symmetric scalar field. Interestingly, similarly to the way TEGR allows for a reformulation of GR in the context of a Weitzenböck spacetime, the Lagrangian that we considered – following [Hohmann, 2018b] – contained a subclass of the Horndeski Lagrangian, written in terms of the teleparallel variables, but allowed for generalisations of these couplings in the context of the Weitzenböck geometry.

Note that, when our other results relied on numerical methods, this work was done by means of analytical arguments. Our main results are of two types.

First, we proved the existence of hairy black holes in the theory by means of an explicit construction of analytical solutions for some particular choices of the non-minimal coupling functions in the Lagrangian. Our main findings include the construction of two types of asymptotically flat solutions for the sector characterised by a quadratic ($\propto \psi^2$) non-minimal coupling between the scalar field ψ and the torsion scalar (1.219).

Second, we derived a no-scalar-hair theorem that allowed us to identify specific choices of the non-minimal coupling functions that provide the existence of non-trivial scalar profiles for asymptotically flat spherically symmetric spacetimes. Our proof is based on arguments similar to those presented here in section 2.3.4.

Without foreseeing the discussion on conclusions and perspectives, let us already point out that this work constitutes a first step in the study of hairy black holes in the context of scalar-torsion theories of gravity that should hopefully open the way to more systematic investigations of compact objects in general for this type of theories – similarly to what has been done for Horndeski gravity.

The detailed presentation of the results of this paper can be found in appendix F. These results were published in Journal of Cosmology and Astroparticle Physics, see [Bahamonde et al., 2022].

Chapter

Colsys in a Nutshell 3

In this chapter, we focus on the numerical method used through this thesis: Colsys.

Colsys is a Fortran routine providing tools to solve numerically a set of differential equations with some boundary conditions. The code is based on collocation method and on use of the Newton-Raphson algorithm.

Despite being unrelated *per se* to the physical considerations of this thesis, this chapter is important. Colsys is the tool underlying all the numerical calculations whose results are presented in this thesis. In this regard, a brief description of it had to be part of the discussion. In addition, taking into account that the tools offered by Colsys may be applied to a variety of problems way beyond the ones presented in this thesis, learning to use it was undeniably one of the most interdisciplinary skills that we have developed through this thesis. This is why we estimate that it deserves its own chapter.

In section 3.1, we present some concepts and introduce some vocabulary useful for the rest of the chapter. In section 3.2, we provide a brief overview of the type of mathematical problems handled by Colsys and fly over the main characteristics of the algorithm. Finally, in section 3.3, we give a summary of the inputs required by Colsys, focussing on the values of those parameters assumed for our projects. This section also contains a description of the expected outputs of the algorithm and how to deal with these.

Before going further, let us emphasise that (despite trying to be self-content) this discussion does not aim to be exhaustive on the theory of differential equations, the aspects of their numerical resolutions or the Colsys algorithm itself. We rather wanted to give a user-friendly overview of this powerful tool encompassing all of our work with the hope that it could be useful to some later reader having to work withColsys.

3.1 Numerical Prerequisites

In this section, for completeness, we review a few concepts related to the numerical resolution of differential equations that will be used in the next sections.

In order to simplify the formulas and focus on the core ideas, we will discuss the different points focussing on the case of a single first order differential equation. All the notions could indeed be extended to systems of equations of generic order with a few caution.

3.1.1 Collocation Methods and Splines

Collocation methods

Collocation methods are methods for solving numerically differential equations.

To fix the ideas, let us discuss the simplest case of an ordinary first order differential equation:

$$y'(t) = f\left(t; y(t)\right) \tag{3.1}$$

subject to the initial condition

$$y(t_0) = y_0 \tag{3.2}$$

on the interval $[t_0, t_0 + h]$ for some $t_0 \in \mathbb{R}$ and $h > 0$.

The general idea of the method is to take a finite dimensional family of functions[3.1] and to approximate the solution $y(t)$ by the unique function in that family that would satisfy equation (3.1) at some given points of the interval called the collocation points. Obviously, in order for the method to select a unique function in the family, one has to select the number of collocation points according to the dimension of our family of functions[3.2] .

Most of the time (and, as we will see, this is the approach used by COLSYS) the method is applied with polynomials up to a certain degree as the finite family of functions and the number of collocation points is taken as the biggest possible degree of a polynomial in this family.

With the choice of N numbers c_k such that $0 \leq c_1 < c_2 < \cdots < c_N \leq 1$, the collocation points are defined as

$$t_k = t_0 + c_k h,\ \ k = 1, 2, \cdots, N \tag{3.3}$$

and $y(t)$ is approximated by the unique polynomial $p(t)$ of degree N (or less) such that

$$\begin{cases} p(t_0) = y_0 \\ p'(t_k) = f\left(t_k; p(t_k)\right) \quad \forall k \in \{\, 1, 2, \ldots, N \,\} \end{cases} . \tag{3.4}$$

Writing $p(t) = \sum_{i=0}^{N} \alpha_i \left(t - t_0\right)^i$, it is easy to show that this kind of algorithm falls into the class of implicit Runge-Kutta methods; it will in fact correspond

[3.1] That is a family of functions which depends on a finite number of free parameters.

[3.2] The number of collocation points should be the dimension of the family minus one, since one parameter will be fixed thanks to the initial condition (3.2).

to a particular case in which all the parameters in the Butcher tableau characterising the method can be defined thanks to the numbers c_k.

Any collocation method is then entirely defined thanks to the choice of the collocation points via the numbers c_k. One choice which has proved to be very interesting (and this will, again, be the one used by COLSYS) is the choice of the so-called Gauss-Legendre points. For the Gauss-Legendre collocation points, the coefficients c_k are defined as

$$c_k = \frac{1+\rho_k}{2}, \tag{3.5}$$

where ρ_k are the Gauss-Legendre points on $[-1,1]$ *i.e.* the N roots of the Legendre polynomial of degree N, P_N, subject to the normalisation condition $P_N(1) = 1$. The reason why this choice is really useful is that the Gauss-Legendre collocation method defined using N collocation points can be proved to be of order $2N$.[3.3]

To conclude this brief introduction to collocation methods, let us emphasise that the method sketched here will then be powerful provided the interval is small enough (*i.e.* as long as h is small) or at least provided the solutions do not vary too much on the interval. Consequently, in practice, if we want to solve the problem (3.1)–(3.2) on an interval $[a,b]$, the collocation method will consist in taking a partition $a = t_0 < t_1 < t_2 < \cdots < t_M = b$ of the interval and applying the above procedure successively on each subinterval $[t_i, t_i + h_i]$, where $h_i = t_{i+1} - t_i$, for $i = 0, 1, 2, \cdots, M-1$.

One key step to apply successfully collocation methods to a given problem will then be to obtain a good discretisation *i.e.* the one that would, in some sense, provide a good fit to the exact solution's behaviour. Indeed, once the discretisation is chosen, everything is determined by the collocation equations (3.4) so the quality of the approximation mainly depends on the one of the discretisation.

If we use the symbol "$\rightarrow$" for "Applying the collocation method with the previous initial condition gives ...", the behaviour of the algorithm can be synthesised as follows:

$$\begin{cases} y_0 = y(t_0) & \rightarrow \text{Approx. } p_0 \text{ on } [t_0, t_0+h_0] \\ y_1 = p_0(t_1) = p_0(t_0+h_0) & \rightarrow \text{Approx. } p_1 \text{ on } [t_1, t_1+h_1] \\ y_2 = p_1(t_2) = p_1(t_1+h_1) & \rightarrow \text{Approx. } p_2 \text{ on } [t_2, t_2+h_2] \\ & \vdots \\ y_{i+1} = p_i(t_{i+1}) = p_i(t_i+h_i) & \rightarrow \text{Approx. } p_{i+1} \text{ on } [t_{i+1}, t_{i+1}+h_{i+1}] \\ & \vdots \end{cases}$$

One then obtains an approximation of the solution y on $[a,b]$ in the form of a piecewise polynomial function v such that

[3.3] In other words, the approximation is valid up to $\mathcal{O}\left(h^{2N}\right)$ terms.

$$v|_{[t_i,t_{i+1}]} = p_i, \ \forall i = 0, 1, \cdots, M-1. \tag{3.6}$$

Splines

Piecewise polynomial functions are also called *splines*. Splines are used in many areas of mathematics such as computer-aided design or curve fitting. One interesting characteristic of splines is that they form a finite dimensional vector space. More precisely, on an interval $[a,b]$, given $N \in \mathbb{N}$, a mesh $\boldsymbol{t}$ of $M+1$ points[3.4] and $\boldsymbol{r} = (r_1, r_2, \cdots, r_{M-1})$ with $r_1, r_2, \cdots, r_{M-1} \in \{ n \in \mathbb{N} \mid n \le N \}$, the set

$$S_N^{\boldsymbol{r}}(\boldsymbol{t}) = \left\{ v : [a,b] \to \mathbb{R} \,\middle|\, \begin{array}{l} v|_{[t_i,t_{i+1}]} \text{ is a polynom of degree} \le N, \ \forall 0 \le i \le M-1, \\ \text{and } v \text{ is } \mathcal{C}^{r_j} \text{ at } t_j, \forall 1 \le j \le M-1 \end{array} \right\}$$

equipped with the pointwise addition of functions and the multiplication of functions by a real is a finite dimensional real vector space.[3.5]

This is an important property for the collocation methods since, given a basis of $S_N^{\boldsymbol{r}}(\boldsymbol{t})$, the evaluation of any spline of this type can be performed at any point in $[a,b]$ by taking linear combinations of the values of the basis functions at that point. The coefficients of the combination are the (constant !) components of the spline in that basis. If one has a way to efficiently encode and evaluate the basis splines, one can then reduce a spline to its (constant) components in this basis. This would then reduce the problem of finding the spline approximation of y to the one of finding its components and provide a fast way to evaluate this approximation once these components are determined. This observation, and the choice of an appropriate basis, is one of the keys that allow COLSYS to be fast, reliable and thus competitive with respect to other types of algorithms.

One example of such bases are the so-called *B-spline* functions (for *basis splines*). This is the type of bases used byCOLSYS. More precisely, once given a mesh $\boldsymbol{t}$ and a vector $\boldsymbol{r}$ (as described above), the term B-spline would encompass a family of functions such that:

1. Any B-spline is an element of $S_N^{\boldsymbol{r}}(\boldsymbol{t})$ for some $N \in \mathbb{N}$,

2. Given $N \in \mathbb{N}$, any $v \in S_N^{\boldsymbol{r}}(\boldsymbol{t})$ can be expressed as a linear combination of the B-splines in $S_N^{\boldsymbol{r}}(\boldsymbol{t})$,

[3.4] A mesh is a vector $\boldsymbol{t} = (t_0, t_1, \cdots, t_M)$, with $a = t_0 < t_1 < \cdots < t_M = b$. A mesh thus defines a partition of the interval $[a,b]$.

[3.5] The dimension of this vector space is

$$\mathrm{Dim}\,(S_N^{\boldsymbol{r}}(\boldsymbol{t})) = M(N+1) - \sum_{j=1}^{M-1}(r_j+1) = MN + 1 - \sum_{j=1}^{M-1} r_j.$$

Note that this expression also gives the correct value for the dimension when no continuity requirement is imposed at a given t_j provided one sets the corresponding $r_j = -1$.

3. The B-splines in $S_0^{\boldsymbol{r}}(\boldsymbol{t})$ are the piecewise constant functions noted $B_{i,1}$ (for $i = 0, \cdots, N-1$) given by

$$B_{i,1}(x) = \begin{cases} 1, & \text{if } t_i \le x < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad ,$$

which obviously form a basis of the piecewise constant functions for the mesh $\boldsymbol{t}$,

4. Given $N \in \mathbb{N}_0$, there is a recurrence formula giving the B-splines in $S_N^{\boldsymbol{r}}(\boldsymbol{t})$ from ones in $S_{N-1}^{\boldsymbol{r}}(\boldsymbol{t})$.

For simplicity, we will not write the explicit recurrence formula from property 4 as we will not explicitly use it afterwards. Let us nevertheless emphasise that properties 3 and 4 above makes B-splines functions particularly appealing for the scope of numerical analysis, especially for collocation methods. Indeed, one can easily implement the family of B-spline functions for any $\boldsymbol{t}$ and $\boldsymbol{r}$ by means of a routine outputting the corresponding $B_{i,1}$'s and a routine applying the recurrence procedure until one gets the desired B-splines. More than that, in case we know in advance the points at which we will need to evaluate those functions[3.6], it is possible to use the above procedure to compute once and for all the values of the desired B-splines at those points and to store them so that we can access them immediately when needed. This "trick" allows COLSYS saving quite some computation time and to remain competitive with respect to other methods.

For a detailed overview on splines see [de Boor, 2001].

3.1.2 Linearised Problem

Consider, on $[a, b] \subset \mathbb{R}$, the ordinary first order differential equation

$$y'(t) = f(t; y(t)) \tag{3.7}$$

subject to the general boundary condition[3.7]

$$G(\zeta; y) = 0, \tag{3.8}$$

where $\zeta \in [a, b]$.

The problem we want to address in this paragraph is to determine under which conditions (3.7)–(3.8) admit different solutions "close to each other". More precisely, let us introduce a (small) control parameter $0 < \varepsilon \ll 1$ and formulate the question in the following way: given a function u solution of (3.7)–(3.8)

[3.6] It will be the case for collocation problems; the points being the collocation and mesh points.

[3.7] Note that in this case the solution to (3.7) that satisfy (3.8) might be non-unique. As an example, imagine that $G(\zeta; y) = (y(\zeta))^2 - (y_0)^2$ for some $y_0 > 0$. In this case, (3.8) will imply that $y(\zeta) = \pm y_0$ which would then give two solutions to (3.7)–(3.8).

and a function w that is bounded such that $\frac{1}{\varepsilon} \gg |w(t)|$ on $[a, b]$, under which conditions can $u + \varepsilon w$ be solution of this same system ?

The problem is then to find under which conditions on w one have

$$\begin{cases} (u + \varepsilon w)' = f\,(\cdot; u + \varepsilon w) \\ G(\zeta; u + \varepsilon w) = 0 \end{cases} .$$

Assuming that f and G are sufficiently smooth, one can write the former system as

$$\begin{cases} u' + \varepsilon w' = f(\cdot; u) + \varepsilon \frac{\partial f}{\partial u}(\cdot; u) w + \mathcal{O}(\varepsilon^2) \\ G(\zeta; u) + \varepsilon \frac{\partial G}{\partial u}(\zeta; u) w(\zeta) + \mathcal{O}(\varepsilon^2) = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} w' - \frac{\partial f}{\partial u}(\cdot; u) w = \mathcal{O}(\varepsilon) \\ \frac{\partial G}{\partial u}(\zeta; u) w(\zeta) = \mathcal{O}(\varepsilon) \end{cases} .$$

Consequently, as long as ε is "sufficiently small to ensure that one term in the expansion is subleading with respect to the former one" so that it then makes sense to address the problem order by order, the first order of this expansion imposes that w should (at least) be solution of the so-called linearised problem at u

$$\begin{cases} \hat{L} w = 0 \\ \hat{B} w = 0 \end{cases} , \tag{3.9}$$

where

$$\hat{L} w = \hat{L}(u) w := w' - \frac{\partial f}{\partial u}(\cdot; u) \cdot w,$$
$$\hat{B} w = \hat{B}(u) w := \frac{\partial G}{\partial u}(\zeta; u) \cdot w(\zeta).$$

One immediate but crucial observation is to realise that $w(t) \equiv 0$ is always a solution of (3.9). From this observation, it follows that, if the system admits a unique solution[3.8], w must identically vanish.

Therefore, one can prove that, if the linearised problem at u admit a unique solution, there will exist a $\sigma > 0$ such that, considering functions whose graph is in some neighbourhood of the graph of u, u is the unique solution to (3.7)–(3.8) within the "sphere of functions that vary mostly like u"

$$B\,(Du, \sigma) = \left\{\, \tilde{u} : [a, b] \to \mathbb{R} \;\middle|\; ||u' - \tilde{u}'||_\infty \leq \sigma \,\right\}.$$

Intuitively, the idea of the argument is to choose $\sigma > 0$ such that for any $\tilde{u} \in B\,(Du, \sigma)$ close enough to u (see remark 3.1), one can write $\tilde{u} = u + \varepsilon w$ with $0 < \varepsilon \ll 1$ and a w that satisfies $\frac{1}{\varepsilon} \gg |w(t)|$ so that we can discuss whether $\tilde{u}$ can solve (3.7)–(3.8) order by order in ε.[3.9]

[3.8] This would be the case if the Green's function of problem (3.9) exists.

[3.9] The all point would be to prove that this is always possible provided one takes an apropriate (sufficently small) σ.

Remark 3.1. *Since the property defining $\tilde{u} \in B(Du, \sigma)$ constrain only the derivative of $\tilde{u}$, clearly, $\tilde{u} + c$ is also in the set for any constant c. This can lead to functions significantly different from u just because they can be written as a function whose derivative and values are close to u plus some gigantic constant term. Consequently, by "sufficiently close" we mean a $\tilde{u}$ such that $||u - \tilde{u}||_\infty$ (and not only $||u' - \tilde{u}'||_\infty$) is small; say such that $||u - \tilde{u}||_\infty < \sigma'$ for some $\sigma' \leq \max(u([a,b])) - \min(u([a,b]))$.*

This can then be seen as restricting the study to elements $\tilde{u} \in B(Du, \sigma) \cap B(u, \sigma')$, i.e. *as restricting the study to functions whose graph is in some σ'-neighbourhood of the graph of u, in order to avoid this irrelevant problem of shift.*

Indeed, in this case, if u solves (3.7)–(3.8), such a $\tilde{u} \in B(Du, \sigma)$ will also be a solution only if w solves (3.9). In this case, the unicity of the solution of (3.9) will impose $w(t) \equiv 0$. This would then prove that, among functions whose graph is in some neighbourhood of the graph of u, u is the only solution to (3.7)–(3.8) in $B(Du, \sigma)$.

3.1.3 Newton's Algorithm(s)

In this paragraph, we review the philosophy of Newton-Raphson algorithm of linearisation and iteration.[3.10] We aim at emphasising the similarity of the algorithm in different situations in order to demonstrate how the procedure applied to differential equations parallels the one known to undergrad students for real-valued functions.

For a function $f : \mathbb{R} \to \mathbb{R}$

In the context of real functions analysis, Newton's method, as taught to undergrad students, is a tool to approximate the roots of a sufficiently regular function $f : \mathbb{R} \to \mathbb{R}$. The idea is to start at a point x_0 and to evaluate $f(x_0)$ with hope that $|f(x_0)| \gtrsim 0$. If so, then there must be a root of f close to x_0. In other words, there must be a small quantity w for which $x_1 = x_0 + w$ satisfies $f(x_1) = 0$.

One way to present the algorithm is to say that one Newton iteration aims at approximating w. According to our previous reasoning, w should be such that

$$f(x_0 + w) = 0 \Leftrightarrow f(x_0) + f'(x_0)w + \mathcal{O}\left(w^2\right) = 0.$$

The prescription of Newton's algorithm is to approximate w by neglecting the $\mathcal{O}\left(w^2\right)$ terms[3.11], *i.e.* to approximate w by the solution of the equation

$$f'(x_0)w = -f(x_0). \tag{3.10}$$

[3.10] This algorithm is also simply refered to as Newton's algorithm.

[3.11] In more usual words, the prescription is to linearize f around x_0 and to approximate the root by the root of the function whose graph is the tangent to the graph of f at x_0.

If $f'(x_0) \neq 0$, in other words, if the (linear) equation $f'(x_0)w = 0$ admits a unique solution, one will obviously get $w = -f(x_0)/f'(x_0)$ so that the prescription will be to approximate the root of f by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

leading to the well-known formula.

As with many schemes of approximation, Newton-Raphson algorithm will rely on an iteration of the above procedure. Given an initial guess x_0, until a given convergence (or stopping) criterion is met, one will iteratively compute

$$x_{i+1} \coloneqq x_i - \frac{f(x_i)}{f'(x_i)},$$

as a hopefully increasingly better approximation of the root of f located next to x_0.

For a function $f : \mathbb{R}^p \to \mathbb{R}^p$

The above procedure can be easily translated to sufficiently regular functions $f : \mathbb{R}^p \to \mathbb{R}^p$. The idea is to start at a point x_0 and to evaluate $f(x_0)$ with hope that $||f(x_0)|| \gtrsim 0$. If so, there must be a small quantity w for which

$$f(x_0 + w) = 0 \Leftrightarrow f(x_0) + J_f(x_0)w + \mathcal{O}\left(||w||^2\right) = 0,$$

where $J_f(x_0)$ is the Jacobian matrix for f at point x_0. The prescription of Newton's algorithm is then to neglect higher order terms and to approximate w by the solution of the equation

$$J_f(x_0)w = -f(x_0). \tag{3.11}$$

If $\det(J_f(x_0)) \neq 0$, in other words, if the (linear) equation $J_f(x_0)w = 0$ admits a unique solution, one will get $w = -J_f^{-1}(x_0)f(x_0)$.

Consequently, given an initial guess x_0, until a given convergence (or stopping) criterion is met, one will iteratively compute

$$x_{i+1} \coloneqq x_i - J_f^{-1}(x_i)f(x_i),$$

as a hopefully increasingly better approximation of the desired root.

Note that, in this case, one will need a linear solver algorithm to solve (3.11) in order to perform the Newton iteration.

For differential equations

On a slightly less trivial scale, one can also define a Newton-Raphson algorithm to approximate the solutions of differential equations. As always in this section, let us focus on the simple case of a single first-order ordinary differential equation

(3.7) on an interval $[a, b]$ subject to the general boundary condition (3.8). The link with the Newton-Raphson algorithm aiming to find roots of functions might be clearer if we rewrite (3.7)–(3.8) as

$$\begin{cases} \hat{\mathcal{F}} y := y' - f(\cdot; y) \equiv 0 \\ \hat{\mathcal{G}} y := G(\zeta; y) = 0 \end{cases} . \tag{3.12}$$

We then try to find a function y which simultaneously vanishes some non-linear differential operator $\hat{\mathcal{F}}$ and some non-linear real-valued operator $\hat{\mathcal{G}}$.

If we have at hand an initial guess y_0 approximately satisfying (3.12), that is such that $\left|\left|\hat{\mathcal{F}} y_0\right|\right|_\infty \gtrsim 0$ and $\left|\hat{\mathcal{G}} y_0\right| \gtrsim 0$, the idea will be that there must be an almost zero function w such that $y_1 = y_0 + w$ obeys (3.12). Consequently, provided f and G are sufficiently regular, one will linearise (3.12) around y_0, getting

$$\begin{cases} w' - \frac{\partial f}{\partial y}(\cdot; y_0) w + y_0' - f(\cdot; y_0) \equiv \mathcal{O}\left(||w||_\infty^2\right) \\ \frac{\partial G}{\partial y}(\zeta; y_0) w(\zeta) + G(\zeta; y_0) = \mathcal{O}\left(||w||_\infty^2\right) \end{cases} .$$

Recognising the linearised operators $\hat{L}$ and $\hat{B}$ from (3.9), one will then try to approximate w by the solution of the system

$$\begin{cases} \hat{L}(y_0) w \equiv -\hat{\mathcal{F}} y_0 \\ \hat{B}(y_0) w = -\hat{\mathcal{G}} y_0 \end{cases} . \tag{3.13}$$

Now, if the linearised problem at y_0, *i.e.* (3.9), admits a unique solution, solving (3.13) will give a hopefully better approximation to the solution of (3.12) via $y_1 = y_0 + w$.

The Newton-Raphson algorithm in this case will then rely on the recurrence relation $y_{i+1} = y_i + w_i$, where w_i is obtained by solving the (linear) system

$$\begin{cases} \hat{L}(y_i) w_i \equiv -\hat{\mathcal{F}} y_i & \text{(3.14a)} \\ \hat{B}(y_i) w_i = -\hat{\mathcal{G}} y_i & \text{(3.14b)} \end{cases} ,$$

to approximate the solution of (3.12) lying in a neighbourhood of the initial guess y_0.

Let us emphasise that, in this case, since the system (3.14) is actually a (linear) differential equation on $[a, b]$ (3.14a) subject to the (linear) boundary condition (3.14b), one will need to choose an algorithm able to solve such a system in order to perform the Newton iteration. For example, anticipating the operation of COLSYS, if the desired functions y_i's and w_i's belong to a given type of splines, one might use a set of basis functions and a collocation method to turn (3.14) into a set of algebraic equations in the splines components and then solve this new system using a linear solver to get those components.

Damped Newton iteration

The pros and cons of Newton-Raphson algorithm are well known. Specifically, it has the appealing advantage that, if the initial guess is close enough to the desired result, the algorithm will converge quadratically. Nevertheless, if the initial guess is too inaccurate, the behaviour of the algorithm is notoriously unpredictable.

As an attempt to overcome this problem, some authors [Deuflhard, 2011] have considered the possibility to slightly modify the previous prescriptions, leading to the so-called Damped Newton method. The idea is to introduce a damping parameter $0 < \lambda_i \leq 1$ that would "temper" the effect of the linear correction w_i at each step via a slight modification of the iteration relation,

$$y_{i+1} = y_i + \lambda_i w_i.$$

This is done in order to prevent some of the bad behaviours of the original (or full) Newton's method.

In order to perform his task, such a modified Newton-Raphson algorithm should come with a prescription on the way to adapt λ_i depending on the behaviour of the Newton correction w_i. We will not cover the details of this question here. The interested reader can refer to [Deuflhard, 2011]. We simply wanted to introduce the principle of Damped Newton methods as it is the variant of Newton's algorithm favoured byColsys.

3.2 What Colsys Does for You

We will now turn to a summary of Colsys' main features. The content of the following discussion comes essentially from [Ascher et al., 1979a] and [Ascher et al., 1979b].

3.2.1 Formulation of the Problem

Consider a set of mixed order (*a priori* non-linear) differential equations. For definiteness, let us call d the number of equations and $1 \leq m_1 \leq m_2 \leq \cdots \leq m_d$, the order of each equation (assumed to be sorted in a suitable manner). Let us also assume that the problem is studied on an interval $[a, b] \subset \mathbb{R}$. More precisely, we will assume that our system can be written as

$$u_n^{(m_n)}(x) = F_n\left(x; \underline{Z}(\underline{u})\right) \tag{3.15}$$

for all $a < x < b$, where $n = 1, 2, \cdots, d$, $\underline{u} = (u_1, u_2, \cdots, u_d)$ is the researched solution, $u_i^{(j)} = \dfrac{\mathrm{d}^j}{\mathrm{d}x^j} u_i$ is the j^{th} derivative of u_i,
$\underline{Z}(\underline{u}) = \left(u_1, u_1^{(1)}, \cdots, u_1^{(m_1-1)}, u_2, u_2^{(1)}, \cdots, u_2^{(m_2-1)}, \cdots, u_d, \cdots, u_d^{(m_d-1)}\right)$ is the vector of unknown that would correspond to the convertion of (3.15) to a first order system and F_n are some (*a priori* non-linear) functions defining the problem.

These equations will also be assumed to be subject to some (again, *a priori* non-linear) boundary conditions

$$G_l\left(\zeta_l; \underline{Z}(\underline{u})\right) = 0, \tag{3.16}$$

where $l = 1, 2, \cdots, m^\star := \sum_{n=1}^{d} m_n$, ζ_l is the location of the l^{th} boundary (or side) condition and G_l are some (*a priori* non-linear) functions. In order to fix the notations, we will consider in the following that $a \leq \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_{m^\star} \leq b$.

What COLSYS does for you is solving, numerically, (3.15)–(3.16), without having to convert it to a first-order problem. The method used by COLSYS is based on collocation method at Gaussian points as introduced in section 3.1.1.

Following the notations of section 3.1.1, the idea of the method is, given a mesh $\boldsymbol{x}$ of $N+1$ points, to construct a collocation approximation, call it $\underline{v} = (v_1, v_2, \cdots, v_d)$, such that for all $n = 1, 2, \cdots, d$

$$v_n \in S^{\boldsymbol{r}_n}_{k+m_n-1}(\boldsymbol{x}), \tag{3.17}$$

where $\boldsymbol{r}_n = (r_{n,1}, r_{n,2}, \cdots, r_{n,N-1})$ is such that $r_{n,1} = r_{n,2} = \cdots = r_{n,N-1} = m_n - 1$, and $k \geq m_d$ is the number of collocation points per subinterval (the same number for all n and all subintervals), see remark 3.2.

Remark 3.2. *Regarding condition* (3.17)*, the idea of choosing the number of collocation points k on each subinterval such that $k \geq m_d$ and the approximation polynom such that its degree is lower or equal to $k + m_n - 1$ (*i.e. *such that it depends* a priori *on $k + m_n$ parameters) corresponds to the intuition that, in order to get an "acceptable" or "rich enough" polynomial approximation of the solution of our equations on a given subinterval, the polynom should depend on "enough" parameters (and then be of sufficently high order). In this remark, we present an intuitive counting of the number of parameters needed to construct a meaningful approximation.*

*Intuitivelly, we can say that, on a given subinterval (*i.e. *for a fixed value of i in* (3.18)*) :*

1. *Each component v_n of the approximation vector $\underline{v}$ is aimed to be fixed by the k conditions* (3.18)*, consequently k parameters are needed in order to be able to satisfy the conditions without arriving at an overdetermined system; so at this stage our polynom v_n should be at least of order k (degree $< k$) and this for all n.*

2. *Furthermore, since the component v_n of the approximation vector $\underline{v}$ is aimed to approximate the solution of a m_n order differential equation subject to the boundary conditions* (3.16)*, m_n parameters in v_n will be fixed by the boundary conditions* (3.16)*. Consequently, in order to avoid an overdetermined system when solving* (3.18)*, we need m_n extra parameters in v_n; which then raises the desired order of v_n to $k + m_n$ (*i.e. *the degreee of v_n should be lower or equal to $k + m_n - 1$).*

3. *Finally, imposing $k \geq m_d$ is related to the differential nature the equations* (3.18)*. In order to approximate the solution of a m_n order differential equation, we should use a polynom whose order is at least $m_n + 1$ so that the $(m_n)^{th}$ derivative of the polynom can be (at least* a priori*) non-trivial. Since the minimal choice for the order of v_n is $k + m_n$, since we should be able to fulfill the former requirement for any component of $\underline{v}$ and since $1 \leq m_1 \leq m_2 \leq \cdots \leq m_d$, this motivate the choice $k \geq m_d$. In this case, for all n the order of v_n automatically satisfies $k + m_n \geq m_d + m_n \geq m_d + 1 \geq m_n + 1$.*

This then comes as a natural generalization of what we described in section 3.1.1, where we had $d = 1$ and $m_d = 1$.

Introducing on each subinterval $[x_i, x_i + h_i]$ the Gauss-Legendre collocation points $x_{ij} = x_i + \frac{1+\rho_j}{2} h_i$, where $h_i = x_{i+1} - x_i$ and ρ_j are the Gauss-Legendre points on $[-1, 1]$, the approximation is constructed by requiring that

$$v_n^{(m_n)}(x_{ij}) = F_n\left(x_{ij}; \underline{Z}(\underline{v})\right), \begin{array}{l} \forall i = 0, 1, \cdots, N-1 \\ \forall n = 1, 2, \cdots, d \\ \forall j = 1, 2, \cdots, k \end{array} \tag{3.18}$$

and that $\underline{v}$ satisfies the boundary conditions (3.16).

In order to be able to apply collocation theory to our problem, we need to have a (sufficently smooth) isolated solution $\underline{u}$ of (3.15)–(3.16). This would be the case[3.12] if the linearized problem at $\underline{u}$ is uniquely solvable. This problem is, given a solution $\underline{u}$ of (3.15)–(3.16), to find $\underline{w} = (w_1, \cdots, w_d)$ such that

$$\begin{cases} L_n \underline{w} = 0, \ \forall n = 1, \cdots, d \\ B_l \underline{w} = 0, \ \forall l = 1, \cdots, m^\star \end{cases}, \tag{3.19}$$

where

$$\begin{aligned} L_n \underline{w} &= L_n\left(\underline{u}\right)\underline{w} \coloneqq w_n^{(m_n)} - \sum_{m=1}^{m^\star} \frac{\partial F_n}{\partial Z_m}\left(\cdot; \underline{Z}(\underline{u})\right) \cdot Z_m(\underline{w}), \\ B_l \underline{w} &= B_l\left(\underline{u}\right)\underline{w} \coloneqq \sum_{m=1}^{m^\star} \frac{\partial G_l}{\partial Z_m}\left(\zeta; \underline{Z}(\underline{u})\right) \cdot Z_m(\underline{w}). \end{aligned} \tag{3.20}$$

Similarilly to what we discussed in section 3.1.2, since $w_n(x) \equiv 0, \forall n = 1, \cdots, d$ is always trivially a solution of (3.19), the uniqueness of the solution to the

[3.12] *i.e.* $\underline{u}$ will be an isolated solution in the sense that, considering functions whose graph is in some neighborhood of the graph of $\underline{u}$, there will exist $\sigma > 0$ such that $\underline{u}$ is the unique solution of (3.15)–(3.16) in the sphere

$$B\left(D^m \underline{u}, \sigma\right) = \left\{ w(x) : \left\|w_n^{(m_n)} - u_n^{(m_n)}\right\|_\infty \leq \sigma, \forall n = 1, \cdots, d \right\}.$$

linearized problem implies that $\underline{w}$ vanishes which, in turn, ensures that $\underline{u}$ is an isolated solution of (3.15)–(3.16). In this case, Newton's method converges quadratically to $\underline{v}$ provided the initial approximation is close enough to $\underline{u}$ (see below).[3.13]

The main challenge in pratice, as for any numerical approximation method, is to be able to estimate and control the error made when approximating $\underline{u}$ by the solution of the collocation problem $\underline{v}$. In other words, for all $n = 1, \cdots, d$ and all $p = 0, \cdots, m_n$, one should be able to control the behaviour of the error terms

$$e_n^{(p)} := u_n^{(p)} - v_n^{(p)}. \tag{3.21}$$

It can be proved, see [Ascher et al., 1979a, Ascher et al., 1979b] and references therein for more details, that the following *a priori* error estimation holds : introducing $h = \max\limits_{0 \le i \le N-1} h_i$ the step of the mesh $\boldsymbol{x}$, one has that

$$\left|\left|e_n^{(p)}\right|\right|_\infty = \mathcal{O}\left(h^{k+m_n-p}\right) \tag{3.22}$$

and that, at the mesh points, superconvergence occurs

$$\left|e_n^{(p)}(x_i)\right| = \mathcal{O}\left(h^{2k}\right), \tag{3.23}$$

for all $i = 0, \cdots, N-1$.

3.2.2 A Posteriori Error Estimation

The precise expression of the error terms $e_n^{(p)}$ requires to evaluate, among other things, the $(m_n + k)^{\text{th}}$ derivative of u_n at the mesh points and to neglect a higher order, but global, term. This makes the "naive" error estimation procedure based on (3.22) unreliable in practice. This come from the fact that the estimation of $u_n^{(m_n+k)}(x_i)$ using high order interpolation of the approximation solution $\underline{v}$ might be quite inacurate. Also, the neglected (global) higher order term is not always of negligible magnitude. To overcome this technical problem, the error estimation implemented in COLSYS is based on a slightly different strategy : an *a posteriori* error estimate obtained by comparison of different approximations.

Say we obtained an approximation $\underline{v}$ using a mesh $\boldsymbol{x}$ of $N+1$ points and an approximation $\underline{\tilde{v}}$ using the mesh $\tilde{\boldsymbol{x}}$ of $2N+1$ points obtained by halving $\boldsymbol{x}$, *i.e.* such that $\tilde{x}_{2i} = x_i$ for $i = 0, \cdots, N$ and $\tilde{x}_{2i+1} = x_i + \frac{h_i}{2}$ for $i = 0, \cdots, N-1$.

Let us introduce for $q \in \mathbb{N}_0$, $p = 0, \cdots, q$ and $i = 0, \cdots, N-1$ the notation $x_{i+\frac{p}{q}} := x_i + \frac{p}{q} h_i = \frac{(q-p)x_i + p x_{i+1}}{q}$. In other words, $x_{i+\frac{p}{q}}$ is the point obtained by performing a displacement from x_i by a fraction (written as p/q) of h_i.

[3.13] Typically, the term "close enough" refers to the idea that the initial approximation is in $B\left(D^m \underline{u}, \sigma\right)$.

If $k > m_d$, it is possible to prove that the following relation hold (see [Ascher et al., 1979a, Ascher et al., 1979b] for more details)

$$\max_{x\in[\tilde{x}_{2i},\tilde{x}_{2i+1}]} \left| u_n^{(p)}(x) - \tilde{v}_n^{(p)}(x) \right| = \omega_{k,k-m_n+p} \left(\Delta_1 + \Delta_2\right) + \mathcal{O}\left(h^{k+m_n-p+1}\right), \tag{3.24}$$

where

$$\Delta_1 = \left| v_n^{(p)}(x_{i+\frac{1}{6}}) - \tilde{v}_n^{(p)}(x_{i+\frac{1}{6}}) \right|, \ \Delta_2 = \left| v_n^{(p)}(x_{i+\frac{1}{3}}) - \tilde{v}_n^{(p)}(x_{i+\frac{1}{3}}) \right|,$$

and where $\omega_{k,\nu}$ are some constants precomptuted and stored in the program.[3.14] What is important to emphasize in (3.24), rather than the exact way to compute $\omega_{k,\nu}$, is that the error made on each interval $[\tilde{x}_{2i}, \tilde{x}_{2i+1}]$ using the approximation $\underline{\tilde{v}}$ instead of the exact solution $\underline{u}$ can be estimated by comparing $\underline{\tilde{v}}$ to an other approximation $\underline{v}$ (obtained using a suitable mesh).

For practical purposes, the error estimation based on (3.24) will be reliable provided the meshes $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ already fit the functions' behaviour. Indeed, roughly speeking, for the dominant term in (3.24) to be relevant one has to assume that the exact solution does not vary too much between the mesh points, or at least between the points of the form $x_{i+\frac{j}{6}}$. In contrast, if the mesh structure does not reflect the solution's behaviour, the procedure might become less accurate. It is then important to have a "suitable" mesh available. The construction of such a mesh is ensured by Colsys via the mesh-selection algorithm.

3.2.3 Mesh Selection Algorithm

First of all, as we already mentionned in section 3.1.1, let us quote that the collocation approximation $\underline{v}$ will converge to the exact solution $\underline{u}$ as the step of the mesh tend to zero. This means that, at least in principle, we could always get what we called a suitable mesh by taking a uniform mesh with an enormous amount of points. The obvious problem of this approach is that, increasing the number of mesh points, one increases the dimension of $S^{\boldsymbol{r}_n}_{k+m_n-1}(\boldsymbol{x})$ (*i.e.* the number of parameters needed to characterise v_n) for any $n = 1, \cdots, d$ and then also the computanional effort required to obtain $\underline{v}$. The aim of the mesh-selection procedure is then to circumvent this problem by finding instead some "optimised" mesh. This concept of an "optimised" mesh needs some clarification.

When working with Colsys, the user may define the desired quality of the approximation by means of a number n_{tol} of pointers l_j and relative tolerences T_j with $1 \leq n_{\text{tol}} \leq m_\star$, $j = 1, 2, \cdots, n_{\text{tol}}$ and such that $\forall j,\ 1 \leq l_j \leq m_\star \wedge T_j > 0$. These informations will be used to define the precision goal of Colsys :

$$\left|\left|Z_{l_j}(\underline{u}) - Z_{l_j}(\underline{v})\right|\right|_\infty \leq \left(1 + \left|\left|Z_{l_j}(\underline{v})\right|\right|_\infty\right) T_j,\ \forall j = 1, 2, \cdots, n_{\text{tol}}. \tag{3.25}$$

[3.14] The precise expression of these coefficients can be found in [Ascher et al., 1979a, Ascher et al., 1979b].

In other words, one can impose a desired precision on some (but not necesarilly all) components of $\underline{v}$ and/or their derivatives.

An "optimised" mesh will then be a mesh $\boldsymbol{x}^*$ which, when halved[3.15], gives a collocation approximation $\underline{v}$ satisfying (3.25) with as few mesh points as possible, say with $N^* + 1$ points. The idea of minimising the number of mesh points naturally corresponds to the whish of minimisation of the computational effort while maintaining the quality of the approximated solution. Put another way, what we call here an optimised mesh is a (most likely non-uniform) mesh that gives the best fit to the solution's behaviour for a given number of mesh points. The aim would then be to determine the smallest N^* allowing to satisfy (3.25) with an optimised mesh and then compute the corresponding mesh $\boldsymbol{x}^*$.

While being intuitively reasonable, as one could expect, finding a procedure to get the best optimised mesh (the one with the smallest number of points) is very difficult in practice. Nevertheless, one can make some progress by trying to equidistribute the error made due to the approximation. This procedure formalises the idea that we do not want an approximation $\underline{v}$ which satisfies (3.25) by performing extremely well[3.16] on some part of the interval $[a, b]$ and much worse on others[3.17].

The idea is, first, given a mesh $\boldsymbol{x}$, to consider the fact that (3.25) will hold (on $[a, b]$) if and only if it holds on each subinterval $[x_i, x_{i+1}]$ individually *i.e.* $\forall i = 0, \cdots, N-1$. One would then get N copies of (3.25) on smaller intervals on which the collocation approximation $\underline{v}$ built thanks to $\boldsymbol{x}$ is simply a polynomial function. The next step in the reasoning is to realise that, using $\underline{Z}(\underline{v})$ and approximating the right hand side of (3.25) with the same procedure as the one giving (3.22), one can construct a positive piecewise constant computable function $\hat{s}$ such that the given conditions will be satisfied up to $\mathcal{O}(h)$ if one imposes

$$\int_{x_i}^{x_{i+1}} \hat{s}(x)\mathrm{d}x = 1, \ \forall i = 0, 1, \cdots, N-1. \tag{3.26}$$

We wont get into the details of how $\hat{s}$ is defined nor how one proves that (3.26) is enough to guaranty that (3.25) is approximately satisfied (the reader can refer to [Ascher et al., 1979a, Ascher et al., 1979b] and references [5] and [12] therein for more details) but, for the understanding of the mesh-selection procedure, we need to stress three things :

Firstly, condition (3.26) is a condition on $\boldsymbol{x}$ to ensure that the corresponding collocation approximation $\underline{v}$, solution from (3.18) and satistying (3.16), will also satisfy (3.25). It is then important to realise that the dependence on the mesh points x_i does not reduces to the integral bounds. Indeed, the function $\hat{s}$ is constructed from $\underline{Z}(\underline{v})$ whose behaviour does also depend on the mesh points via (3.18). Given a number of mesh points, (3.26) might then be used as target for an algorithm attempting to make the mesh points fit the solution's behaviour but it would be more subtle than just ajusting the integral bounds.

[3.15] Remember that the mesh should be halved in order to approximate the error via (3.24).
[3.16] *i.e.* with some of the relative tolerences way below the corresponding T_j.
[3.17] *i.e.* with some of the relative tolerences very close to the corresponding T_j.

Secondly, since $\hat{s}$ is a piecewise constant function,

$$\int_{x_i}^{x_{i+1}} \hat{s}(x)\mathrm{d}x = \hat{s}(x_i)h_i, \ \forall i = 0, \cdots, N-1.$$

Thirdly, from condition (3.26), one immediatly gets that

$$\int_a^b \hat{s}(x)\mathrm{d}x = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \hat{s}(x)\mathrm{d}x = N,$$

or equivalently

$$\int_a^b \hat{s}(x)\mathrm{d}x = \sum_{i=0}^{N-1} \hat{s}(x_i)h_i = N. \tag{3.27}$$

Conditions (3.26)–(3.27) formalise the idea of an equidistribution of the error. The function $\hat{s}$, constructed by manipulation of the quantities in (3.25), might be seen as a "mesure of error" function. The intuition is that, according to (3.27), the whished approximation should have a "total error score" of N spread equaly, according to (3.26), among the N subintervals, one "error point" each.

Equation (3.27) then suggests, given a mesh $\boldsymbol{x}$ of $N+1$ points (which maybe does not fulfill (3.26)), to use the integral of $\hat{s}$ on $[a, b]$ to approximate the whished number of points N^*. Unfortunately, this may be unreliable in practice, leading to a very large N^* compared to N, especially in the early steps of the algorithm since the approximation $\underline{v}$ obtained from $\boldsymbol{x}$ might be pretty innacurate.

The mesh-selection algorithm implemented in Colsys is then based on a variation of the above proposal. The idea is to appproach an optimised mesh by choosing between two options : redistributing the mesh points on the interval or halving the mesh. It can be summarized as follows :

Let $\bar{N}$ indicate the maximum possible mesh size according to the storage specification of the program. Given a current mesh $\boldsymbol{x}$ of $N+1$ points,

1. Try to compute a collocation approximation $\underline{v}$.
   - (a) If the non-linear iteration supposed to give $\underline{v}$ does not converge, try to halve the current mesh.
     - i. If the new mesh is such that $N > \bar{N}$, exit (unsuccesfully)
     - ii. Else, let the new mesh become the current mesh and go back to start of step 1.
   - (b) Else, go to the next step.
2. Try to estimate the quality of the solution.
   - (a) If the current mesh has been obtained by halving a former one and convergence occured on both meshes, estimate the error using (3.24) and check if (3.25) is satified.

i. If yes, exit (successfully).
ii. Else, go to the next step.

(b) Else, go to the next step.

3. Using $\underline{Z}(\underline{v})$, compute the indicators

$$r_1 = \max_{i=0}^{N-1} \hat{s}(x_i)h_i, \;\; r_2 = \sum_{i=0}^{N-1} \hat{s}(x_i)h_i \text{ and } \;\; r_3 = \frac{r_2}{N}.$$

(a) If $r_1 < 2r_3$, halve the curent mesh, let the refined mesh becomes the new mesh and go back to step 1.

(b) Else, redistribute the points :
Define $N^* = \min\left\{ \frac{1}{2}\bar{N}, N, \frac{1}{2}\max\{ N, r_2 \} \right\}$,
Determine the new mesh $\boldsymbol{x}^*$ according to

$$\int_{x_i^*}^{x_{i+1}^*} \hat{s}(x)\mathrm{d}x = \frac{r_2}{N^*}, \;\; \forall i = 0, \cdots, N^* - 1,$$

Let this mesh becomes the current mesh and go back to step 1.

This procedure (especially step 3) calls for some comments :

- About the indicators r_1, r_2 and r_3 : By construction
  - r_1 is the maximal value of the integral of $\hat{s}$ on a subinterval $[x_i, x_{i+1}]$. If the error was equidistribute, this value would be 1 according to (3.26).
  - r_2 is the integral of $\hat{s}$ on $[a, b]$. If the error was equidistribute, this value would be N according to (3.27). Consequently, r_3 should give another way to approximate the integral of $\hat{s}$ on any of the subintervals $[x_i, x_{i+1}]$.
  - The deviation of r_1/r_3 from 1 is then a mesure of how the mesh $\boldsymbol{x}$ fails to equidistribute the error.
- About the threshold in step 3a : The idea is that the code estimates that it would be as advantageous, to reduce the error, to redistribute the points with $N^* = N$ than to take $N^* = (r_1/r_3)N$ with the same point distribution. This is why the threshold is placed at $r_1/r_3 = 2$, so that the mesh is redistributed only when it would cost more to halve the mesh.
- About the definition of N^* in step 3b : This definition is taken so that $N/2 \leq N^* \leq N$ and so that it will still be possible to halve the mesh later ($N^* \leq \bar{N}/2$) if needed to estimate the error.

- About the redistribution procedure in step 3b : This condition is adapted from (3.26) to tend to equidistribute the error since both r_2 and N^* are supposed to give an approximation of the optimal number of points. Note that here every quantity is computed using the mesh $\boldsymbol{x}$ so that the dependance in the mesh points of $\boldsymbol{x}^*$ is really limited to the integral bounds. We can then use this to find the mesh points. For example, one could iteratively determine x^*_{i+1} by finding the root of the increasing function $f_i(x) \coloneqq \int_{x_i^*}^{x} \hat{s}(t)\mathrm{d}t - r_2/N^*$.

As a final comment on this procedure, we already mentionned that, in order to perform well, the mesh-selection algorithm should start with the best possible initial guess since the error estimation based on (3.24) or the estimations done in step 3 might become a bit inaccurate if the quality of the apprximation $\underline{v}$ is low. The authors of the algorithm then claimed to only trust it to some extent and, to overcome this apparent loophole in the algorithm, an extra matter of precaution is added on top of the above procedure so that the current mesh will automatically be halved if either one of the following happens :

1. The size of the current mesh is smaller than that of the former one,
2. There have been 3 consecutive mesh selections resulting in the same size N,
3. There have been 3 consecutive pairs of mesh selection followed by mesh halving resulting in the same mesh size N.

In other words, the mesh size is increased automatically in several cases in order to improve the quality of the available approximation so that the mesh-selection algorithm is more or less ensured to perform well.

3.2.4 Approximating the Collocation Approximation

Up to now, we have covered the main features of the way Colsys proceeds, assuming that it can obtain a spline approximation $\underline{v}$. We should now turn to the problem of obtaining such a spline $\underline{v}$ in the first place via (3.18)–(3.16).

The procedure followed by Colsys to approximate the solution of (3.18)–(3.16) is a Damped Newton method as introcuded in section 3.1.3. Given (*a priori* from the user) an intial guess $\underline{v}_0$ over a mesh $\boldsymbol{x}$, untill a convergence criterion[3.18] is satisfied, it will apply the recurrence procedure

$$\underline{v}_{s+1} = \underline{v}_s + \lambda_s \underline{w}_s, \tag{3.28}$$

where $\underline{w}_s$ is determined as solution of the linearised problem

$$\begin{cases} L_n(\underline{v}_s)\underline{w}_s = -f_n(\underline{v}_s), \ \forall n = 1, \cdots, d \\ B_l(\underline{v}_s)\underline{w}_s = -g_l(\underline{v}_s), \ \forall l = 1, \cdots, m^\star \end{cases}, \tag{3.29}$$

[3.18] This criterion will be identical to (3.25) provided one replaces $\underline{u} \to \underline{v}_{s+1}$ and $\underline{v} \to \underline{v}_s$ in the left-hand side and $\underline{v} \to \underline{v}_{s+1}$ in the right-hand side.

with L_n and B_l defined in (3.20),

$$\begin{aligned} f_n(\underline{v}_s) &:= (v_s)_n^{(m_n)} - F_n(\cdot;\underline{Z}(\underline{v}_s)) \\ g_l(\underline{v}_s) &:= G_l\left(\zeta_l;\underline{Z}(\underline{v}_s)\right) \end{aligned}, \tag{3.30}$$

and where $0 < \lambda_s \leq 1$ is the relaxation factor. Here again we will skip the question of the diagnostic performed to fix λ_s. We will just quote that, within the algorithm used in Colsys, a minimal value $\lambda_{\min} > 0$ is defined for the relaxation factor so that, if the diagnotic calls for a $\lambda_s < \lambda_{\min}$, this part of the procedure will return with no convergence and require to halve the current mesh before restarting the Newton procedure with $\lambda_s = \lambda_{\min}$.

As usually, assuming we have a suitable λ_s at hand, the problem then becomes to solve (3.29) in order to perform the Newton iteration. As briefly mentionned in section 3.1.3, one can circumvent the difficulty of solving (3.29) as a system of differential equations with some boundary conditions by means of the properties of $\underline{v}_s$ and $\underline{w}_s$. Indeed, for any s and $\forall n = 1, \cdots, d$, we want that $(\underline{v}_s)_n, (\underline{w}_s)_n \in S^{\boldsymbol{r}_n}_{k+m_n-1}(\boldsymbol{x})$, where the vectors $\boldsymbol{r}_n$ are as in (3.17).

Similarilly to what we described in section 3.1.1, using an appropriated set of B-splines functions and evaluating the equations at the Gauss-Legendre collocation points x_{ij}, it is then possible to convert (3.29) into a set of $Nkd+m_\star$ (linear) equations whose unknown are the components of $\underline{w}_s$ in the corresponding B-spline basis. If the collocation matrix (the matirc obtained on the left-hand side of the system) is non-singular, it is then possible to solve the system using linear methods. Once obtained the components of $\underline{w}_s$, it is then possible to perform the Newton iteration at the level of the components of $\underline{v}_s$ and, if needed, to evaluate this spline using the B-splines functions.

Note that, since the result of Newton iteration that would satisfy the convergence criterion, say $\underline{v}_{s+1}$, gives an approximation to the collocation approximation $\underline{v}$ of the exact solution $\underline{u}$ of the initial problem (3.15)–(3.16), there are then two components in the error : the difference between $\underline{v}_{s+1}$ and $\underline{v}$ and the one between $\underline{v}$ and $\underline{u}$. This may lead to the "fear" of an accumulation of errors.

It is then important to see one interesting feature of the combination of Newton algorithm with the mesh selection procedure described in the previous section. As we already saw, the evaluation of the error always requires, after a convergence was obtained, to halve the mesh and to compare the approximation obtained with this new refined mesh with the former one. Both aproximations will be obtained using Newton's method but the first one will give an excelent initial guess to compute the second one so that, if we can obtain convergence of the Newton algorithm once, we are in good conditions to ensure subsequent convergences. The conjuction of the mesh selection and error estimation procedure should then feed Newton algorithm with increasingly better initial guesses on icreasingly better meshes.

This then gives a reliable procedure provided the very first guess $\underline{v}_0^{\text{guess}}$ on the initial mesh $\boldsymbol{x}_0$ is close enough from the collocation solution for this mesh, say $\underline{v}_0$. In this case, Newton algorithm will provide a better approximation to $\underline{v}_0$, say $\underline{v}_0^{\text{Newt}}$. The subsequent uses of the mesh selection procedure should give increasingly better meshes $\boldsymbol{x}_i$ on which good guesses for Newton algorithm $\underline{v}_i^{\text{guess}} = \underline{v}_{i-1}^{\text{Newt}}$ will then be available. Consequently, the next uses of Newton's procedure will provide better approximations $\underline{v}_i^{\text{Newt}}$ to the (better) collocation aproximations $\underline{v}_i$ over these increasingly better meshes $\boldsymbol{x}_i$. This then ensures that, when all the precision requirements are satisfied, the result of Newton iteration $\underline{v}_{i+1}^{\text{Newt}}$ on the mesh $\boldsymbol{x}_{i+1} = \tilde{\boldsymbol{x}}_i$ (*i.e.* obtained by halving the last selected mesh $\boldsymbol{x}_i$) with $\underline{v}_i^{\text{Newt}}$ as initial guess will be a good approximation to $\underline{u}$ within the defined range of tolerence.

3.3 How to Use Colsys?

In this section, we list the inputs and outputs of Colsys and we mention, when it is necessary, the default value that we always used for those parameters.

In order to work correctely, Colsys needs several things as one can see from the definition of the subroutine COLSYS :

SUBROUTINE *COLSYS* (NCOMP, M, ALEFT, ARIGHT, ZETA, IPAR, LTOL, TOL, FIXPNT, ISPACE, FSPACE, IFLAG, FSUB, DFSUB, GSUB, DGSUB, SOLUTN)

Note that the subroutine COLSYS in itself is just an interface between the user and the package of subroutines called collectivelly Colsys. It serves only to test some of the inputs parameters, to rename some of the parameters (to make understanding of the code easier), to do some initializations and to break FSPACE and ISPACE up into the arrays needed by the program to work.

3.3.1 Inputs for Colsys

NCOMP

An integer corresponding to the number of differential equations to solve. In notations of the previous sections, it is d.

NOTE : The program is tought such that NCOMP ≤ 20.

M

A list of integers corresponding to the order of each of the NCOMP differential equations. In notations of the previous sections, it is a list containing the m_n's.

NOTE 2 : Similarilly to what we assume in section 3.2.1, *the components of* M *should be given such that*

$$\forall j = 1, \cdots, \text{NCOMP} - 1,\ \text{M}(j) \leq \text{M}(j+1).$$

NOTE 3 : The program is tought such that M(NCOMP) ≤ 4 and such that

$$\sum_{j=1}^{\text{NCOMP}} \text{M}(j) =: \text{MSTAR} \leq 40.$$

In notations of the previous section, MSTAR $= m^\star$.

ALEFT and ARIGHT

ALEFT and ARIGHT are respectively the left and right end of the interval on which we want to solve our equations. In notations of the previous sections, ALEFT $= a$ and ARIGHT $= b$.

ZETA

A list of real numbers corresponding to the position of the different boundary conditions. In notations of the previous sections, it is a list containing the ζ_j. Its length should then be MSTAR.

NOTE 4 : Similarilly to what we assume in section 3.2.1, *the components of* ZETA *should be given such that*

$$\forall j = 1, \cdots, \text{MSTAR} - 1,\ \text{ZETA}(j) \leq \text{ZETA}(j+1).$$

IPAR

A list of 11 integers containing parameters used to specify options to personalize the behaviour of Colsys.[3.19]

1. IPAR(1) : Specifies if the problem is linear or not.

   $$\text{IPAR}(1) = \begin{cases} 0 \text{ if the problem is linear} \\ 1 \text{ if the problem is non-linear} \end{cases}$$

   $\longrightarrow$ Through this thesis, we then always used IPAR(1) $= 1$.

2. IPAR(2) : Specifies the number of collocation points per subinterval. In notations of the previous section, IPAR(2) $= k$.

   NOTE 5 : The program was built such that M(NCOMP) $\leq$ IPAR(2) ≤ 7.

   NOTE 6 : If IPAR(2) $= 0$, Colsys will automatically choose this number to be max$\{$ M(NCOMP) $+ 1,\ 5 -$ M(NCOMP) $\}$.

   $\longrightarrow$ Through this thesis, we usually set IPAR(2) $= 0$ so that we let Colsys choose the number of points per subinterval. In addition, since the equations we had to solve usually (if not always) had $m_d =$ M(NCOMP) $=$ 2, we then had M(NCOMP) $+ 1 =\ 5 -$ M(NCOMP) $= 3$, so in practice IPAR(2) $= 3 = k$ for our projects.

3. IPAR(3) : Specifies the number of subintervals in the initial mesh. In notations of the previous sections, IPAR(3) $= N$ *wihch is then the number of points in the mesh* ***minus 1***.

   NOTE 7 : If IPAR(3) $= 0$, Colsys will arbitrarilly set this number to 5.

[3.19] Note that its allowed size in the program should then be at least 11.

4. IPAR(4) : Specifies for how much fields we give a constraint on tolerence. In order to be consistent, one should then have IPAR(4) $\leq$ MSTAR. In notations of the previous sections, IPAR(4) $= n_{\text{tol}}$.

   $\longrightarrow$ Through this thesis, usually, we defined a tolerence for all the fields and their derivatives so IPAR(4) = MSTAR.

5. IPAR(5) : Specifies the dimension of FSPACE (see below).

   $\longrightarrow$ Through this thesis, usually, we gave a gigantic value for IPAR(5) in order to be sure that there will be enough space in memory for FSPACE.

6. IPAR(6) : Specifies the dimension of ISPACE (see below).

   $\longrightarrow$ Through this thesis, usually, we gave a gigantic value for IPAR(6) in order to be sure that there will be enough space in memory for ISPACE.

7. IPAR(7) : Controls the output of Colsys.

$$\text{IPAR}(7) = \begin{cases} -1 \text{ for a full printout diagnostic} \\ 0 \text{ for selected printout} \\ 1 \text{ for no printout} \end{cases}$$

8. IPAR(8) : Controls the definition of the initial mesh

$$\text{IPAR}(8) = \begin{cases} 0 \text{ Colsys choose an uniform initial mesh} \\ 1 \text{ the initial mesh is specified by the user (see FSPACE)} \\ 2 \text{ idem IPAR(8)=1 } + \textbf{ no} \text{ adaptive mesh selection is done} \end{cases}$$

   $\longrightarrow$ Through this thesis, usually, we set IPAR(8)=0.

9. IPAR(9) : Controls the definition of the initial guess provided by the user

$$\text{IPAR}(9) = \begin{cases} 0 & \text{if no initial guess is provided} \\ 1 & \text{if an initial guess is provided in} \\ & \text{subroutine SOLUTN (see below)} \\ 2 & \text{if an initial mesh } \textbf{and} \text{ approximate solution} \\ & \text{coefficients are provided in FSPACE} \\ 3 & \text{idem IPAR(9)=2 } \textbf{but} \text{ the desired new mesh} \\ & \text{is obtained by halving the given one} \\ 4 & \text{idem IPAR(9)=2 } \textbf{and} \text{ a former initial mesh is also given} \end{cases}$$

   $\longrightarrow$ Through this thesis, usually, we set IPAR(9)=1 and used SOLUTN to reuse a previously obtained solution.

10. IPAR(10) : Specifies to which extent the problem is sensitive

$$\text{IPAR}(10) = \begin{cases} 0 & \text{if the problem is regular} \\ 1 & \text{to be used for an extrasensitive non linear problem } \textbf{only} \\ 2 & \text{if Colsys should return immediatly upon} \\ & \text{(a) two succesive non-convergences or} \\ & \text{(b) after obtaining error estimate for the first time} \end{cases}$$

$\longrightarrow$ Through this thesis, usually, we set IPAR(10)=0 as the problem was regular in the sense that the equations were defined by algebraic expressions in the different fields.

11. IPAR(11) : Specifies the number of fixed points in the mesh *other* than ALEFT and ARIGHT. These points are then specified in FIXPNT (see below).

$\longrightarrow$ Through this thesis, usually, we set IPAR(11)=0 since there was no reason to impose fixed points in the mesh other than the limits of the interval.

LTOL

An array providing IPAR(4) pointers to specify on which fields we want to impose a tolerence. LTOL(j) = L means that the j^{th} component of LTOL will constrain the L^{th} field (*i.e.* the L^{th} component of $\underline{Z}(\underline{u})$). Consequently, any LTOL(j) should be smaller than MSTAR. In notations of the previous section, LTOL(j) = l_j.

NOTE 8 : The program is tought such that

$$\forall j = 1, \cdots, \text{IPAR}(4) - 1, \quad \text{LTOL}(j) \leq \text{LTOL}(j+1).$$

$\longrightarrow$ Through this thesis, since we usually set IPAR(4) = MSTAR, we set LTOL(j) = j, for simplicity, so that we just specified the tolerence of the fields in natural order.

TOL

An array of dimension IPAR(4). For each $j = 1, \cdots, \text{IPAR}(4)$, TOL($j$) corresponds to the tolerence imposed on the field whose number is LTOL(j). In notations of the previous section, TOL(j) = T_j.

$\longrightarrow$ Through this thesis, for simplicity, we usually imposed the same tolerence for all the fields.

FIXPNT

An array of dimension IPAR(11) specifying the points (other than ALEFT and ARIGHT) which should be present in every mesh computed by Colsys.

$\longrightarrow$ Through this thesis, since we set IPAR(11)=0, FIXPNT was empty.

ISPACE

An integer workspace array of dimension IPAR(6).

Colsys will use ISPACE to store some parameters needed to call the approximate solution (see below). ISPACE will then be explicitly modified before Colsys returns.

NOTE 9 : Its size provides a constraint on the maximum number of subintervals.

$\longrightarrow$ This is why through this thesis we ususally gave a gigantic value for IPAR(6). The modern computers have enough memory space to be sure that there would be no problem for the kind of systems that we had to study.

FSPACE

A real workspace array of dimension IPAR(5).

If IPAR(8)=1, FSPACE is used to specify the initial mesh : if $\boldsymbol{x}$ is the initial mesh, its components ALEFT $= x_0 < x_1 < \cdots < x_N =$ ARIGHT will occupy FSPACE(1), FSPACE(2), $\cdots$, FSPACE($N+1$). Note that, in this case, the user only needs to supply the interior mesh points, *i.e.* the elements that will occupy FSPACE(2), $\cdots$, FSPACE(N) (since ALEFT and ARIGHT are already known).

In cases where IPAR(9) ≥ 2, FSPACE is also be used to store the initial mesh provided by the user followed by the approximate solution coefficients.

FSPACE will be explicitly modified by Colsys as it will be used to store the final mesh and coefficients of the approximated solution before Colsys returns (see below).

NOTE 10 : The size of FSPACE provides a constraint on the maximum number of subintervals.

$\longrightarrow$ This is why through this thesis we ususally gave a gigantic value for IPAR(5). The modern computers have enough memory space to be sure that there would be no problem for the kind of system that we had to study.

IFLAG

An integer that Colsys will modify to express his mode of return (see below).

FSUB

This is a subroutine used by Colsys and **provided by the user**. The goal of this routine is to evaluate the functions F_n from (3.15) at a given point. This is basically where we define the equations that Colsys will solve.

It should have the following heading (otherwise Colsys will not be able to use it correctly) :

SUBROUTINE *FSUB* (X , Z , F)

where X stands for the point of the evaluation, Z is an array of size MSTAR that corresponds to $\underline{Z}(\underline{u})$ and F is an array of size NCOMP whose components are the F_n.

DFSUB

This is a subroutine used by Colsys and **provided by the user**. The goal of this routine is to evaluate the gradients of the functions F_n from (3.15) at a given point; these are used in the linearized problem.

It should have the following heading (otherwise Colsys will not be able to use it correctly) :

SUBROUTINE *DFSUB* (X , Z , DF)

where X stands for the point of the evaluation, Z is an array that corresponds to $\underline{Z}(\underline{u})$ and DF is a NCOMP by MSTAR array whose components are the partial derivatives of F_n with respect to Z_l; *i.e.* for a given call to the subroutine, one will compute $\mathrm{DF}(n,l) = \dfrac{\partial F_n}{\partial Z_l}$ for $n = 1, \cdots, \mathrm{NCOMP}$ and $l = 1 \cdots, \mathrm{MSTAR}$.

GSUB

This is a subroutine used by Colsys and **provided by the user**. The goal of this routine is to evaluate the functions G_l from (3.16). This is basically where we define the boundary conditions for Colsys.

It should have the following heading (otherwise Colsys will not be able to use it correctly) :

SUBROUTINE *GSUB* (I , Z , G)

where I stands for the number of the desired boundary condition (an integer between 1 and MSTAR), Z is an array of size MSTAR that corresponds to $\underline{Z}(\underline{u})$ and G is an array of size MSTAR whose components are the G_l. The subroutine will then compute $G_{\mathrm{I}}(\mathrm{ZETA(I)}; \underline{Z}(\underline{u}(\mathrm{ZETA(I)})))$

DGSUB

This is a subroutine used by Colsys and **provided by the user**. The goal of this routine is to evaluate the gradients of the functions G_l from (3.16); these are used in the linearised problem. More precisely, it is aimed at evaluating one given row of the jacobian of $G(X; \underline{Z}(\underline{u}(X)))$.

It should have the following heading (otherwise Colsys will not be able to use it correctly) :

SUBROUTINE *DGSUB* (I , Z , DG)

where I stands for the number of the boundary condition (an integer between 1 and MSTAR), Z is an array of size MSTAR that corresponds to $\underline{Z}(\underline{u})$ and DG is a MSTAR array whose components are the partial derivatives of G_{I} with respect to Z_l; *i.e.* for given call of the subroutine, one will compute $\mathrm{DG}(\mathrm{I}, l) = \dfrac{\partial G_{\mathrm{I}}}{\partial Z_l}$ for $l = 1, \cdots, \mathrm{MSTAR}$.

SOLUTN

This is a subroutine used by Colsys and **provided by the user**. It is used only when IPAR(9)=1.

The goal of this routine is to evaluate the initial approximation for $\underline{Z}(\underline{u})$ and for the $u_n^{(m_n)}$ at a given point.

It should have the following heading (otherwise Colsys will not be able to use it correctly) :

SUBROUTINE *SOLUTN* (X , Z , DMVAL)

where X stands for the point of the evaluation, Z is an array of size MSTAR that corresponds to $\underline{Z}(\underline{u})$ and DMVAL is an array of size NCOMP whose j^{th} component is the $\mathrm{M}(j)^{\mathrm{th}}$ derivative of u_j.

NOTE 11 : Since this should be used to evaluate the functions $u_n^{(i)}$ for all $n = 1, \cdots, d$ and all $i = 0, \cdots, m_n$ at a given point in the interval, all the MSTAR components of Z and NCOMP components of DMVAL should be specified for any X between ALEFT and ARIGHT (and not just at the level of the mesh points).

$\longrightarrow$ Through this thesis, there was usually two versions of SOLUTN for a given project. One was aimed to simply import the content of an external file (solu.new) containing a previous solution that will be used as initial guess for the new problem. The other one was used to specify the initial guess in the form of "explicit" functions. The second version was basically used to produce the very first initial solution for one given project. After that, we always took the last obtained solution as initial guess for the next step.

3.3.2 Ouputs from Colsys

As we already mentionned, the natural IFLAG and the workspace arrays FSPACE and ISPACE will be used by Colsys to store the results of the calculation.

IFLAG

IFLAG will be set to one of the following values, depending on how the simulation went :

$$\text{IFLAG} = \begin{cases} 1 & \text{for normal (successful) return} \\ 0 & \text{if the collocation matrix is singular} \\ -1 & \text{if the expected number of subintervals} \\ & \text{exceeds the storage specification} \\ -2 & \text{if the non-linear iteration has not converged} \\ -3 & \text{if there is an input data error} \end{cases}$$

The smaller IFLAG is, the worst it is.

FSPACE

Before returning, Colsys will use the workspace array FSPACE to store the mesh points and coefficients of the approximate solution.

The precise way in which these datas are stored is not really important for the user since one can evaluate the approximate solution using the APPSLN subroutine (see below) so, in principle, one does not need to directly manipulate entries from FSPACE.

Let us nevertheless mention (since this might be used to save a solution on an external file for later use) that the mesh points will be stored in the first entries of FSPACE, *i.e.* in notations of the previous section

$$\text{FSPACE}(1) = x_0, ..., \text{FSPACE}(N+1) = x_N.$$

ISPACE

Before returning, Colsys will use the workspace array ISPACE to store integers used as pointers to refer to the approximate solution. Roughly speekeing, ISPACE contains informations about what kind of information is stored at which place in FSPACE. FSPACE contains several informations stored "in a row" and ISPACE provides informations to access the correct blocks in FSPACE.

Here again, the precise way in which these data are stored is not really important for the user since one can evaluate the approximate solution using the APPSLN subroutine (see below) so, in principle, one does not need to manipulate entries from ISPACE manually.

Let us nevertheless mention (since this might be used by the user to save a solution on an external file for later use) that the number of mesh points will be stored in the first entry of ISPACE, *i.e.* in notations of the previous section

$$\text{ISPACE}(1) = N + 1.$$

APPSLN

When Colsys returns succesfully, the user can access and use the approximate solution $\underline{Z}(\underline{v}(x))$ at a given point $a \leq x \leq b$ by means of the subroutine APPSLN.

The syntax to call the routine is

CALL *APPSLN* (X, Z, FSPACE, ISPACE)

where X stands for the point x where the solution should be evaluated, Z is an array of size MSTAR that corresponds to $\underline{Z}(\underline{v})$ and FSPACE and ISPACE are the workspace arrays in which COLSYS stored the informations necessary to reconstruct the approximate solution.

Calling APPLSN does in fact correspond to a standard call to a more general and more sophisticated subroutine called APPROX (whose purpose is to evaluate a piecewise polynomial function). This means that it is possible to have a more sophisticated way to recover the solution thanks to APPROX. This also means that it is not necessary in practice since APPSLN provides an autonomous way to call the solution that takes care of anything for the user.

$\longrightarrow$ Through this thesis, we always accessed the approximate solutions using the APPSLN subroutine.

Conclusion

Our dissertation already comes to an end... While doing our best to not be (too) emotional, it is time to wrap everything up. Hereinbelow, we propose a brief summary of the results of this thesis as well as an overview of possible directions to extend the present work.

1 Conclusions

As an important point of this conclusion, we should first acknowledge the following: research never stops its progression. Consequently, since we were lucky enough to see our research endeavour supported for six years, some of the results presented here might have been more relevant at the time we developed them that they are at the time we write this summary (and this should then be even more true for later readers). In this respect, we have in mind, for instance, the status of Horndeski gravity.

Back in 2016 (and even a bit before that), Horndeski gravity attracted a lot of attention from both the point of view of cosmology and compact objects in view of the motivations outlined in section 2.4. Throughout the last six years, the recent developments of gravitational wave astronomy and black hole imaging astronomy started to put stringent constraints on Horndeski gravity, especially at the cosmological level; one can see *e.g.* the reviews [Langlois, 2019, Kobayashi, 2019] which include a discussion on this point.

In this respect, we should emphasise once more that the tensions on Horndeski gravity coming from the cosmological perspective do not necessarily affect its relevance for the study of compact objects – which was the main interest of this thesis. Also, from the theoretical perspective, since Horndeski gravity provides the most general scalar-tensor theory of gravity in 4 dimensions, based on the same geometrical setup as general relativity[1] and leading to second order field equations, even if the theory is ruled out, it might be interesting to know as much as possible on its features. Indeed, this might help to get more insight on what does not work and this could still help to derive constraints leading

[1]In other words, one assumes a theory based on a manifold endowed with a metric and the associated Levi-Civita connection, supplemented by a single real scalar field.

to most successful theories. That being said, we should acknowledge that these tensions inevitably put into question the status of Horndeski gravity as a viable candidate to make a breakthrough regarding the puzzles of dark matter and dark energy.

Despite this cautionary note, we can still point out that, on a much more personal scale, the work undertaken during this thesis gave us the opportunity to develop, improve and share skills whose applicability goes way beyond the study of Horndeski gravity. These more indirect contributions to our research field should, nevertheless, last independently of the status of Horndeski gravity itself.

Also, to not leave you with a distorted state of the art, we should acknowledge that research regarding compact objects in Horndeski gravity is still a living topic as emphasised by very recent (at the moment we write this sentence) papers such as [Vandeev and Semenova, 2022] or [Papageorgiou et al., 2022].

Finally, we should obviously comment on the results themselves. We will not give a detailed overview of our results here as it is already the subject of section 2.5. Let us nonetheless summarise the main contributions of this work:

→ In the paper presented in appendix B, we studied deformations of the hairy black hole solutions from [Sotiriou and Zhou, 2014b], constructed assuming a linear non-minimal coupling between a real scalar field and the Gauss-Bonnet invariant, see (2.38), in the presence of a NUT charge. We also investigated the geodesic motions in the corresponding spacetime.

→ In the paper presented in appendix C, we obtained 5 dimensional hairy black hole solutions corresponding to a family of rotating black holes[2] endowed with a minimally coupled doublet of complex, massive and electrically charged scalar fields. This is analogous to the 5 dimensional equivalent of a Kerr-Newman black hole endowed with a boublet of massive and charged scalar fields.

→ In the paper presented in appendix D, we found a family of hairy black hole solutions in a theory presenting a non-minimal coupling between a real scalar field and the Gauss-Bonnet invariant, see (2.38) but were $\alpha\phi\mathscr{L}_{\text{GB}} \to \left(\gamma_1\phi + \gamma_2\phi^2\right)\mathscr{L}_{\text{GB}}$, that smoothly extrapolates between two families of hairy black holes previously known in the literature and whose behaviour was drastically different. Our solutions extrapolate between the shift-symmetric hairy black holes from [Sotiriou and Zhou, 2014b] and the spontaneously scalarised black holes from [Silva et al., 2018].

In this same paper, we also provided an analysis of boson stars constructed with the same kind of non-minimal coupling as the one leading to spontaneously scalarised black holes. Our results demonstrate that this type of non-minimal coupling has an important influence on the classical stability of these solutions.

[2]The two angular momenta were equal by construction.

→ In the paper presented in appendix E, we extended previous analysis on neutron stars in a theory presenting a non-minimal coupling between the Einstein tensor and the first derivatives of a real scalar field, see (2.39) with $\beta = 0$. We also investigated the behaviour of boson stars in this same theory. Our results have shown a similar behaviour for both types of objects. Namely, we obtained that the sign of the non-minimal coupling has an important influence on the deformation of solutions with respect to the ones known in general relativity. We also obtained that, assuming a negative non-minimal coupling constant, one could obtain a new branch of neutron star solutions that were previously unnoticed in the literature.

→ Finally, in the paper presented in appendix F, we proposed the first explicit construction of (exact) hairy black hole solutions in a class of scalar-torsion theories[3] proposed in [Hohmann, 2018b].

In this same paper, we also derived a no-scalar-hair theorem for this type of scalar-torsion theories. Our construction was similar to what is outlined in section 2.3.4.

If we had to class them in terms of their importance, we would consider the results from appendixes D and F as the two main contributions of this thesis; the results from appendix D since they offer a connection between two, seemingly unrelated, important types of black hole solutions known in Horndeski gravity and those from appendix F since they offer a first study of hairy black holes in a mostly unstudied branch of scalar-to-gravity couplings and hence open the way to several extensions in the future (see next section).

2 Perspectives

Time and financial constraints force every adventure to eventually come to an end. Yet, baring the risk to repeat ourselve, research never stops. In this respect, let us point here toward some directions that could extend the work of this thesis and that we hope we could address in the future.

The study of Horndeski gravity – and the new experimental constraints that starts to limit its viability – slowly started to motivate the search for alternative frameworks for classical modifications of general relativity. One is thus encouraged to question the hypothesis underlying the construction of Horndeski theory.

Questioning the key hypothesis of Horndeski theory that field equations should remain second order, some attention have been devoted in the last few years to the construction of viable scalar-tensor theories presenting higher-order equations. This led to the construction of the so-called Degenerate Higher Order Scalar-Tensor (DHOST) theories. This is a first potentially interesting direction that one could consider, even though this class of theories seems to suffer

[3]Remember that we use this terminology to denote theories based on the teleparallel equivalent to general relativity to which one add a non-minimally coupled scalar field.

from similar problems as Horndeski gravity from a cosmological perspective; see [Langlois, 2019] for a review.

If DHOST theories call into question the hypothesis constraining the dynamical content of Horndeski gravity, the other key hypothesis of Horndeski's construction lies in the, say, kinematical content of the theory. Indeed, Horndeski gravity assumes the same geometrical framework as general relativity: a 4 dimensional differential manifold endowed with a Lorentzian metric and the associated Levi-Civita connection. This geometrical setup has been used so far as a standard in the study of classical alternatives to general relativity, for obvious reasons, but, on account of the tensions suffered by Horndeski gravity, it is now interesting to acknowledge that alternatives are possible.

Of course, the modification of the geometrical framework of spacetime is not an end in itself if it is not physically motivated. In this respect, teleparallel equivalents to general relativity (especially TEGR) present promising alternatives since, as detailed in section 1.6, they allow constructing theories of gravity whose phenomenological content matches that of general relativity. These theories thus provide an interesting starting point for constructions of classical modifications of general relativity including a scalar field that by-pass Horndeski's construction. This is what we were interested in for our last paper (see appendix F). Of course, the results presented in this paper only constitute a preliminary study of what happens for compact objects in this type of theories. This thus opens quite directly a rather broad direction to extend the work undertaken in this thesis.

Following the same idea, in a probably slightly more distant future, one could consider pushing on this direction and studying the influence of scalar fields in more sophisticated gauge theoretical frameworks of gravity such as the Einstein-Cartan theory or, later on, the full Poincaré gauge theory.

Finally, one could also acknowledge that the study of non-minimally coupled scalar fields in itself can be a first step and look into the direction of non-minimal couplings for vector fields or other physical fields.

The last word

To finally conclude this conclusion by a conclusive paragraph, I would like to thank once more my supervisor Yves Brihaye for his constant kindness, for always pushing me forward and for teaching me or encouraging me to learn much more than I would have expected at the beginning of this adventure. I also would like to thank my supervisor Claude Semay for his valuable advices ever since I entered the university, for the many stimulating discussions on various subjects and for taking care of the balance between my teaching and research duties.

I would also like to express my gratitude to all of you who helped, directly or indirectly, to the realisation of this fascinating and quite demanding project. You are too many to list here (I already did my best to include as much of you as possible in the acknowledgement section) but you should recognise yourself.

Finally, I should keep the last thank to you, dear reader, for reaching this sentence. I hope to see you again for other interesting projects in the future.

Cheers!

Ludovic Ducobu,
Mons, Belgium, 24 April 2022.

Appendix

Building Blocks for the Galileon Lagrangian A

In this appendix, we present the explicit form of the 5 "building blocks" $\mathscr{L}_i$ for the Galileon Lagrangian density (2.28) as well as a sketch of the steps necessary to construct those terms. A more detailed analysis can be found in [Deffayet and Steer, 2013].

A.1 Building Blocks

The 5 indepent building blocks of the Galileon Lagrangian (2.28) are

$$\begin{aligned}
\mathscr{L}_1 &= \pi, \\
\mathscr{L}_2 &= (\partial\pi)^2, \\
\mathscr{L}_3 &= (\partial\pi)^2 \;\Box\pi, \\
\mathscr{L}_4 &= (\partial\pi)^2 \left[(\Box\pi)^2 - (\partial\partial\pi)^2\right], \\
\mathscr{L}_5 &= (\partial\pi)^2 \left[(\Box\pi)^3 - 3\Box\pi \;(\partial\partial\pi)^2 + 2\,(\partial\partial\pi)^3\right],
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
\Box\pi &= \partial_\mu\partial^\mu\pi, \\
(\partial\pi)^2 &= \partial_\mu\pi\partial^\mu\pi, \\
(\partial\partial\pi)^2 &= \partial^\alpha\partial_\beta\pi \;\partial^\beta\partial_\alpha\pi, \\
(\partial\partial\pi)^3 &= \partial^\alpha\partial_\beta\pi \;\partial^\beta\partial_\gamma\pi \;\partial^\gamma\partial_\alpha\pi.
\end{aligned}$$

Let us mention that the form expressed in (A.1) is distinct from (but equivalent to) the one given in the original paper [Nicolis et al., 2009]. This form is nevertheless more suitable to express the link between the Galileon theory and its different generalisations. Also, it has the nice property that, for $i \geq 2$, the Lagrangian densities have the form $\mathscr{L}_i = (\partial\pi)^2 \, \mathcal{E}_{i-1}$, where $\mathcal{E}_i = \delta\mathscr{L}_i/\delta\pi$.

Note also that the first piece $\mathscr{L}_1$, referred to as the "tadpole" in the original paper, has a different status than the other 4 terms. Indeed, due to its very specific form the associated "field equation" would be $\mathcal{E}_1 = 1 \approx 0$, which can obviously not be satisfied on its own. Also, since $\mathcal{E}_1$ is a constant, it technically does not contradict the fact that it leads to a field equation in which every appearing π is acted upon by exactly two derivatives. Yet, the tadpole might be seen as a trivial example and some authors then simply ignore it despite the fact that it is present in the original paper's classification.

A.2 Sketch of Construction

In this paragraph we present two toy models that aim to help the reader to understand the main steps in the construction of the Galileon Lagrangian.

The first step in finding the most general Lagrangian density for a scalar field π in Minkowski spacetime presenting a second order field equation is to realise that this Lagrangian density will have to explicitly depend on the second-order derivatives of the scalar field $\partial_\mu\partial_\nu\pi$. This can be rather counter-intuitive considering the fact that, for a Lagrangian density of the form $\mathscr{L}\left(\pi, \partial_\mu\pi, \partial_\mu\partial_\nu\pi\right)$, the Euler-Lagrange equation is given by

$$\left(\frac{\partial}{\partial\pi} - \partial_\mu\left[\frac{\partial}{\partial(\partial_\mu\pi)}\right] + \partial_\mu\partial_\nu\left[\frac{\partial}{\partial(\partial_\mu\partial_\nu\pi)}\right]\right)\mathscr{L} = 0. \tag{A.2}$$

In general, such an equation will involve third and fourth orders in π's derivatives thanks to the last term. It is yet possible to maintain a second order field equation. As a proof of concept, let us consider the toy model

$$\tilde{\mathscr{L}} = \frac{1}{2}\pi \, \Box\pi - V(\pi) = \frac{1}{2}\pi \; \partial_\mu\partial_\nu\pi \; \eta^{\mu\nu} - V(\pi).$$

In this case, (A.2) takes the form

$$\begin{aligned}\left(\frac{1}{2}\Box\pi - V'(\pi) - \partial_\mu\left[0\right] + \partial_\mu\partial_\nu\left[\frac{1}{2}\pi \; \eta^{\mu\nu}\right]\right) &= 0 \\ \Leftrightarrow \quad \Box\pi - V'(\pi) &= 0.\end{aligned}$$

The field equation (which is actually the Klein-Gordon equation (2.8)) is then clearly second order. This becomes obvious if we realise that

$$\tilde{\mathscr{L}} = \frac{1}{2}\pi \; \partial_\mu\partial_\nu\pi \; \eta^{\mu\nu} - V(\pi) = -\frac{1}{2}\partial_\mu\pi \; \partial_\nu\pi \; \eta^{\mu\nu} + \partial_\mu\left(\frac{1}{2}\pi \; \partial_\nu\pi \; \eta^{\mu\nu}\right) - V(\pi).$$

Now that we are convinced that theories depending explicitly on the second-order derivatives of the scalar field can have second order field equations, a less trivial example would be to find a way to maintain second order field equations for theories involving terms which are polynomial in $\partial_\mu\partial_\nu\pi$. Again, as a proof of concept, let us consider the following toy model :

$$\bar{\mathscr{L}} = \mathcal{T}^{\mu_1\mu_2\nu_1\nu_2}\ \partial_{\mu_1}\partial_{\nu_1}\pi\ \partial_{\mu_2}\partial_{\nu_2}\pi,$$

where we assume $\mathcal{T} = \mathcal{T}(\pi)$.

Here, (A.2) is given by

$$\left(\frac{\partial}{\partial\pi} + \partial_\mu\partial_\nu\left[\frac{\partial}{\partial(\partial_\mu\partial_\nu\pi)}\right]\right)\bar{\mathscr{L}} = 0.$$

Since

$$\frac{\partial\bar{\mathscr{L}}}{\partial\pi} = \partial_\pi\mathcal{T}^{\mu_1\mu_2\nu_1\nu_2}\ \partial_{\mu_1}\partial_{\nu_1}\pi\ \partial_{\mu_2}\partial_{\nu_2}\pi,$$

the first term cannot induce higher order terms but

$$\frac{\partial\bar{\mathscr{L}}}{\partial(\partial_\mu\partial_\nu\pi)} = \mathcal{T}^{\mu\mu_2\nu\nu_2}\ \partial_{\mu_2}\partial_{\nu_2}\pi + \mathcal{T}^{\mu_1\mu\nu_1\nu}\ \partial_{\mu_1}\partial_{\nu_1}\pi,$$

and so

$$\begin{aligned}\partial_\mu\partial_\nu\left[\frac{\partial\bar{\mathscr{L}}}{\partial(\partial_\mu\partial_\nu\pi)}\right] &= \partial_\mu\partial_\nu\mathcal{T}^{\mu\mu_2\nu\nu_2}\ \partial_{\mu_2}\partial_{\nu_2}\pi + \mathcal{T}^{\mu\mu_2\nu\nu_2}\ \partial_\mu\partial_\nu\partial_{\mu_2}\partial_{\nu_2}\pi \\ &\quad + \partial_\mu\partial_\nu\mathcal{T}^{\mu_1\mu\nu_1\nu}\ \partial_{\mu_1}\partial_{\nu_1}\pi + \mathcal{T}^{\mu_1\mu\nu_1\nu}\ \partial_\mu\partial_\nu\partial_{\mu_1}\partial_{\nu_1}\pi.\end{aligned}$$

This will then lead, as expected, to a higher order equation unless we impose some (anti-)symmetry condition on $\mathcal{T}$. For example, since partial derivatives always commute with each other, the higher order terms will vanish if we assume $\mathcal{T}^{\mu\nu\rho\sigma} = -\mathcal{T}^{\nu\mu\rho\sigma}$ and $\mathcal{T}^{\mu\nu\rho\sigma} = -\mathcal{T}^{\mu\nu\sigma\rho}$. Further conditions might be necessary depending on the requirements. For example, for the Galileon model, one wants that, more than being a second-order equation, the field equation depends only on $\partial_\mu\partial_\nu\pi$. This might require to impose different conditions on $\mathcal{T}$. In any case, the key conditions to maintain second order field equations will have to be put on the possible symmetries of the tensor $\mathcal{T}$.

To complete the real general classification, one needs to consider more general polynomial expressions (with a generic number of $\partial_\mu\partial_\nu\pi$ and where the equivalent of the tensor $\mathcal{T}$ can depend on both π and $\partial_\mu\pi$). We will not go into those details here and redirect the reader to [Deffayet and Steer, 2013] or [Nicolis et al., 2009] for more details.

Appendix

B Nutty Black Holes in Galileon Scalar-Tensor Gravity

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Nutty black holes in galileon scalar-tensor gravity

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Abstract

Einstein gravity supplemented by a scalar field non-minimally coupled to a Gauss-Bonnet term provides an example of model of scalar-tensor gravity where hairy black holes do exist. We consider the classical equations within a metric endowed with a NUT-charge and obtain a two-parameter family of nutty-hairy black holes. The pattern of these solutions in the exterior and the interior of their horizon is studied in some details. The influence of both – the hairs and the NUT-charge – on the lightlike and timelike geodesics is emphasized.

1 Introduction

Evading the "No-Hair-Theorem" for black holes in General Relativity – and its numerous extended versions – has constituted a challenge for a long time. One issue consists in supplementing gravity by an appropriate matter sector like the Skyrme Lagrangian [1]. Recently several kinds of hairy black holes have been constructed with a simpler matter sector : scalar fields.

Both cosmological and astrophysical observations suggest the presence of scalar fields in the models attempting to describe the Universe in its early (inflaton, dilaton) or actual stage (dark matter). These scalar fields could be fundamental (although not yet directly observed) or effective, modelling the effects of more involved – but still unknown – phenomena on space-time and standard particles. These considerations, namely, motivate the extension of standard formulation of General Relativity (also called tensor gravity) to the most general scalar-tensor gravity theory.

The first general construction in this direction was achieved by G. Horndeski in [2] where the condition of second order equations is imposed throughout. Recently, new families of scalar-tensor theories, the so-called Galileon [3] and generalized Galileon [4], have been proposed with different motivations and contexts (for a review see e.g. [5]). In particular, these theories require a symmetry of the Lagrangian under the shift $\phi \to \phi + C$ where ϕ denotes the scalar field and C a constant. In four dimensions [6], the generalized Galileon theory has been shown to be equivalent to the Horndeski theory. The generalized Galileon theory is quite general, involving the different geometric invariants and depending on several arbitrary functions of the standard kinetic terms $\partial_\mu\phi\partial^\mu\phi$. A no-hair theorem for black holes in generic forms of the Galileon theory, assuming static spherically symmetric space-time and scalar field, was established [7]. However, as shown in [9, 10] a few specific choices of the Galileon Lagrangian allow for hairy black holes to exist.

The hairy black holes constructed in the framework of Galileon gravity have a real and massless scalar field and can be static. Independently of the shift symmetry of the scalar field hypothesis, another class of models that retained a lot of attention is the Einstein-Gauss-Bonnet-Dilaton theory where hairy black holes can be constructed as well [11, 12]. Hairy black holes have been constructed within Einstein gravity coupled to a complex scalar field in [13] (see also [14] for a review). In this case, the scalar field needs to have a mass and the black hole exists only when it rotates quickly enough.

In the long history of classical solutions of General Relativity, the so-called "NUT solution" [15] is certainly one of the most intriguing. In the absence of matter fields, the NUT solution is a generalization of the Schwarzschild black hole characterized by a new parameter : the so-called NUT charge n. Although purely analytic, the NUT space-time presents peculiarities [16, 17] that makes that its physical interpretation is, till now, a matter of debate. In particular, the solution presents a Misner string singularity on the polar axis and the corresponding space-time contains closed timelike curves. Various arguments rehabilitating space-time with a NUT charge are proposed in [18]. In spite of the difficulty of finding a global definition of the NUT space-time, the solution possesses many remarquable properties, namely : (i) like the Kerr solution, it is stationary but non-static due to non-vanishing $g_{t\varphi}$ metric terms ; (ii) it can be extended analytically (i.e. without curvature singularity) in the interior region by means of a TAUB solution.

Likely for these reasons, several authors (see namely [19, 20]) have considered the NUT parameter as a possible ingredient of some astrophysical object and have studied its effect on geodesics in NUT space-times. Another application of the NUT parameter was proposed recently in [21] to obtain families of non-trivial, spherically symmetric solutions of the Einstein-Chern-Simons gravity coupled to a scalar field. Such a construction was possible by taking advantage of the stationary character of the underlying metric.

In this paper we extend the construction of the hairy black holes of [10] by including a NUT parameter in the metric. We show that Nutty-hairy-black holes exist in a specific domain of the NUT charge and Gauss-Bonnet parameter. A special emphasis is set on the way the NUT charge affects the solution in the interior of the black hole. Also, we study the influence on the light-like geodesic of both the presence of the scalar field and of the NUT charge. It is found in particular that, mimicking a rotation, the NUT charge leads to a non-planar drift of the trajectories.

The paper is organized as follows. In Sect. 2 we present the model, the ansatz for the metric, the boundary conditions of the ensuing classical equations and sketch the form of a perturbative solution. The non-perturbative solutions, obtained with a numerical method, are reported in Sect. 3. The influence of the Gauss-Bonnet gravity term and of the NUT parameter on the light-like geodesics are emphasized in Sect. 4 and illustrated by some figures. Conclusion and perspectives are given in Sect. 5.

2 The model

2.1 The Gauss-Bonnet modified gravity

The modified theory that we want to emphasize was first studied in [10] as a particular case of the general scalar-tensor-galileon-gravity. It can be defined in terms of its action:

$$S \coloneqq S_{\mathrm{EH}} + S_{\mathrm{GB}} + S_{\phi}, \tag{2.1}$$

where the Einstein-Hilbert term is given by

$$S_{\mathrm{EH}} = \kappa \int_{\mathcal{V}} \mathrm{d}^4x \sqrt{-g} R, \tag{2.2}$$

the Gauss-Bonnet term is given by

$$S_{\mathrm{GB}} = \frac{\gamma}{2} \int_{\mathcal{V}} \mathrm{d}^4x \sqrt{-g} \phi \mathcal{G} \quad , \quad \mathcal{G} = (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2), \tag{2.3}$$

the scalar field term is given by

$$S_{\phi} = -\beta \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}^4x \sqrt{-g} \left[g^{\rho\sigma} \left(\nabla_{\rho}\phi\right) \left(\nabla_{\sigma}\phi\right) \right]. \tag{2.4}$$

In these equations, $\kappa^{-1} = 16\pi G$, γ and β are dimensional coupling constants, g is the determinant of the metric, ∇_{μ} is the covariant derivative associated with $g_{\mu\nu}$, R is the Ricci scalar and the volume integrals are on the manifold $\mathcal{V}$.

The scalar field ϕ is in principle a function of space-time. In the case ϕ = const., the Gauss-Bonnet modified gravity would reduce identically to GR since, as well known (see e.g. Appendix B in [10]), the Gauss-Bonnet action density (2.3) can be expressed as a divergence.

The equations of motion for this model read

$$G_{\mu\nu} + \frac{\gamma}{4\kappa} K_{\mu\nu} = \frac{\beta}{2\kappa} T_{\mu\nu}, \tag{2.5}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, the tensor

$$K_{\mu\nu} = (g_{\mu\lambda} g_{\nu\delta} + g_{\mu\delta} g_{\nu\lambda}) \nabla_\rho (\partial_\sigma \phi \epsilon^{\sigma\delta\xi\chi} \epsilon^{\lambda\rho\omega\eta} R_{\omega\eta\xi\chi}), \tag{2.6}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor, results from the variation of the Gauss-Bonnet term and $T_{\mu\nu}$ is the energy-momentum tensor of the scalar field :

$$T_{\mu\nu} = \left[(\nabla_\mu \phi) (\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\nabla_\rho \phi) (\nabla^\rho \phi) \right]. \tag{2.7}$$

The vanishing of the variation of the action also leads to an extra equation of motion for the Gauss-Bonnet coupling field, namely

$$\beta \, \Box \phi = -\frac{\gamma}{2} \mathcal{G}, \tag{2.8}$$

where $\Box = \nabla^\mu \nabla_\mu$, which we recognize as the Klein-Gordon equation in the presence of a sourcing term.

2.2 The metric

We consider NUT-charged space-times [15, 16] whose metric can be written locally in the form

$$ds^2 = \frac{dr^2}{N(r)} + P^2(r)(d\theta^2 + \sin^2\theta d\varphi^2) - N(r)A^2(r)(dt + 4n \sin^2(\frac{\theta}{2}) d\varphi)^2, \tag{2.9}$$

the NUT parameter n being defined as usual in terms of the coefficient appearing in the differential $dt + 4n \sin^2(\theta/2) d\varphi$. Here θ and φ are the standard angles parametrizing an S^2 sphere with ranges $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. Apart from the Killing vector $K_0 = \partial_t$, this line element possesses three more Killing vectors characterizing the NUT symmetries :

$$\begin{aligned} K_1 &= \sin\varphi \partial_\theta + \cos\varphi \cot\theta \partial_\varphi + 2n \cos\varphi \tan\frac{\theta}{2} \partial_t, \\ K_2 &= \cos\varphi \partial_\theta - \sin\varphi \cot\theta \partial_\varphi - 2n \sin\varphi \tan\frac{\theta}{2} \partial_t, \\ K_3 &= \partial_\varphi - 2n \partial_t. \end{aligned} \tag{2.10}$$

Unexpectedly, these Killing vectors form a subgroup with the same structure constants that are obeyed by spherically symmetric solutions $[K_i, K_j] = \varepsilon_{ijk} K_k$.

The $n \sin^2(\theta/2)$ term in the metric means that a small loop around the $z-$axis does not shrink to zero at $\theta = \pi$. This singularity can be regarded as the analogue of a Dirac string in electrodynamics and is not related to the usual degeneracies of spherical coordinates on the two-sphere. This problem was first encountered in the vacuum NUT metric. One way to deal with this singularity has been proposed by Misner [17]. His argument holds also independently of the precise functional form of N and A. In this construction, one considers one coordinate patch in which the string runs off to infinity along the north axis. A new coordinate system can then be found with the string running off to infinity along the south axis with $t' = t + 4n\varphi$, the string becoming an artifact resulting from a poor choice of coordinates. It is clear that the t coordinate is also periodic with period $8\pi n$ and essentially becomes an Euler angle coordinate on S^3. Thus an observer with $(r, \theta, \varphi) = const.$ follows a closed timelike curve. These lines cannot be removed by going to a covering space and there is no reasonable spacelike surface. One finds also that surfaces of constant radius have the topology of a three-sphere, in which there is a Hopf fibration of the S^1 of time over the spatial S^2 [17].

Therefore for n different from zero, the metric structure (2.9) generically shares the same troubles exhibited by the vacuum Taub-NUT gravitational field [8], and the solutions cannot be interpreted properly as black holes.

The vacuum Taub-NUT one corresponds to

$$A(r) = 1 \quad , \quad \phi(r) = 0 \quad , \quad P(r)^2 = n^2 + r^2 \quad , \quad N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2}. \tag{2.11}$$

where $M = (r_h^2 - n^2)/(2r_h)$. This solution presents an horizon at $r = r_h$. In the following section, we will study how these closed form solutions get deformed by the Gauss-Bonnet term.

2.3 Gauge fixing and boundary conditions

Up to our knowledge, the system above does not admit closed form solutions for $\gamma > 0$. The solutions can then be constructed either perturbatively (for instance using γ as a perturbative parameter) either non perturbatively by solving the underlying boundary-value-differential equations numerically.

For the numerical integration, the "gauge" freedom associated with the redefinition of the radial coordinate r has to be fixed. We found it convenient to fix this freedom by setting $P(x)^2 = x^2 + n^2$ and to note x the radial coordinate defined this way. The ansatz is then completed by assuming the scalar field of the form $\phi(x^\mu) = \phi(x)$.

The Einstein-Gauss-Bonnet-Klein-Gordon equations can then be transformed into a system of three coupled differential equations for the functions $N(x)$, $A(x)$ and $\phi(x)$. The equations for the metric functions are of the first order while the Klein-Gordon equation is, as usual, of the second order.

Our goal is to construct regular solutions presenting (like black holes) a horizon at $x = x_h$, the regularity of the solution at the horizon needs the following condition to be imposed :

$$N(x_h) = 0 \quad , \quad \left[\gamma(\phi')^2 + x\phi' + 3\gamma\frac{x^2 - n^2A^2}{(x^2 + n^2)^2}\right]_{x=x_h} = 0 \quad , \quad \phi(x_h) = 0 \ , \tag{2.12}$$

(the last relation is imposed by using the invariance of the theory under the translations of the scalar field); these three relations are completed by a fourth condition at infinity, namely $A(\infty) = 1$.

Also, the equations are invariant under the following rescaling by the parameter λ :

$$x \to \lambda x \quad , \quad n \to \lambda n \quad , \quad P \to \lambda^2 P \quad , \quad \gamma \to \lambda^2 \gamma. \tag{2.13}$$

This symmetry can be exploited to fix one of the three parameters x_h, n, γ to a particular value, reducing by one unit the number of parameters to vary. We will use it by setting $x_h = 1$ throughout the rest of the paper.

In the asymptotic region, the fields obey the following form :

$$\begin{aligned} N(x) &= 1 - \frac{2M}{x} + \frac{Q^2 - 4n^2}{2x^2} + \frac{M(Q^2 + 4n^2)}{2x^3} + o(1/x^4) \ , \\ A(x) &= 1 - \frac{Q^2}{4x^2} - \frac{2MQ^2}{3x^3} + o(1/x^4) \ , \\ \phi(x) &= \phi_\infty + \frac{Q}{x} + \frac{QM}{x^2} + \frac{Q(4n^2 + 16M^2 - Q^2)}{12x^3} + o(1/x^4) \ . \end{aligned} \tag{2.14}$$

The perturbative expansion depends on the two "charges" M and Q which are determined numerically.

2.4 Invariants

For later use, we mention the Ricci and Kretchmann invariants for the metric (2.9)

$$R = -\frac{P^3\left(2NPA'' + A'\left(3PN' + 4NP'\right)\right) - 2n^2A^3N + AP^2\left(P^2N'' + 4P\left(N'P' + NP''\right) + 2NP'^2 - 2\right)}{AP^4}, \tag{2.15}$$

where the prime denotes the derivative with respect to x. The expression for the Kretschmann invariant is much longer and we do not write it.

In the case of the vacuum NUT solution (2.11) the Ricci scalar is identically zero while for the Kretschmann invariant we find

$$K = \frac{K_{num}}{x_h^2(n^2+x^2)^6} \ , \tag{2.16}$$

with

$$K_{num} = 12n^6(n^4-6n^2x_h^2+x_h^4)(x^6-15n^2x^4+15n^5x^2-n^6)+96x_hn^4(x_h^2-n^2)(3x^5-10n^2x^3+3n^4x). \tag{2.17}$$

2.5 Perturbative expansion.

The Einstein-Gauss-Bonnet equations can be attempted to be solved perturbatively in powers of the Gauss-Bonnet coupling constant γ. The perturbation has the same form as in the $n=0$ case [10] :

$$N(x) = 1 - \frac{2(Mx+n^2)}{x^2+n^2} + \gamma^2(\Delta N)_1 + o(\gamma^4) \ \ , \ \ A(x) = 1+\gamma^2(\Delta A)_1 + o(\gamma^4) \ \ , \ \ \phi(x) = \gamma\phi_1(x) + o(\gamma^3) \ . \tag{2.18}$$

Already at the first order, the form of the scalar field is rather involved although easy to construct :

$$\phi_1' = \frac{P(x,n)}{(n^2+x^2)^4(n^2+x)} \ , \tag{2.19}$$

with

$$P(x,n) = -x^7-x^6-x^5(1+4n^2)-x^4n^2(10-n^2)+x^3n^2(8-23n^2)+x^2n^4(25-8n^2)-xn^4(3-16n^2)+n^6(3n^2-2). \tag{2.20}$$

For simplicity we have written the derivative ϕ' which is the function entering effectively in the Lagrangian. The integration constant was fixed in such a way that the field is regular at $x=x_h$. The above expression is regular at $x=0$ for $n>0$ and coincides with [10] in the limit $n \to 0$. Confirming the non-pertubative results discussed in the next section, the Taub-Nut parameter n regularizes the scalar field at the origin. The form of ΔN and ΔA is much more involved and will not be reported here. However, for a crosscheck of the numerical results, we mention that these functions are singular in the limit $x \to 0$. For instance, we find $\Delta A = \Gamma/x^2 + o(1/x)$ where Γ is a function of n.

3 Numerical Results

The system of differential equations above cannot be solved explicitly for generic value of the three external parameters x_h, γ, n. We therefore used a numerical routine to construct the solutions. The subroutine COLSYS [22] based on collocation method with a self-adapting mesh has been used for the computation.

The integration proceeds in two steps. First, we solve the equations for $x \in [x_h, \infty]$ with a particular value for the three parameters x_h, γ, n. This provides, in particular, the values of the fields and their derivatives at $x = x_h$ with the accuracy demanded. Then a second integration is performed to determine the form of the solutions in the interior region by using the data at $x = x_h$ as an initial value.

As already stated, the symmetry (2.13) will be exploited to set $x_h = 1$. This choice of the scale allows for the two known limits $\gamma \to 0$ and $n \to 0$ to exist continuously.

3.1 Case $n = 0$

The first problem is to determine how the vacuum solution (2.11) is affected by the scalar field through the non-minimal coupling to the Gauss-Bonnet term. This was the object of [10] but we briefly summarize this result for completeness. Setting $\gamma \neq 0$, the non-homogeneous part in the equation for the scalar field ϕ enforces this function to be non-trivial. Remark that, since the initial Lagrangian depends only on the product $\gamma\phi$, only the case $\gamma > 0$ needs to be emphasized.

The corresponding "Hairy-solution" is characterized, namely, by the values $\phi'(x_h)$, $A(x_h)$, $N'(x_h)$, ϕ_∞ as well as by the mass M and the charge Q. These parameters are determined numerically. M and Q read off from the asymptotic decay of the fields (2.14). Some of these parameters are reported as functions of γ on the left side of Fig. 1.

In this case the regularity of the solutions on the horizon requires

$$\phi'(x_h) = \frac{-x_h^2 \pm \sqrt{x_h^4 - 12\gamma^2}}{2x_h\gamma} \quad , \tag{3.21}$$

implying in particular that real solutions will exist only for $\gamma \leq \gamma_{\max} = 1/\sqrt{12}$. The branch of solutions connecting to the vacuum in the limit $\gamma \to 0$ corresponds to the $+1$ sign. A branch of solutions corresponding to the -1 sign exists as well and is represented partly on the figure. It is very likely that this branch can be continued for smaller values of γ but the numerical computation of the second branch appeared to be tricky, alterating the numerical accuracy of the results. On the figure we limited the data of this branch to the values with a reliable accuracy.

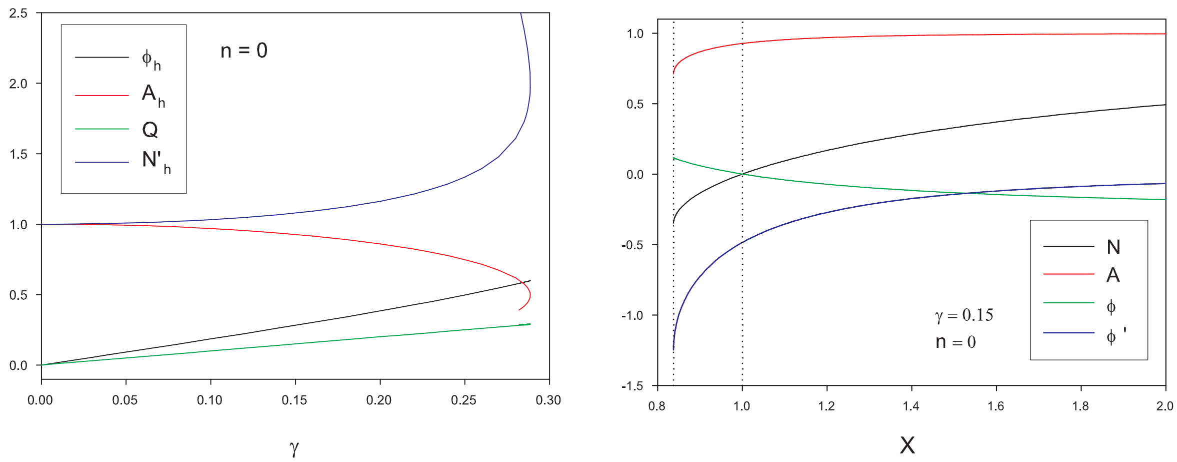


Figure 1: *Left :* The dependence of some parameters on the Gauss-Bonnet constant γ for $n = 0.0$ and $x_h = 1.0$. *Right :* The profile of the solution with $\gamma = 0.15$ and $n = 0$.

The profile of the solutions for $n = 0$ and $\gamma = 0.15$ is presented on the right side of Fig. 1. As pointed out in [10], the integration of the equations in the interior region shows that the metric and scalar functions are limited in a region $x_c < x < \infty$. The value x_c corresponds to a critical radius where the derivatives of the fields and – by a consequence – the metric invariants R and K diverges. In the case of Fig. 1, we find $x_c \approx 0.8375$. The dependence of x_c on γ was presented in [10] but it is reproduced by means of the black line on Fig. 2 for the sake of comparison with the case $n > 0$.

3.2 Case $n > 0$

We now discuss the solutions obtained for $n > 0$.

Exterior : In the exterior region, the family of solutions corresponding to a fixed n and varying γ present qualitatively the same features as in the case $n = 0$. One quantitative difference with respect to $n = 0$ solutions is related to the fact that the condition of regularity (2.12) now depends on the value $A(x_h)$. Since this is determined numerically, the value $\gamma_{\max}$ cannot be determined analytically for $n > 0$. The value $\gamma_{\max}$ increases slightly with n; we have $\gamma_{\max} = 1/\sqrt{12} \approx 0.2886$ for $n = 0$ and we find $\gamma_{\max} \approx 0.3$ and $\gamma_{\max} \approx 0.36$ respectively for $n = 0.25$ and $n = 0.50$, as sketched on Fig. 2. The value $\gamma_{\max}$ corresponding to the value of γ for which the critical radius x_c becomes equal to the horizon radius x_h (see discussion below). The solution then stop existing before exhibiting a naked singularity.

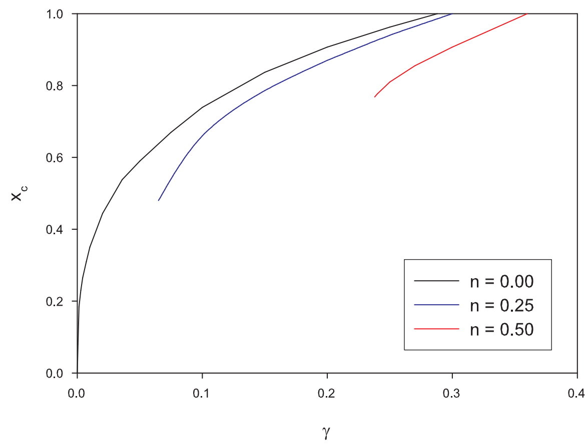


Figure 2: Dependence of the critical radius x_c on γ for several values of n.

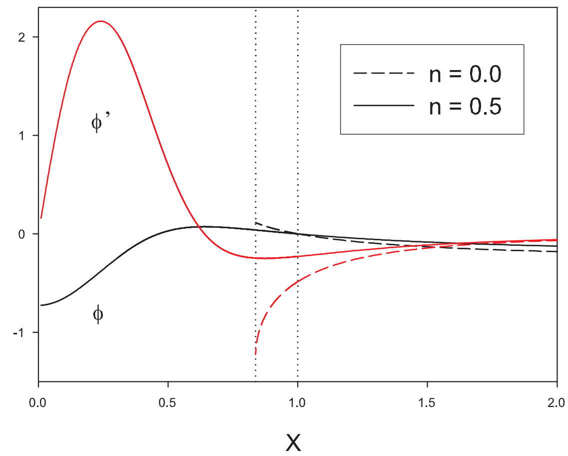
Note : the reason the lines do not reach $\gamma = 0$ for $n > 0$ will be discussed below.

Interior : Because the vacuum (2.11) NUT solution is everywhere regular in the interior region $r < r_h$, the question of the structure of the Nutty-hairy black holes for $r < r_h$ raises naturally. The pattern is not so simple due to a double critical phenomenon which we now explain.

Fixing $n > 0$ we know that the vacuum solution (2.11) is regular for $0 \leq x < \infty$. Increasing the coupling constant gradually γ, with $n > 0$ fixed, the numerical results strongly suggest that the hairy-nutty-black hole is regular for $0 < x < \infty$. In the limit $x \to 0$, our results indicate in particular $N(x) \to -\infty$ while the scalar field remains finite in agreement with the perturbative expression (2.18). The corresponding Ricci scalar (and the other invariants) also diverge to infinity, confirming the occurrence of a singularity at the origin. This is illustrated on Fig. 3 where the profiles of the solutions corresponding to $\gamma = 0.15$ are superposed for $n = 0$ (dashed lines) and $n = 0.5$ (solid lines).

We now discuss the second singularity: again with n fixed, an increase of γ reveals another peculiarity of the nutty black holes in the interior region. The results indicate that, apart from the singularity at the origin, a second singularity appears at an intermediate radius, say at $x = x_c$ (with $0 < x_c < x_h$) when the Gauss-Bonnet coupling constant approaches a critical value, $\gamma \to \gamma_c$. For example, for $n = 0.25$ and $n = 0.5$, we find respectively $\gamma_c \approx 0.065$ and $\gamma_c \approx 0.238$; this explains why the lines on Fig. 2 stop suddenly without reaching $\gamma = 0$.

This phenomenon is not so easy to detect from the numerical solutions and is manifest only through a careful examination of the derivatives of the metric fields A and N. As an illustration we plot on Fig. 4. the function A'' for $n = 0.5$ and for three values of γ approaching γ_c. In this case we find $\gamma_c \approx 0.238$, with the corresponding critical radius $x_c\,(\gamma_c) \approx 0.768$, and $\gamma_{\max} \approx 0.36$. One can appreciate on this plot that, while γ approach γ_c, a "bump" appears in the profile of A''. This bump grows quickly – and possibly tend to infinity, although it is hard to verify numerically – in the limit $\gamma \to \gamma_c$ and is roughly centred around $x_c(\gamma_c)$. For $\gamma > \gamma_c$, the pattern of [10] is recovered : the solutions can be continued only for $0 < x_c < \infty$. As illustrated on Fig. 2, x_c increases with γ and the solution stops to exist in the limit $\gamma \to \gamma_{\max}$ for which $x_c\,(\gamma \to \gamma_{\max}) \to x_h$. Then, as we discussed above, the solution stops before reaching a naked singularity.



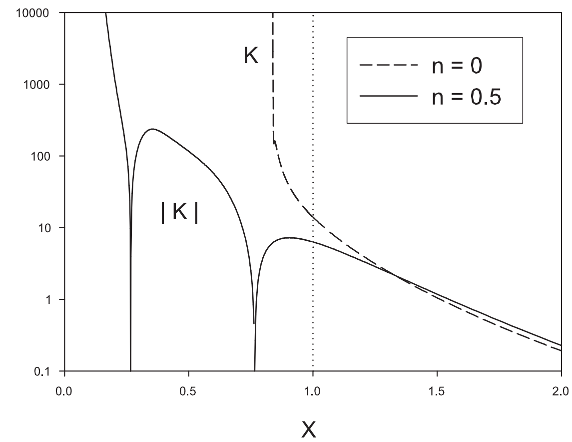


Figure 3: *Left*: The profiles of the scalar field ϕ and its derivative ϕ' for $\gamma = 0.15$ and two values of n. *Right:* The corresponding Kretschmann scalar K. Note the spikes of the solid line correspond to zeros of K because of the logarithmic scale on the y axis.

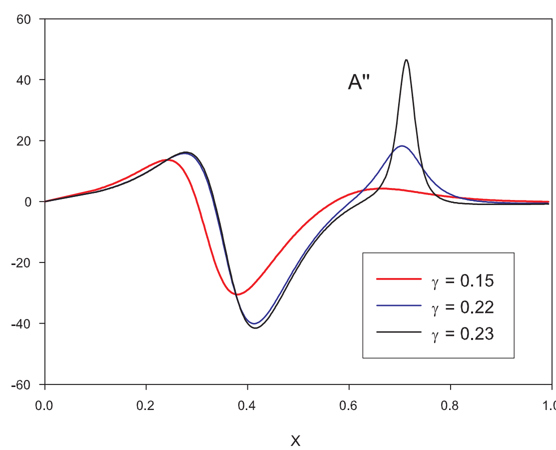


Figure 4: Profile of the second derivative A'' for $n = 0.5$ and three values of γ.

4 Geodesics around hairy-nutty black holes

In this section we study the geodesics in a nutty-black hole space-time with a special emphasis on the effects of the NUT charge and of the Gauss-Bonnet coupling term. We first describe the generic properties of the curves and then study the possible motions in the equatorial plane.

4.1 Equations and constants

We parametrize the geodesics by means of the functions

$$X(\lambda) = (T(\lambda), R(\lambda), \Theta(\lambda), \Phi(\lambda)) \ ,$$

where λ is an affine parameter. The equations of geodesics in the space-time of a black hole such as constructed in the previous section can be set as follows :

$$\begin{aligned}
\ddot{T}(\lambda) = & \frac{-2\dot{A}\dot{R}(\dot{T} - 2n(\cos(\Theta) - 1)\dot{\Phi})}{A} + \frac{\dot{N}\dot{R}(2n(\cos(\Theta) - 1)\dot{\Phi} - \dot{T})}{N} \\
& + \frac{n}{\Sigma}\left(-4\sin^2(\frac{\Theta}{2})\left\{-2R\dot{R} + \Sigma\tan(\frac{\Theta}{2})\dot{\Theta}\right\}\dot{\Phi}\right. \\
& \left. -4nA^2N\dot{\Theta}\left\{\tan(\frac{\Theta}{2})\dot{T} + 8n\csc(\Theta)\sin^4(\frac{\Theta}{2})\dot{\Phi}\right\}\right) \ , \\
\ddot{R}(\lambda) = & -A\dot{A}N^2\Delta + NR(\dot{\Theta}^2 + \sin^2(\Theta)\dot{\Phi}^2) - \frac{\dot{N}(A^2N^2\Delta^2 - \dot{R}^2)}{2N} \ , \\
\Sigma\ddot{\Theta}(\lambda) = & -2R\dot{R}\dot{\Theta} - 2nA^2N\sin(\Theta)\dot{T}\dot{\Phi} + \dot{\Phi}^2\left[\Sigma\sin(\Theta)\cos(\Theta) + n^2A^2N(2\sin(2\Theta) - 4\sin(\Theta))\right] \ , \\
\Sigma\ddot{\Phi}(\lambda) = & -2R\dot{R}\dot{\Phi} + \dot{\Theta}\left[2\cot(\Theta)\Sigma\dot{\Phi} + nA^2N\left\{2\csc(\Theta)\dot{T} - (4n(\cot\Theta - \csc\Theta)\dot{\Phi})\right\}\right] \ ,
\end{aligned}$$

where $\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}\lambda}$ and

$$\Delta \equiv \left(\dot{T} - 2n\dot{\Phi}\left(\cos\Theta - 1\right)\right) \quad , \quad \Sigma \equiv n^2 + R^2 \ .$$

The metric functions $N(R(\lambda)), A(R(\lambda)), \cdots$ are obtained numerically (see the previous sections).

The above equations are solved with the initial conditions:

$$X(\lambda = 0) = (0, r_0, \theta_0, \varphi_0) \, , \, \dot{X}(\lambda = 0) = (k_t, k_r, k_\theta, k_\varphi), \tag{4.22}$$

where k_t is related to the energy of the particle along the geodesic, k_r is essentially its radial velocity, and k_θ, k_φ are angular velocities.

The quadrivector $\dot{X}$ is subject to the the condition $\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu = -\epsilon$ with $\epsilon = 0, 1, -1$ respectively for lightlike, time-like and space-like geodesics. Note that $\dot{X}$ is nothing else than the wave vector when the affine parameter is the proper time along the geodesic. The constraint fixes the energy of the particle once the 3-momentum is fixed.

The case $\epsilon = 0$ describes the propagation of light rays and the case $\epsilon = 1$ corresponds to massive particles. The case $\epsilon = -1$, corresponding to tachyonic motions, is unphysical and would not be considered in the following.

Using the fact that ∂_t and ∂_φ are Killing vectors, one can find two constants of motion along the geodesics. Denoting $\dot{X} = \left(\dot{T}, \dot{R}, \dot{\Theta}, \dot{\Phi}\right)$ the 4-velocity along a geodesic, the constants obtained with ∂_t and ∂_φ respectively read :

$$E = (\partial_t)_\mu \dot{X}^\mu = g_{0\mu}\dot{X}^\mu = -A^2(R)N(R)\left(\dot{T} + 4n\sin^2\left(\frac{\Theta}{2}\right)\dot{\Phi}\right) \ , \tag{4.23}$$

$$\begin{aligned}
L &= -\left(\partial_\varphi\right)_\mu \dot{X}^\mu = -g_{3\mu}\dot{X}^\mu \\
&= 4n\sin^2\left(\frac{\Theta}{2}\right)A^2(R)N(R)\dot{T} + \left(16n^2A^2(R)\sin^4\left(\frac{\Theta}{2}\right)N(R) - \sin^2(\Theta)\left(n^2 + R^2\right)\right)\dot{\Phi}.
\end{aligned} \tag{4.24}$$

The constant E might be interpreted as the energy of the particle moving along the geodesic and L as an analogue to the third component of the angular momentum of the particle. The above relations can then be used to express the quantities $\dot{T}$ and $\dot{\Phi}$ in terms of the functions R, Θ and the constants E, L :

$$\dot{T} = \frac{n\left(\sec^2\left(\frac{\Theta}{2}\right)(L+4nE) - 4nE\right)}{n^2+R^2} - \frac{E}{A^2(R)N(R)} \ , \tag{4.25}$$

$$\dot{\Phi} = -\frac{\csc^2(\Theta)(L+2nE-2nE\cos(\Theta))}{n^2+R^2} \quad . \tag{4.26}$$

As a consequence, the system of four equations above reduces to the two equations corresponding to the functions R and Θ. These equations are lengthy and we do not write them here.

4.2 Generic motions

We haved solved numerically the geodesic equations for several values of the parameters n, γ and of the initial conditions (4.22). Let us first present families of geodesics highlighting the effects of the Gauss-Bonnet parameter γ and of the NUT charge n. Examples of light-like geodesics with fixed wave-vector are shown in Fig. 5 for two values of n and several values of γ.

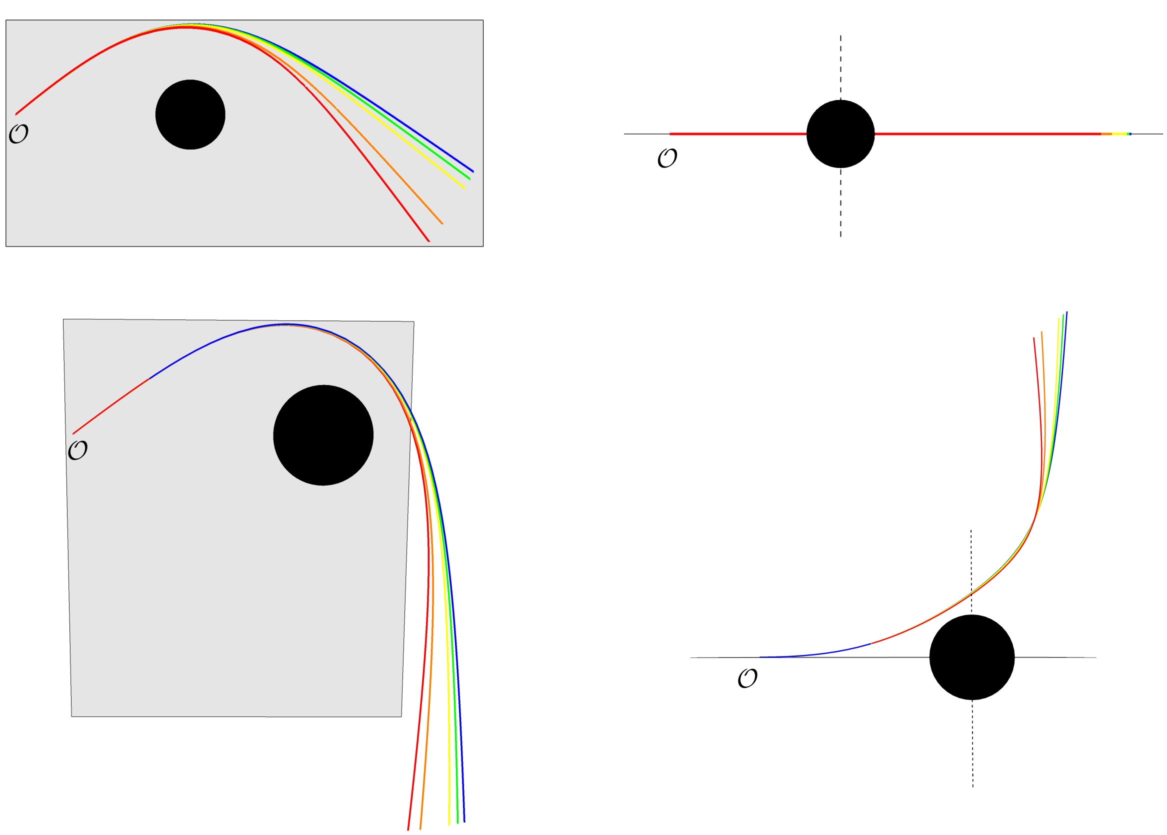


Figure 5: *Top (Left and Right)* : $n = 0$. *Bottom (Left and Right)* : $n = 1$. In the two cases, the considered values of the coupling constant are $\gamma = 0$, $\gamma = 0.1$, $\gamma = 0.15$, $\gamma = 0.25$, $\gamma = 0.28$, (respectively in blue, green, yellow, orange and red). The black sphere corresponds to the horizon $r = r_h = 1$. The observer is located at a distance $r = 5r_h$ and lies in the equatorial plane. It's position is denoted by $\mathcal{O}$. The photons are sent with identical starting values, so their different geodesics are only function of γ.

Let us first discuss the **top** part of Fig. 5 corresponding to $n = 0$. On this figure, we show the same set of geodesics seen from different points of view : the left side represents the equatorial (or XY) plane, in grey, while the right side represents the YZ plane with the OZ axis figured out by the dashed line. We see in particular that the photons evolve in a plane for all values of γ. The Gauss-Bonnet coupling constant just changes the curvature of the different lines; increasing γ the black hole becomes "more and more attractive" since the lines become more and more curved.

The **bottom** part of Fig. 5 corresponds to $n = 1.0$. Here the trajectories cease to be planar. This could be expected since, turning on the NUT parameter one also turns on the $(t\varphi)$ component of the metric. Consequently, for $n \neq 0$, we deal with a stationary non-static space-time. This case is similar to the case of a rotating black hole for which frame-dragging effects are well known.

Fig. 6 confirms that trajectories do not lie in a plane for $n \neq 0$ and that increasing the NUT parameter causes an increase of the geodesics curvature and torsion (defined as usual for curves).

The above statement is further illustrated in Fig. 7 where various trajectories are shown for two values of n and $\gamma = 0.1$. For $n \neq 0$, the frame-dragging effect is clearly seen on the right part of the figure. Let us highlight that the purpose of this figure is to reveal the global frame-dragging feature rather than quantitative details.

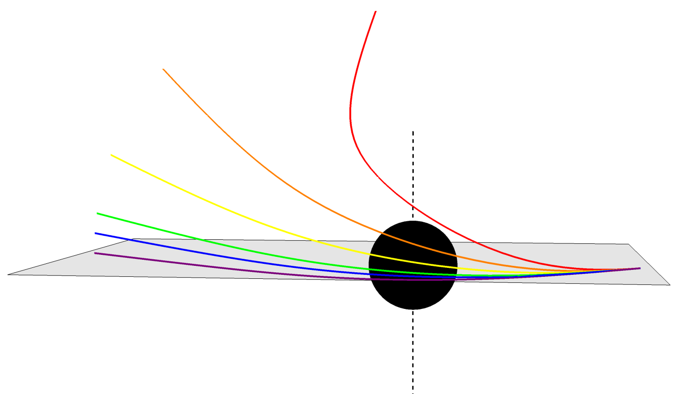


Figure 6: Geodesics for $\gamma = 0$ and for $n = 0$, $n = 0.05$, $n = 0.1$, $n = 0.25$, $n = 0.5$, $n = 1$, respectively represented in purple, blue, green, yellow, orange and red. The setup is identical as in figure 5.

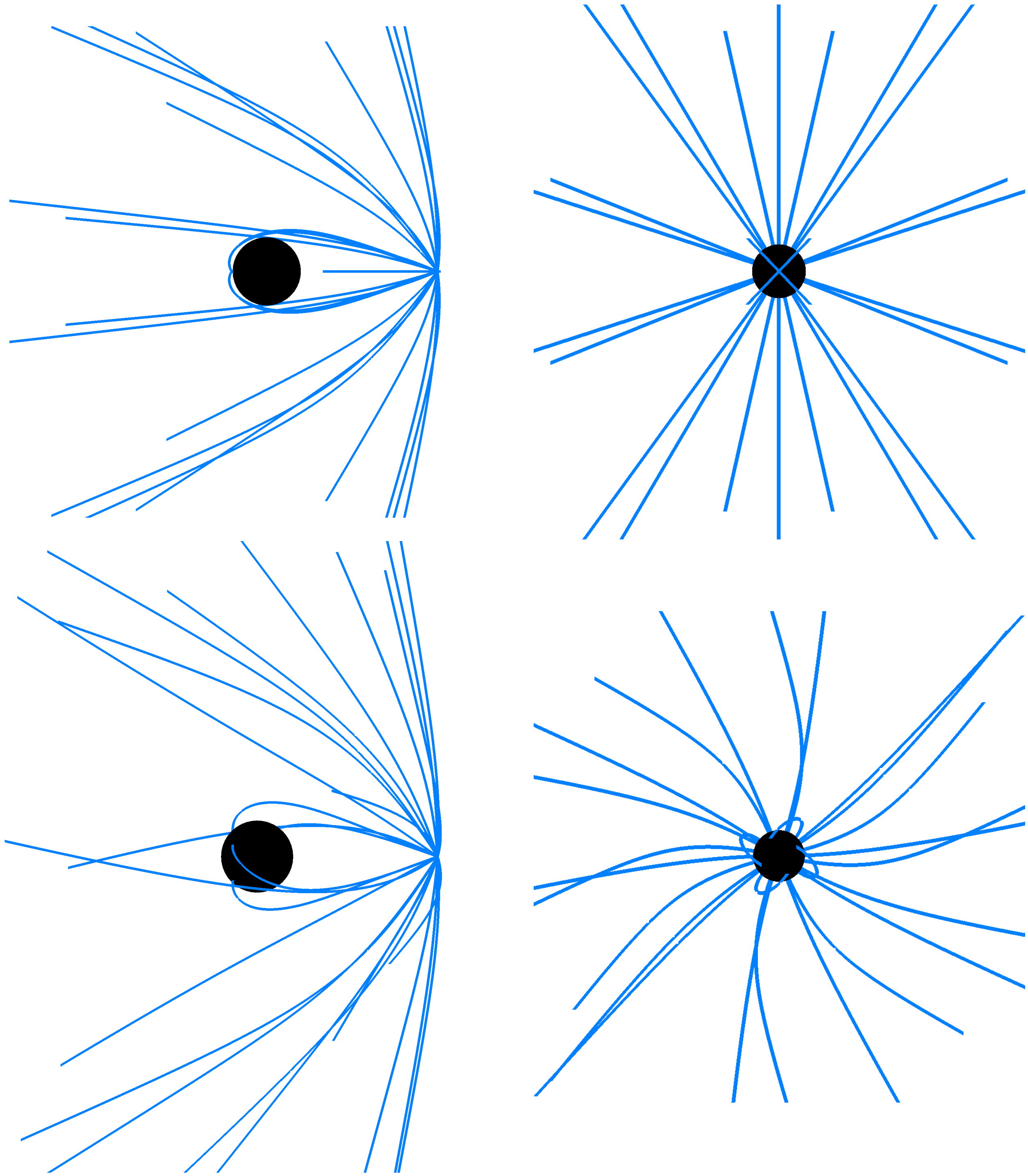


Figure 7: *Top (left and right)* : $n = 0$. *Bottom (left and right)* : $n = 1$. Light rays for various initial directions and $\gamma = 0.1$ are represented. The observer lies where the trajectories meet. On the top and bottom figures, the left and right situations are the same, but from a different point of view so the behaviour of the geodesics becomes clearer.

In order to be as exhaustive as possible, analogous results for other setups are shown in the appendix.

4.3 Motions in the equatorial plane

We now investigate the possibility of geodesic motion in the equatorial plane (i.e. with $\Theta(\lambda) = \frac{\pi}{2}, \forall\lambda$). For this purpose, we fix as initial conditions $\Theta(0) = \frac{\pi}{2}$ and $\dot{\Theta}(0) = 0$. The relevant conditions to guarantee that these initial conditions will lead to a constant value of Θ for all values of λ are then obtained through the equation fixing the $\ddot{\Theta}$ function which turns out to be :

$$\ddot{\Theta}(\lambda) = -\frac{2En(L+2nE)}{\left(n^2+R(\lambda)^2\right)^2} .$$

Then, a given geodesic would stay in the equatorial plane **iff** $\ddot{\Theta}(\lambda) = 0$ for all λ, namely **iff**

$$n = 0 \vee E = 0 \vee L_3 = 0, \tag{4.27}$$

where $L_3 \equiv L + 2nE$.

The planarity of trajectories for $n = 0$ was already pointed out ; the other two solutions, which somehow are *a priori* unexpected, are worth being examined. In order to understand what happens in these cases, we have to examine the last equation of geodesics : the equation of the R function. One can look directly at the $\ddot{R}$ equation or, equivalently, use the condition $\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu = -\epsilon$ together with $\Theta(\lambda) = \frac{\pi}{2}$ and the equations (4.25) and (4.26) to obtain :

$$\dot{R}(\lambda)^2 = U\left(R; n, E, L_3\right) \ , \tag{4.28}$$

where

$$U\left(R; n, E, L_3\right) = \frac{E^2}{A(R)^2} - N(R)\left(\frac{(L_3)^2}{n^2+R^2} + \epsilon\right). \tag{4.29}$$

The equation above can be studied as a potential-like equation : the motion is only possible in the regions where $U\left(R; n, E, L_3\right) \geq 0$ so a study of the properties of the "potential" $U\left(R; n, E, L_3\right)$ will give us all the necessary information to classify the geodesics living in the equatorial plane. Let us emphasize that Eq. (4.28) is relevant **iff** $(n = 0 \vee E = 0 \vee L_3 = 0)$.

As a consistency check, note that when $n = 0$ and $\gamma = 0$, the situation should reduce to the Schwarzschild scenario ; that is $A(r) = 1$ and $N(r) = 1 - \frac{2M}{r}$. Using those expressions for A and N, one can easily verify that (4.28) reduces to the equation for geodesics in Schwarzschild background, see for example [23]. The influence of n and γ on $U(R; n, E, L_3)$ is mostly "*hidden*" in the metric functions A and N – especially for γ since one should obtain A and N numerically for $\gamma \neq 0$. Actually, according to our numerical results, the behaviour of these functions remains more or less the same for all values of n and γ :

As pointed out in Sect. 2.3, we have constructed our solutions such that $N(r_h) = 0$, $N(r) \underset{r\to\infty}{\longrightarrow} 1$ and $A(r) \underset{r\to\infty}{\longrightarrow} 1$. For all values of n and γ, it turns out that $1 \geq A(r_h) > 0$ and that N is a strictly increasing function on the exterior space-time (i.e. for $r \geq r_h$). N would then smoothly grow from 0 to 1 as r increase from r_h to infinity (see Fig. 8 left side).

The situation for the function A is a bit different. For $\gamma = 0$, A is constant (and then $A(r) = 1$ according to the boundary conditions). For $\gamma \neq 0$, A acquires a non-trivial behaviour and becomes, as N, a strictly increasing function, growing from $A(r_h)$ to 1 as r increase from r_h to infinity. However, unlike N, A tends extremely quickly to 1 (see Fig. 8 right side). Our numeric indicates that, for all values of n and γ, one typically have $A(2r_h) > 0.9$. Then, as a good approximation, $A(r) \approx 1$ for all values of n and γ.

As a consequence, we have

$$\infty > U\left(r_h; n, E, L_3\right) = \frac{E^2}{A(r_h)^2} > 0 \ ,$$

and

$$U\left(R; n, E, L_3\right) \approx E^2 - N(R)\left(\frac{(L_3)^2}{n^2+R^2} + \epsilon\right) \equiv E^2 - V(R).$$

Then the shape of the curve is due to $V(R)$ and the energy acts like a shift constant.

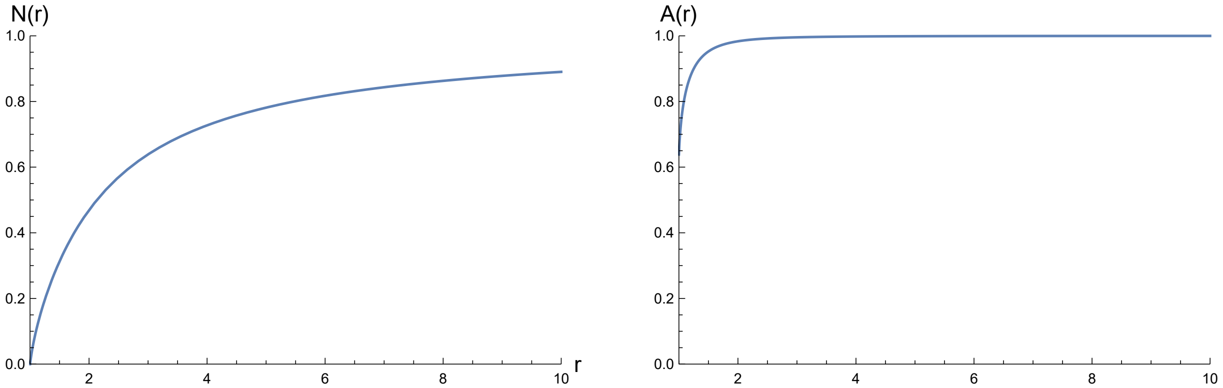


Figure 8: Profile of the metric functions N (left side) and A (right side) in the exterior region for $n = 0.1$ and $\gamma = 0.28$. Here $A(r_h) = 0.639$ and $A(2r_h) = 0.983$ ($r_h = 1$). For the sake of comparison, we used the same scale on both plots.

Since the relevance of (4.28) require condition (4.27) to be satisfied, three cases might appear :

4.3.1 Case $E = 0$:

In this case the "potential" reduces to

$$U\left(R; n, 0, L_3\right) = -N(R)\left(\frac{(L_3)^2}{n^2 + R^2} + \epsilon\right) \leq 0,$$

so there is no possible motion. Note that this case is twice disfavoured since, imposing $E = 0$, motions are only possible with $L_3 = L \neq 0$. Indeed, when $E = 0 = L$, Eq.(4.25) would lead to $\dot{T} = 0$; this would correspond to a particle which does not propagate in time.

4.3.2 Case $L_3 = 0$:

This case corresponds to purely radial motion since, together with $\Theta(\lambda) = \frac{\pi}{2}$, $L_3 = 0$ lead to $\dot{\Phi} = 0$ via (4.26). The "potential" is

$$U\left(R; n, E, 0\right) = \frac{E^2}{A(R)^2} - N(R)\epsilon \ ,$$

and its derivative with respect to R

$$U'\left(R; n, E, 0\right) \approx -N'(R)\epsilon.$$

- **Case $\epsilon = 0$:**

When $\epsilon = 0$ (i.e. for light rays), $U\left(R; n, E, 0\right)$ is always positive and almost constant. So the motion is always possible for all values of R and the light rays might fall in the black hole (if $\dot{R}(0) < 0$), or be diffused (if $\dot{R}(0) > 0$), but no bounded trajectory is possible.

- **Case $\epsilon = 1$:**

When $\epsilon = 1$ (i.e. for massive particles), one has $U\left(r_h; n, E, 0\right) > 0$ and $U'\left(R; n, E, 0\right) \approx -N'(R) < 0$ then the "potential" would be strictly decreasing. Consequently, there would exist one unique $r_\star$ such that $U\left(r_\star; n, E, 0\right) = 0$ and the function would be positive on $[r_h, r_\star]$ and negative elsewhere. Massive particles would then always be absorbed in this case.

4.3.3 Case $n = 0$:

This case is, in some sense, the most relevant one since one recovers the case of a spherically symmetric space-time for which geodesic motion always occur in a plane which (in an appropriate coordinate system) can be chosen to be the equatorial one.

The function $U(R; 0, E, L_3)$ is given by

$$U(R; 0, E, L_3) = \frac{E^2}{A(R)^2} - N(R)\left(\frac{L^2}{R^2} + \epsilon\right).$$

This case is the only one for which motions in the equatorial plane are possible with both $E \neq 0$ and $L_3 \neq 0$.

The analysis of the situations where $E = 0$ or $L_3 = 0$ was already pointed out and was valid for all n. We can then focus on the geodesics with $E \neq 0$ **and** $L_3 \neq 0$.

• **Case** $\epsilon = 0$ **:**

Fig. 9 shows the shape of $U(R; 0, E, L_3)$, which is the same for all non-vanishing values of E and L_3 when $\epsilon = 0$.

For a given value of $L_3 = L$, if E^2 is sufficiently small (smaller than $V(x_m)$, where x_m is the location of the maximum of V[1]), $U(R; 0, E, L_3)$ possesses two zeros, x_1 and x_2 with $x_1 < x_2$, is negative between those values and positive elsewhere, just as in Fig. 9. Consequently, there are two possible types of motions : if $R(\lambda = 0) < x_1$, the light ray will be absorbed by the black hole (even if $\dot{R}(0) > 0$), and if $R(\lambda = 0) > x_2$, the particle will diffuse (even if $\dot{R}(0) < 0$).

Conversely, if $E^2 \gg V(x_m)$, the curve has the same shape but is always positive. The photon will then be absorbed if $\dot{R}(0) < 0$ and diffused if $\dot{R}(0) > 0$.

Between those situations (when $E^2 \approx V(x_m)$), fine-tuning the energy, there is a limit such that the "potential" admits a unique zero equal to its minimum. In this limit, *unstable* circular motions are possible.

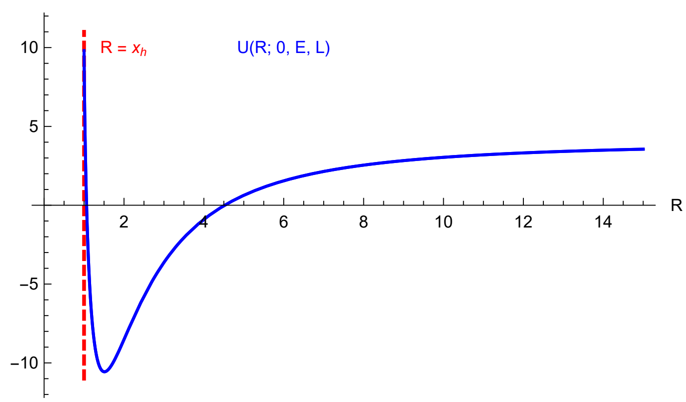


Figure 9: Shape of the "potential" in the exterior space-time for $\epsilon = 0$ and $n = 0$.

• **Case** $\epsilon = 1$ **:**

One of the distinguished properties of the massive case with respect to the massless case is the existence of stable circular orbits for large enough values of the angular momentum L. This is a well-known property in the Schwarzschild case where, in our units, stable circular orbits exist for $L^2 \geq 3$. In this case, the smallest stable circular orbit is then reached for $L = \sqrt{3}$, corresponding to $R_c = 3$.

We have checked the influence of γ on this property. As we already pointed out, the shape of the curve is due to $V(R)$ which depends on γ, via the metric function N, and on L_3. Nevertheless, since γ does not change the qualitative feature of N, its influence on the shape of $V(R)$ would be negligible.

[1] That is the minimum of $U(R; 0, E, L_3) \approx E^2 - V(R)$.

Consequently, since the existence of stable circular orbits and, more generally, of bounded trajectories require the existence of extrema of $V(R)$, existence of such kinds of motions would be controlled by L_3 as in the Schwarzschild case and the critical value of L_3, say L_c, would not vary significantly when the parameter γ increases.

Nevertheless, γ would have an influence on the position of the extrema of $V(R)$ and the value of $V(R)$ at these extrema. The effective potential in the region of the stable circular orbit is shown on Fig. 10 for $L_3 = 2$ and two values of the parameters n and γ ; we here focus on the solid lines (corresponding to $n = 0$). The variation of the potential valley due to the changes of γ can be appreciated on the picture.

For a given value of $L \geq L_c$, if we note $x_{\min}$ and $x_{\max}$ the position of the minimum and the maximum of $U(R; 0, E, L_3)$, one has $x_{\min} < x_{\max}$. Our numerical results indicate that, when γ increases, $x_{\min}$ increases and $x_{\max}$ decreases while the depth of the potential valley $|U(x_{\max}; 0, E, L_3) - U(x_{\min}; 0, E, L_3)|$ also decreases. Consequently, increasing γ, the range of energy for which bounded trajectories would exist[2] also decreases. This point is in agreement with our interpretation of section 4.2 (see top-left part of Fig. 5 and discussion in the text) : when γ is increased "the black hole becomes more attractive", since it would be able to absorb particles with significantly higher energy.

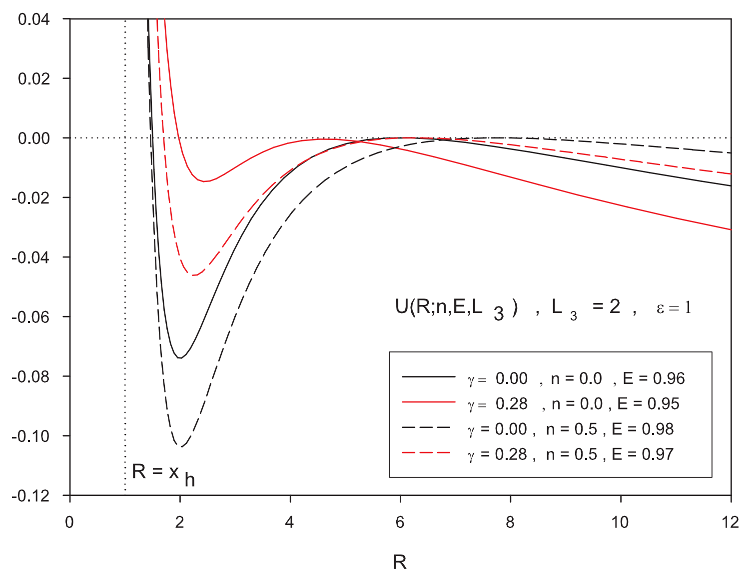


Figure 10: Shape of the "potential" in the exterior space-time for $\epsilon = 1$ and several values of γ and n. The parameter E was tuned as to coincide with the stable circular orbit. Note : Pay attention that the two dotted curves are unphysical; see discussion in the text.

4.3.4 Discussion

In this paragraph, we have studied particle motions in the equatorial plane. In conclusion to this discussion, let us here summarize and talk about the interpretation of our results.

We saw that, to guarantee a motion confined in the equatorial plane, one have to fulfil condition (4.27). Since no motion is possible with $E = 0$, this reduces to

$$n = 0 \vee L_3 = 0.$$

[2]The values of E for which $U(x_{\max}; 0, E, L_3) > 0$ and $U(x_{\min}; 0, E, L_3) < 0$.

When $n = 0$, the pattern is qualitatively the same as in the Schwarzschild case for all the allowed values of γ. Increasing γ would just quantitatively increase the black hole attraction (see discussion above). In terms of bounded trajectories, massless particles admit only unstable circular orbits, while massive ones admit stable and unstable circular orbits and bounded trajectories, assuming that they have a sufficiently high angular momentum L_3.

Our most important result concerns the case $n \neq 0$. In this case, in order to satisfy (4.27), one must necessarily have $L_3 = 0$. Then the only possible motions enclosed in the equatorial plane are the purely radial ones. Consequently, there is no possible bounded trajectory in the equatorial plane for $n \neq 0$. This further reinforces the idea that the NUT charge mimic a rotation and produces frame-dragging-like effects.

Actually, when $n \neq 0$, if one wants to obtain a "potential" ensuring the existence of stable circular orbit (see dotted curves of Fig. 10), one have to impose $L_3 \geq L_c(n)$. For example, setting $\gamma = 0$, we find that $L_c(n)$ slightly decreases while n increases (e.g. $L_c(n = 0) = \sqrt{3}$, $L_c(n = 1) \approx \sqrt{(}2))$, the radius R_c increases and diverges for $n \to 1$. These values do not vary significantly when the parameter γ is increased. But, since $L_c(n)$ remains always strictly positive, such a case is not physically possible since it would violate (4.27).

5 Conclusion

In this paper we have investigated the effects of a NUT-charge on the family of hairy black holes existing in the Einstein-Gauss-Bonnet gravity extended by a real scalar field coupled to the Gauss-Bonnet term. The underlying solutions of the equations form galileon.

In Sect. 3, we have seen that the NUT-charge n smoothly deforms the solutions of [10], characterized by the Gauss-Bonnet coupling constant γ, but affects non-trivially the singularity structure in the interior of the solution. We have put a special attention on the structure of this interior solution and shown that, while it presents two singularities (one located at $x = 0$ and another one at $x = x_c(\gamma) > 0$) for any $\gamma \neq 0$ when $n = 0$, a non-vanishing NUT-charge tend to regularize the solution for small values of γ. In particular we have seen that for $n \neq 0$, there exists a critical value of the Gauss-Bonnet coupling constant, say $\gamma_c(n)$, such that for $\gamma < \gamma_c$ the interior solution presents only one singularity at $x = 0$, while for $\gamma > \gamma_c$ a second singularity occurs at a critical radius $x_c > 0$. Our numerical results indicate that γ_c increases with n.

Existence of the critical radius x_c is essential to understand the bound in the domain of existence for the solutions in the (γ, n) plane. Our numerical results tend to proof that for a fixed γ, when it exists, x_c slightly decreases with n while, for a fixed n, it increases with γ. For a given n, this increase of x_c with γ was responsible for the existence of a maximal value $\gamma_{\max}(n)$ above which solutions cannot exist. This $\gamma_{\max}$ corresponding to the value of γ for which x_c tends to the black-hole horizon radius. The solutions would then stop existing before exhibiting a naked singularity.

Finally, in Sect. 4, we have characterized the geodesics of massless and massive test particles in the space-time of the underlying galileon, finding that, mimicking frame-dragging effects, a non-vanishing NUT-charge gives rise to non-planar geodesics. More than this, we have established that the NUT-charge avoids the existence of motion confined in the equatorial plane. The Gauss-Bonnet parameter haves a quantitative influence on the geodesics but cannot re-establish the properties known in the "minimal" Schwarzschild limit in the presence of a NUT-charge.

Appendix

This appendix provides several plots to complete the illustrations of the situation described in Sect. 4 for the geodesic motions.

Fig. 11 emphasizes the fact that the particles evolve in a plane if and only if $n = 0$, for all values of γ.

On Fig. 12, we show various trajectories for a fixed value of φ_s and for several values of θ_s, where φ_s and θ_s are respectively polar and azimuthal angle parametrizing the initial direction of the geodesics on the local celestial sphere of the observer. We can clearly see in this plot that when $n \neq 0$ the space-time is not symmetric under the transformation $\theta_s \to \pi - \theta_s$.

Completing Fig. 7, Fig. 13 illustrates the influence of the Gauss-Bonnet parameter on the curvature of light rays for fixed value of the NUT parameter. As in Fig. 7 the purpose of the picture is not to show in detail where go each geodesic. The two cases look qualitatively similar and are analogous to the lower plots in Fig. 7, confirming that the presence of the NUT charge mimic a rotation. Nevertheless, even if the two plots are qualitatively similar, looking at the right parts, one can see that the frame-dragging effect is influenced by γ since the four photons absorbed by the black hole near the centre of the picture are not absorbed at the same spot.

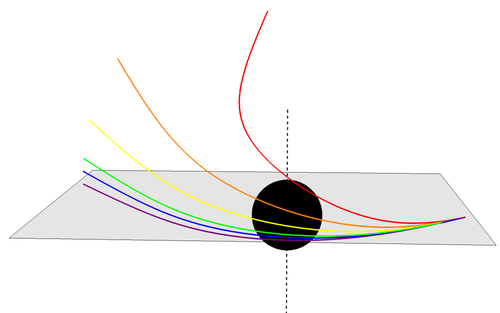


Figure 11: Geodesics for $\gamma = 0.28$ and for $n = 0$, $n = 0.05$, $n = 0.1$, $n = 0.25$, $n = 0.5$, $n = 1$, respectively represented in purple, blue, green, yellow, orange and red. The setup is identical as in Fig. 5.

Figure 12: Trajectories for $\gamma = 0.15$ and $n = 0.5$. Photons are emitted in the direction $\varphi_s = 0.5$ and for various $\theta_s \in [0, \pi]$. *Left* and *Right* present the same setup from different points of view.

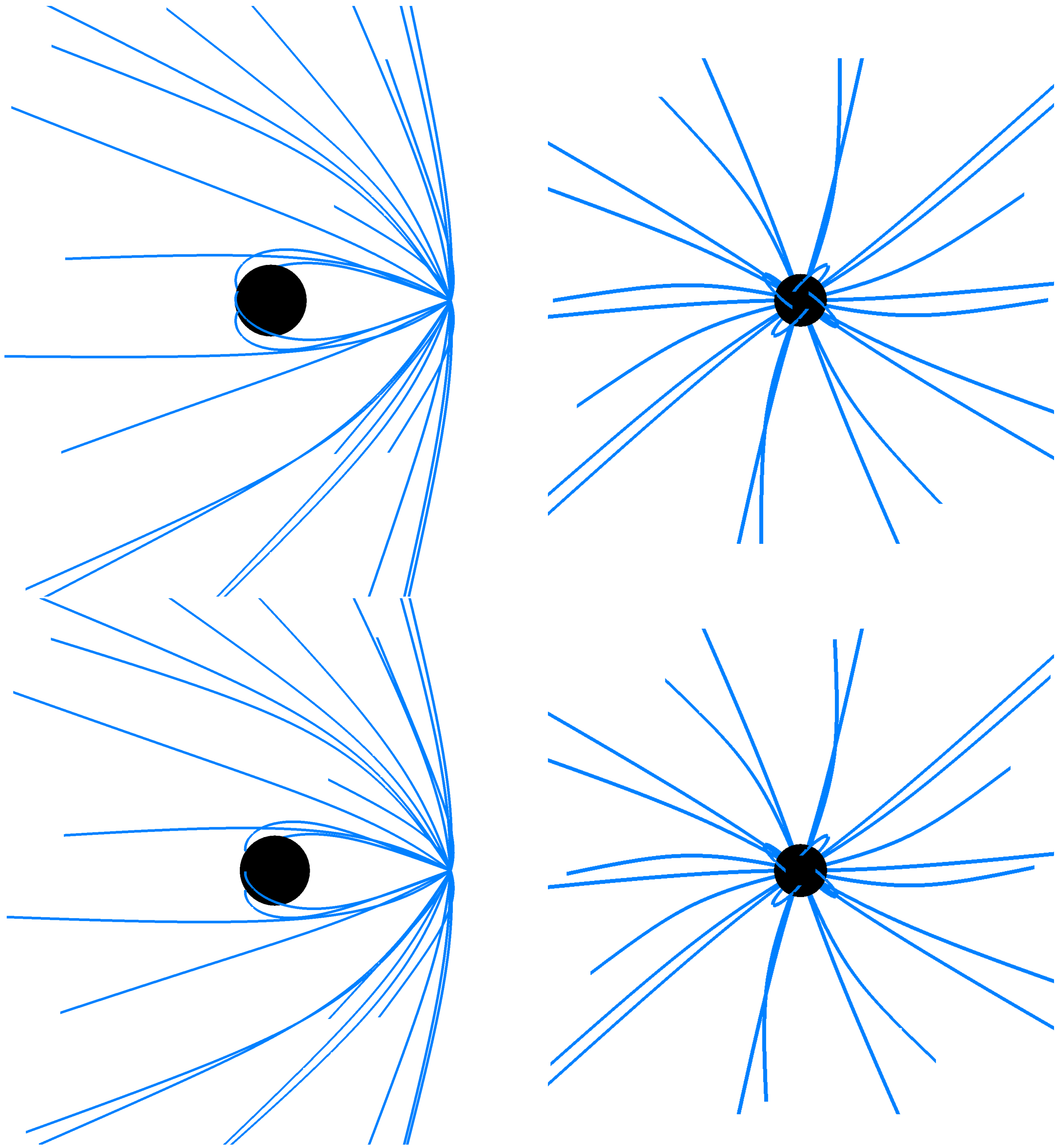


Figure 13: *Top (left and right)* : $\gamma = 0.1$. *Bottom (left and right)* : $\gamma = 0.28$. Light rays for various initial directions and $n = 0.25$ are represented. The observer lies where the trajectories meet. On the top and bottom figures, the left and right situations are the same, but from a different point of view so the behaviour of the geodesics becomes clearer.

References

[1] H. Luckock and I. Moss, Phys. Lett. B 176, 341 (1986).

[2] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).

[3] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D **79** (2009) 064036

[4] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D **84** (2011) 064039

[5] C. Deffayet and D. A. Steer, Class. Quant. Grav. **30** (2013) 214006

[6] T. Kobayashi, M. Yamaguchi and J. 'i. Yokoyama, Prog. Theor. Phys. **126** 511 (2011).

[7] L. Hui and A. Nicolis, Phys. Rev. Lett. **110** (2013) 241104

[8] M. T. Mueller and M. J. Perry, Class. Quant. Grav. **3** (1986) 65. doi:10.1088/0264-9381/3/1/009

[9] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. Lett. **112** (2014) 251102

[10] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. D **90** (2014) 124063

[11] B. Kleihaus, J. Kunz and E. Radu, Phys. Rev. Lett. **106** (2011) 151104

[12] J. L. Blázquez-Salcedo *et al.*, IAU Symp. **324** (2016) 265 doi:10.1017/S1743921316012965 [arXiv:1610.09214 [gr-qc]].

[13] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112** (2014) 221101

[14] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D **24** (2015) no.09, 1542014

[15] E. T. Newman, L.Tamburino and T. Unti, J. Math. Phys. **4** (1963) 915.

[16] C. W. Misner, J. Math. Phys. **4** (1963) 924;
C. W. Misner and A. H. Taub, Sov. Phys. JETP **28** (1969) 122.

[17] C. W. Misner, in *Relativity Theory and Astrophysics I: Relativity and Cosmology*, edited by J. Ehlers, Lectures in Applied Mathematics, Volume 8 (American Mathematical Society, Providence, RI, 1967), p. 160.

[18] G. Clément, D. Gal'tsov and M. Guenouche, Phys. Lett. B **750** (2015) 591

[19] V. Kagramanova, J. Kunz, E. Hackmann and C. Lammerzahl, Phys. Rev. D **81** (2010) 124044

[20] P. Jefremov and V. Perlick, Class. Quant. Grav. **33** (2016) no.24, 245014

[21] Y. Brihaye and E. Radu, Phys. Lett. B **764** (2017) 300

[22] U. Ascher, J. Christiansen, R. D. Russell, Math. of Comp. **33** (1979) 659;
U. Ascher, J. Christiansen, R. D. Russell, ACM Trans. **7** (1981) 209.

[23] A. Riazuelo, "Seeing relativity – I. Basics of a raytracing code in a Schwarzschild metric", arXiv:1511.06025 [gr-qc].

Appendix

C Spinning-Charged-Hairy Black Holes in 5-d Einstein Gravity

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Spinning-Charged-Hairy Black Holes in 5-d Einstein gravity

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Abstract

The spinning-hairy black holes that occur in Einstein's gravity supplemented by a doublet of complex scalar fields are constructed within an extension of the model by a $U(1)$ gauge symmetry involving a massless vector potential. The hairy black holes then acquire an electric charge and a magnetic moment; their domain of existence is discussed in terms of the gauge coupling constant.

1 Introduction

One century after the discovery of General Relativity (GR) by Einstein in 1915, the research in this topic has never been so intensive and exciting. This is mainly boosted by two antinomic phenomena: the direct observation of gravitational waves and the desperate lack of evidence of dark matter. For obvious reasons, the main effort is realized essentially in four-dimensional space-time but there are numerous reasons to study GR in higher dimensions (see e.g. [1] and references therein). Besides looking at our Universe with a String theory or Brane-World point of view, the study of GR in $d > 4$ dimensions provides an inexhaustible domain of research and a fertile ground for new discoveries and innovating techniques. First the basic Einstein-Hilbert action can be enlarged by a hierarchy of Lovelock actions involving higher powers in the Rieman tensor [2]; the first term of which being the Gauss-Bonnet action. Second, while the basic classical solutions in 4-d gravity are drastically featureless [3] (the electro vacuum black holes being described by a few macroscopic degrees of freedom), higher dimensional gravity admits families of new classical solutions, black holes with horizon topologies other than the sphere [4],[1].

Escaping the rigidity of the "No-hair" conjecture [5] has been a long fight which (up to our knowledge) was been broken first time in [6] (see also [7]). Attracting a lot of interest, families of four-dimensional hairy black holes endowed by scalar hairs have been constructed in [8],[9]. The model consists of the standard Einstein gravity minimally coupled to a massive, complex, scalar field. Among ingredients entering crucially in this construction let us point out that: (i) the black hole has to spin sufficiently fast, (ii) the solution is synchronized in the sense that the spinning velocity of the black hole on the horizon coincides with the frequency, say ω, parametrizing the harmonic time dependence of the scalar field, (iii) the solutions bifurcate for a peculiar subfamily of (hairless) Kerr solutions, (iv) the solution involves the numerical integration of a set non-linear partial differential equations.

The discovery of these solutions motivated research of similar solutions in different extensions of the model and/or in different gravity frameworks, see e.g. [10]. Also it is natural to emphasize that hairy black holes (HBH) exist in higher dimensional gravity extended by an appropriate matter sector. In space-time with odd dimensions the construction of spinning black holes is technically simpler since a suitable ansatz of the metric leads to differential equations (instead of partial derivative ones). As a consequence, some aspects of HBH can be studied in such space-time.

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In the absence of matter field, the generic spinning black holes of $d > 4$ space-time gravity are the Myers-Perry (MP) solutions [11]. Supplementing the Einstein-Hilbert action by a single complex scalar leads essentially to non-spinning boson stars. To our knowledge, no parametrization of the metric and matter functions can be implemented to obtain spinning solutions or black holes. One possible key ingredient for obtaining spinning boson stars and black holes consists of the inclusion of a doublet of complex scalar fields. This has been proposed in [12] for the construction of spinning boson stars in $d = 5$ and, using a similar ansatz for the scalar fields, spinning black holes in $d = 5$ gravity were constructed in [13] and [14]. One main feature of these results is that, unlike the $d = 4$ case, the spinning HBH remain decoupled from the family of MP solutions. Interestingly, the family of $d = 5$ HBH is "enveloped" by the boson stars on the one side and by a family of extremal solution on the second side. Asymptotically AdS black holes and boson stars were constructed in [15] with the same matter contain but with a slightly different parametrization. Similar to the $d = 4$ case, these black holes bifurcate from the AdS-Myers-Perry solutions.

On the other hand, supplementing gravity by gauge fields often results in families of charged objects (black holes or solitons) presenting new physically interesting properties. The oldest and most famous example is the occurrence of extremal solutions of Reisner-Nordstrom black holes. Recently, it was revealed that charged black holes possess the property of superradiance [16, 17, 18]. The coupling of the d=4 HBH solutions of [8] to electromagnetism has been investigated in details [19].

Recently it was shown [20] that, in d=5, non-spinning hairy black holes exist in the Einstein-Gauss-Bonnet-Maxwell theory provided both -the gauge coupling constant and the Gauss-Bonnet parameters- are sufficiently large. Several properties of these solutions have been discussed in [21]. To our knowledge, the gauge version of d=5 spinning black holes has not yet been studied and this problem in emphasized in this paper.

The ingredients of the model, the ansatz and the boundary conditions are presented in the second section. In Sect. 3 we review the properties of the non-hairy solutions: the Myers-Perry and Reissner-Nordstrom solutions. For completeness, the uncharged spinning hairy black holes are briefly summarized in Sect. 4 and the new results are presented in Sect. 5. Because of their rotation, the family of black holes in the full model acquire an electric charge as well as a magnetic moment. Finally, in section 6 we summarize our results and conclude.

2 The field equations

2.1 The model

Following the conventions of [12], we consider the action of the self-interacting complex doublet scalar field Φ coupled minimally to Einstein gravity in 5 dimensional spacetime and supplemented by a Maxwell field

$$S = \int \left[\frac{R}{16\pi G} - (D_\mu \Phi)^\dagger (D^\mu \Phi) - U(|\Phi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \sqrt{-g} \mathrm{d}^5 x. \tag{2.1}$$

In the first term, R represents the curvature scalar and G the Newton's constant (in 5 dimensions). The second term is the kinetic part of the scalar field with $\Phi = (\phi_1, \phi_2)^t$ and $A^\dagger$ denotes the complex transpose of A. The third term U is a self-interaction potential depending on the norm $|\Phi|^2 = \Phi^\dagger \Phi$. With this choice, the scalar sector possesses an $U(2)$ global symmetry whose any $U(1)$ subgroup can be gauged. In this work, we will gauge the diagonal $U(1) \times U(1)$ subgroup. Then the covariant derivative and Faraday tensor take the form

$$D_\mu = (\partial_\mu - iqA_\mu \mathbb{I}_2) \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where $\mathbb{I}_2$ is the 2×2 identity matrix, and q denotes the gauge coupling constant. We will assume $q > 0$ since the sign of q can be reabsorbed in the gauge fields.

Variation of the action with respect to the metric leads to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T^s_{\mu\nu} + T^v_{\mu\nu}) \tag{2.2}$$

with stress-energy tensor

$$T^s_{\mu\nu} = (D_\mu\Phi)^\dagger(D_\nu\Phi) + (D_\nu\Phi)^\dagger(D_\mu\Phi) - \frac{1}{2}g_{\mu\nu}\Big[(D_\alpha\Phi)^\dagger(D_\beta\Phi) + (D_\beta\Phi)^\dagger(D_\alpha\Phi)\Big]g^{\alpha\beta} - g_{\mu\nu}U(|\Phi|), \quad (2.3)$$

$$T^v_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \ , \quad (2.4)$$

where the upper-script s stand for *scalar* while the upper-script v stand for *vector*, *i.e.* the Maxwell field.

The variation with respect to the matter fields leads respectively to the equations,

$$\frac{1}{\sqrt{-g}}D_\mu\left(\sqrt{-g}D^\mu\Phi\right) = \frac{\partial U}{\partial|\Phi|^2}\Phi \ , \ \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}F^{\mu\nu} = J^\nu \ , \ J^\nu = iq((D^\nu\Phi)^\dagger\Phi - \Phi^\dagger(D^\nu\Phi)). \quad (2.5)$$

2.2 The Ansatz

While rotating EKG black holes will generically possess two independent angular momenta and a more general topology of the event horizon we restrict here to configurations with equal-magnitude angular momenta and a spherical horizon topology.

Thus the solutions possess bi-azimuthal symmetry, implying the existence of three commuting Killing vectors, $\xi = \partial_t$, $\eta_1 = \partial_{\varphi_1}$, and $\eta_2 = \partial_{\varphi_2}$.

A suitable metric ansatz in this case reads

$$\begin{aligned} \mathrm{d}s^2 = &\frac{\mathrm{d}r^2}{f(r)} + g(r)\mathrm{d}\theta^2 + h(r)\sin^2\theta\left(\mathrm{d}\varphi_1 - W(r)\mathrm{d}t\right)^2 + h(r)\cos^2\theta\left(\mathrm{d}\varphi_2 - W(r)\mathrm{d}t\right)^2 \\ &+ (g(r) - h(r))\sin^2\theta\cos^2\theta(\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)^2 - b(r)\mathrm{d}t^2, \end{aligned} \quad (2.6)$$

where $\theta \in [0, \pi/2]$, $(\varphi_1, \varphi_2) \in [0, 2\pi]$, and r and t denote the radial and time coordinate, respectively.

For such solutions the isometry group is enhanced from $\mathbb{R}_t \times U(1)^2$ to $\mathbb{R}_t \times U(2)$, where $\mathbb{R}_t$ denotes the time translation. This symmetry enhancement allows to factorize the angular dependence and thus leads to ordinary differential equations.

Completing the metric (2.6), the scalar field is taken in the form used in [12] :

$$\Phi = F(r)e^{i\omega t}\begin{pmatrix} \sin\theta e^{i\varphi_1} \\ \cos\theta e^{i\varphi_2}\end{pmatrix}, \quad (2.7)$$

where the frequency ω parametrized the harmonic time dependence. For the scalar field potential we restrict our study to the simplest case

$$U(|\Phi|) = \mu^2\Phi^\dagger\Phi = \mu^2F(r)^2 \ , \quad (2.8)$$

where μ corresponds to the scalar field mass.

Finally, the electromagnetic potential is chosen in the form

$$A_\mu\mathrm{d}x^\mu = V(r)\mathrm{d}t + A(r)(\sin^2(\theta)\mathrm{d}\varphi_1 + \cos^2(\theta)\mathrm{d}\varphi_2) \ , \quad (2.9)$$

which turns out to be consistent with the symmetries of the metric and scalar fields. The whole ansatz leads to a consistent set of differential equations for the radial functions f, b, h, g, W, V, A and F.

Without fixing a metric gauge, a straightforward computation leads to the following reduced action for the system

$$A_{\mathrm{eff}} = \int \mathrm{d}r\mathrm{d}t\ L_{\mathrm{eff}}, \quad \text{with} \quad L_{\mathrm{eff}} = L_g + 16\pi G(L_s + L_v), \quad (2.10)$$

with

$$L_g = \sqrt{\frac{fh}{b}}\Big(b'g' + \frac{g}{2h}b'h' + \frac{b}{2g}g'^2 + \frac{b}{h}g'h' + \frac{1}{2}ghW'^2 + \frac{2b}{f}(4-\frac{h}{g})\Big), \tag{2.11}$$

$$L_s = g\sqrt{\frac{bh}{f}}\left(fF'^2 + (\frac{2}{g} + \frac{(1-qA)^2}{h} - \frac{(\omega - W + q(V+WA))^2}{b} + \mu^2)F^2\right), \tag{2.12}$$

$$L_v = g\sqrt{\frac{bh}{f}}\left(\frac{2A^2}{g^2} + \frac{f}{2h}(A')^2 - \frac{f}{2b}(V'+WA')^2\right), \tag{2.13}$$

where a prime denotes a derivative with respect to r.

It can be checked that the full Einstein equations (2.2) are recovered by taking the variation of $A_{\rm eff}$ with respect to h, b, f, g and W. The Klein-Gordon and Maxwell equations are found by taking the variation with respect to F, V and A.

The metric gauge freedom can be fixed afterwards, leading to a system of seven independent equations plus a constraint which is a consequence of the other equations. For the construction of the solutions, we have fixed the metric gauge by taking

$$g(r) = r^2 \tag{2.14}$$

consistently with the standard analytic form of the Myers-Perry solution. Appropriate combinations of the equations can be used in such a way that only the first derivative of f appears in the system. Accordingly, the equation for $f(r)$ is a first order equation while the equations of the six other fields are of the second order.

2.3 Asymptotics

The submanifold of space-time characterized by a fixed value of the radial and time coordinates, $r = r_H > 0$ and $t = t_0$ respectively, is a squashed S^3 sphere. Imposing the horizon of the metric by means of the conditions $f(r_H) = b(r_H) = 0$ therefore leads to black holes with the same horizon topology.

Restricting to nonextremal solutions, the following expansion holds near the event horizon:

$$\begin{aligned} &f(r) = f_1(r-r_H) + O(r-r_H)^2, \quad h(r) = h_H + O(r-r_H), \\ &b(r) = b_1(r-r_H) + O(r-r_H)^2, \quad w(r) = \Omega_H + w_1(r-r_H) + O(r-r_H)^2, \\ &V(r) = V_H + V_1(r-r_H) + O(r-r_H)^2, \quad A(r) = A_H + V_1(r-r_H) + O(r-r_H)^2, \\ &F(r) = F_0 + F_1(r-r_H) + O(r-r_H)^2. \end{aligned} \tag{2.15}$$

A straightforward calculation gives the following asymptotic expansion for the solution

$$\begin{aligned} &b(r) = 1 + \frac{\mathcal{U}}{r^2} + \cdots, \quad f(r) = 1 + \frac{\mathcal{U}}{r^2} + \cdots, \quad h(r) = r^2 + \frac{\mathcal{V}}{r^2} + \cdots, W(r) = \frac{\mathcal{W}}{r^4} + \cdots, \\ &V(r) = V_0 + \frac{q_e}{r^2} + \cdots, A(r) = \frac{q_m}{r^2} + \cdots, \quad F(r) = c_0\frac{e^{-\sqrt{\mu^2-(\omega-qV_0)^2}r}}{r^{3/2}} + \cdots, \end{aligned} \tag{2.16}$$

which guarantees Minkowski spacetime background to be approached at infinity. In this expansion, $\mathcal{U}, \mathcal{V}, \mathcal{W}$, q_e, q_m, V_0 and c_0 are free parameters that can be reconstructed from the result of the numerical integration on $[r_H, \infty[$.

Note that, in the last equation (2.16), one can see that the scalar field acquires a squared effective mass $M_{\rm eff}^2 = \mu^2 - (\omega - qV_0)^2$. The condition

$$\mu - |\omega - qV_0| > 0 \tag{2.17}$$

should therefore be obeyed to guarantee bound-state (or localized) solutions.

This condition generalize the bound-state condition known for uncharged solutions : that is $\omega < \mu$, see fig. 1 (where we put $\mu = 1$ without loss of generality).

2.4 Boundary conditions

We will not write the field equations explicitly which are lengthy and non-illuminating. However, one can easily see from (2.12) that the regularity of the solutions at the horizon implies a generalized synchronization condition between the frequency ω of the scalar field and the rotation velocity $\Omega_H \equiv W(r_H)$ of the black hole at the horizon

$$\omega - \Omega_H + q(V_H + A_H \Omega_H) = 0.$$

This condition can be used to determine the frequency ω in terms of the other parameters.

Because the electric potential $V(r)$ can be shifted up to a redefinition of quantity $A_H \Omega_H$ and the frequency ω, we will take advantage of this freedom to set $V_H = 0$. The condition would then become

$$\omega - \Omega_H + q A_H \Omega_H = 0. \tag{2.18}$$

This condition generalizes the so-called "synchronization condition" ($\omega = \Omega_H$) known for the uncharged case. We clearly see here that, for a given value of $\Omega_H \neq 0$, the synchronization condition hold **iff** $q = 0$ or $A_H = 0$.

The regularity of the equations for the fields $h(r), W(r)$ and the Maxwell equations imply three independent non-trivial conditions to be obeyed at the horizon, we will note them symbolically $\Gamma_h, \Gamma_W, \Gamma_V$; these are lengthy polynomials in the various fields and their first derivative, we do not write them explicitly because they are not illuminating. Summarizing, we have eight conditions at the horizon

$$f(r_H) = 0 \ , \ b(r_H) = 0 \ , \ W(r_H) = \Omega_H \ , \ V(r_H) = 0 \ , \ A(r_H) = A_H \ , \ \Gamma_{h,W,V} = 0 \ . \tag{2.19}$$

where A_H is an undetermined constant.

The boundary value problem is then fully specified by imposing the conditions (2.19) at the horizon and the asymptotic conditions (2.16) for b, h, W, A and F.

2.5 Rescaling

The theory is determined by three parameters : G, μ, q. The constants G and μ can be rescaled into the matter fields and the radial variable. We will use rescaled quantities such that $\mu = 1$ and $8\pi G = 1$. The gauge coupling q is therefore the only intrinsic parameter of the model.

The hairy black holes will be specified by the event horizon radius r_H, the horizon velocity Ω_H and the value A_H.

The (constant) horizon angular velocity Ω_H is defined in terms of the Killing vector $\chi = \partial/\partial_t + \Omega_1 \partial/\partial\varphi_1 + \Omega_2 \partial/\partial\varphi_2$ which is null at the horizon. For the solutions within the ansatz (2.6), the horizon angular velocities are equal, $\Omega_1 = \Omega_2 = \Omega_H$.

2.6 Quantities of interest

The mass and angular momentum of the solutions are given by

$$M = -\frac{3S_3}{16\pi G}\mathcal{U}, \quad J = \frac{S_3}{8\pi G}\mathcal{W}, \tag{2.20}$$

where $S_3 = 2\pi^2$ denotes the area of the unit three-dimensional sphere.

The solution is further characterized by the electric charge Q_e and magnetic moment Q_m, these quantities are related to the parameters q_e and q_m by

$$Q_e = \frac{2S_3}{4\pi G} q_e \quad , \quad Q_m = \frac{2S_3 \pi^2}{4\pi G} q_m \quad . \tag{2.21}$$

Finally, the gyromagnetic ratio $\tilde{g}$ can be computed : $\tilde{g} = 2MQ_m/(Q_e J)$. Other quantities of interest are the Hawking temperature T_H and the area $\mathcal{A}_H$ of the black hole horizon

$$T_H = \frac{\sqrt{b_1 f_1}}{4\pi}, \quad \mathcal{A}_H = \sqrt{h_H} r_H^2 S_3. \tag{2.22}$$

The horizon mass and angular momentum M_H, J_H can further be defined (see e.g. [22]); these quantities obey a Smarr relation of the form

$$M_H = \frac{3}{2}(\frac{\mathcal{A}_H T_H}{4} + 2\Omega_H J_H). \tag{2.23}$$

This has been used as a numerical test of our numerical results.

In order to figure out the way the distribution of the energy of the scalar field is affected by the electromagnetic field, we will present the energy density of some solutions. For completeness, we write these formulas

$$\mathcal{E}_v = g\sqrt{\frac{bh}{f}}(\frac{f}{2bh}(A')^2(b-hW^2) + \frac{2}{g^2}A^2 + \frac{f}{2b}(V')^2),$$

$$\mathcal{E}_s = g\sqrt{\frac{bh}{f}}\left(f(F')^2 + \mu^2 F^2 + F^2\left(\frac{2}{g} + \frac{(1-qA)^2}{h}\right) + \frac{F^2}{b}\left[(qV-\omega)^2 - W^2(qA-1)^2\right]\right).$$

3 Non-hairy solutions

In the absence of the scalar field ($\Phi = 0$), two types of solutions exist that are known in analytic form.

3.1 Uncharged, spinning black holes

In the absence of the Maxwell field one recovers the MP (vacuum) black holes [11] with equal-magnitude angular momenta. Expressed in terms of the event horizon radius and the horizon angular velocity[1] (which are the control parameters in our numerical approach), this solution reads

$$\begin{aligned} f(r) &= 1 - \frac{1}{1-r_H^2\Omega_H^2}\left(\frac{r_H}{r}\right)^2 + \frac{r_H^2\Omega_H^2}{1-r_H^2\Omega_H^2}\left(\frac{r_H}{r}\right)^4, \quad h(r) = r^2\left(1 + \left(\frac{r_H}{r}\right)^4 \frac{r_H^2\Omega_H^2}{1-r_H^2\Omega_H^2}\right), \\ b(r) &= 1 - \left(\frac{r_H}{r}\right)^2 \frac{1}{1-\left(1-(\frac{r_H}{r})^4\right)r_H^2\Omega_H^2}, \quad W(r) = (\frac{r_H}{r})^4 \frac{\Omega_H}{1-\left(1-(\frac{r_H}{r})^4\right)r_H^2\Omega_H^2}. \end{aligned} \tag{3.24}$$

Therefore, for a MP black hole, the relevant parameters in the event horizon expansion (2.15) are

$$f_1 = \frac{2(1-2r_H^2\Omega_H^2)}{r_H(1-r_H^2\Omega_H^2)}, \quad b_1 = \frac{2}{r_H}(1-2r_H^2\Omega_H^2), \quad w_1 = \frac{4\Omega_H}{r_H}(r_H^2\Omega_H^2 - 1), \tag{3.25}$$

while the constants $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ in the far field expansion (2.16) have the following expression

$$\mathcal{V} = \frac{r_H^6\Omega_H^2}{1-r_H^2\Omega_H^2}, \quad \mathcal{U} = -\frac{r_H^2}{1-r_H^2\Omega_H^2}, \quad \mathcal{W} = \frac{r_H^4\Omega_H}{1-r_H^2\Omega_H^2}. \tag{3.26}$$

The $d = 5$ MP black holes with equal-magnitude angular momenta emerge smoothly from the static Schwarzschild-Tangherlini black hole when the event horizon velocity Ω_H is increased from zero. For a given event horizon radius, the solutions exist up to a maximal value of the horizon angular velocity $\Omega_H^{(c)} = 1/\sqrt{2}r_H$ for which the black hole becomes extremal ($T_H \to 0$). Expressed in terms of the mass-energy M and the equal-magnitude angular momenta $|J_1| = |J_2| = J$, this bound reads $27\pi J^2/8G < M^2$. The extremal solution saturating this bound has a regular but degenerate horizon.

[1] This metric is usually expressed in terms of the mass M and the angular momentum parameter a, with $M = r_H^2/2(1-r_H^2\Omega_H^2)$ and $a = r_H^2\Omega_H$.

3.2 Charged, non-spinning black holes

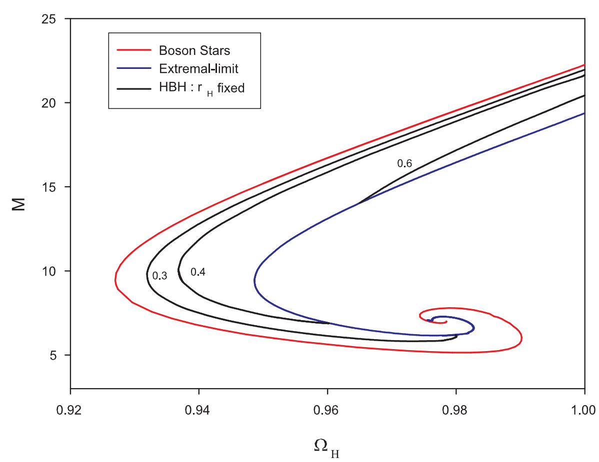
In the absence of rotation, charged black holes exist generalizing the Reisner-Nordstrom solutions. The equations are considerably simpler since $f(r) = b(r), h(r) = r^2$ and $W(r) = A(r) = 0$ while the non-trivial fields read

$$f(r) = 1 - \frac{2M}{r^2} + \frac{Q^2}{3r^4} \quad , \quad V(r) = V_0 + \frac{Q}{r^2} \quad , \quad M = \frac{r_H^2}{2} + \frac{Q^2}{6r_H^2} \tag{3.27}$$

These black holes exist for $0 \leq Q^2 < 3r_H^4$ and become extremal in the limit $Q^2 \to 3r_H^4$.

4 Uncharged Hairy Black Holes

In the absence of the electromagnetic field, a family of hairy black holes exist on a specific domain of the r_H, Ω_H parameter space. These solutions were obtained in [13] but we shortly discuss them again for completeness. The $\Omega_H - M$ domain of these solutions is represented in Fig. 1 (left side) where a few values of r_H are supplemented. The solutions exist for $0.924 < \Omega_H < 1$ in the region limited by boson stars (red line, reached in the limit $r_H \to 0$) and by a family of singular extremal solutions (blue line: $T_H = 0$). The intermediate constitutes the regular black holes. On the right side of Fig. 1 The domain of Myers-Perry and hairy black holes are superposed, demonstrating that both families of solutions are disjoined.



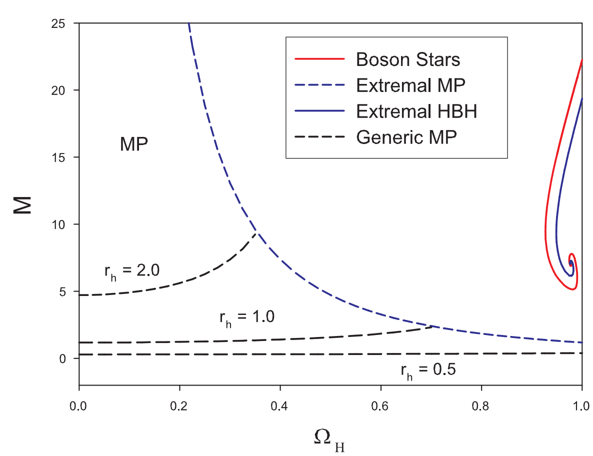


Figure 1: Left: Mass-frequency relation for boson star solutions (red line), for hairy black holes with three values of $r_H = 0.3, 0.4, 0.65$ (black lines) and for extremal solutions (blue line). Right: Sketch of the domains of Myers-Perry and hairy black holes in the $\Omega_H - M$ plane.

5 Charged-hairy-solutions

Any hairy black holes, e.g. any point of Fig. 1 (left side) characterized by a couple (r_H, Ω_H) gets deformed in the presence of an electromagnetic field. The purpose of this section is to study the pattern of solutions in dependence of the gauge coupling q. Unlike the Myers-Perry solutions, the equations with the matter fields have no closed form solutions and have to be solved numerically. We used the numerical routine COLSYS [23] to perform the integration. The radial interval was discretized by means of a mesh of 300 points and the equations were solved with a relative error of less than 10^{-6}.

The frequency of the scalar field ω is fixed via the condition (2.18). Accordingly, synchronized solutions (that is with $\omega = \Omega_H$) occur **iff** $q = 0$ or $A_H = 0$. These cases will be discussed separately.

5.1 Synchronized solutions: Case $q = 0$

This case corresponds to the ungauged model, whose $d = 4$ counterpart is studied in details in Sect. 2 of [19]. In the absence of a direct coupling of the scalar fields to the electromagnetic field (i.e. for $q = 0$), the spinning solution gets deformed through a non-trivial magnetic potential $A(r)$ through the influence of the rotating space-time. The parameter controlling the magnetic potential is $A_H \equiv A(r_H)$. The electromagnetic field vanishes identically for $A_H = 0$ and becomes non-trivial for $A_H \neq 0$. The effect is demonstrated on

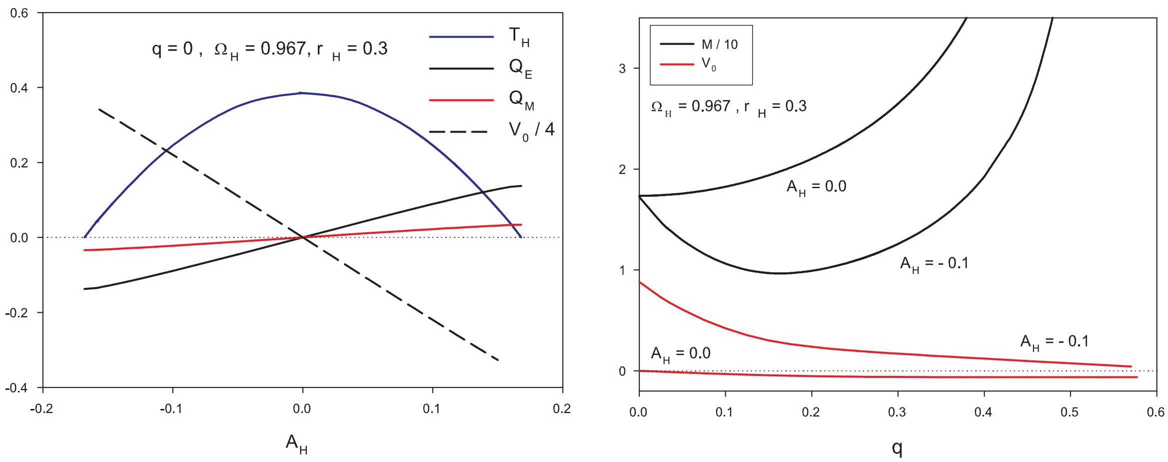


Figure 2: *Left*: Several quantities characterizing the black holes for $q = 0$ as functions of the parameter A_H. *Right:* The mass M and electric potential V_0 as function of q for the solutions corresponding to $A_H = 0.0$ and $A_H = -0.1$.

Fig. 2 (left side) : the electric and magnetic parameters Q_e, Q_m and the electric potential $V_0 \equiv V(\infty)$ are reported as functions of A_H. The temperature T_H of the black hole is supplemented and reveals that a limiting configuration with zero temperature is approached for a maximal value of $|A_H|$. In the ungauged case, since $q = 0$, the equations simplify drastically. In particular they become symmetric under $A_\mu \to -A_\mu$, explaining the symmetry of Fig. 2 (left) under the sign reversal of A_H. The mass and angular momentum of these solutions depend weakly on A_H and have not been reported. The qualitative features of Fig. 2 seem to be generic, it was checked for several values of the parameters r_H and Ω_H.

5.2 Synchronized solutions : case $A_H = 0$

Synchronized black holes strongly coupled to the electromagnetic field are obtained by increasing the gauge coupling parameter gradually q while imposing $A_H = 0$ as boundary condition. The electric potential V_0 is negative and decreases monotonically while q increases; this is shown on the right side of Fig.2. Our numerical results strongly suggest that the effective squared mass $\mu^2 - (\omega - qV_0)^2$ approaches zero for $q \to q_{\max}$ (see Fig. 3 (left side)); as a consequence localized solution does not exist for $q > q_{\max}$. The value $q_{\max}$ in fact corresponds to the value of the coupling constant for which the bound state condition (2.17) ceases to be satisfied. On the example of Fig. 3 , i.e. with $r_H = 0.3, \Omega_H = 0.967$, we find $q_{\max} \approx 0.577$. The temperature corresponding to this set of solutions has been reported on the right side of Fig. 3.

[2]On Fig. 4 (left side) the effective energy density of the scalar part and vector part of the matter fields are reported for $q = 0.0$ and $q = 0.4$. For $q = 0$, the energy density of the vector field is identically null. One can also see that the maximum of the scalar energy density is closer to the horizon in this case compared to the case $q \neq 0$. This might be interpreted as follows : The case $q = 0$ corresponds to an uncharged massive scalar field. In this case, the existence of bound state is insured by the gravitational interaction

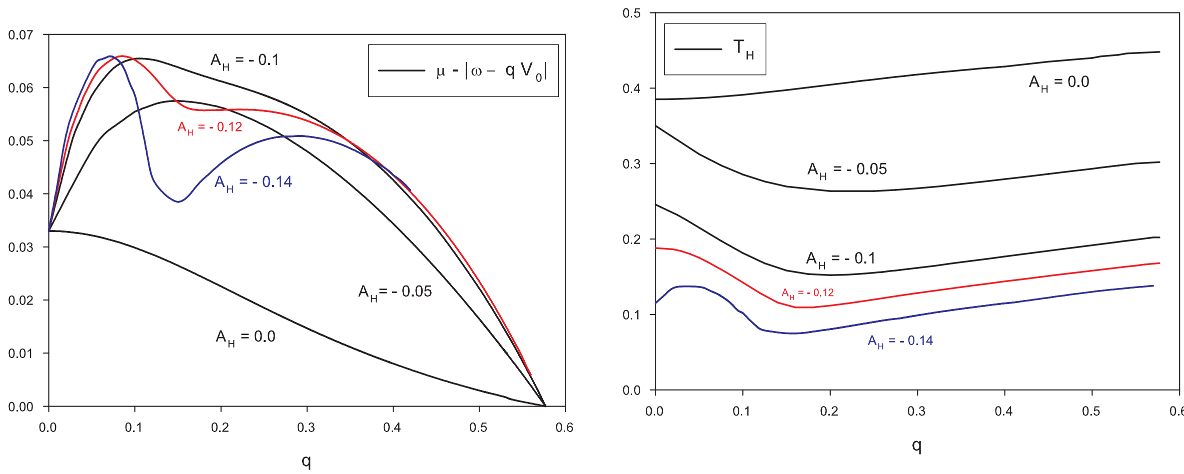


Figure 3: *Left*: The combination $\mu - |\omega - qV_0|$ as a function of q for several values of $A_H \leq 0$. *Right*: The Temperature for the same set of solutions. Both plots are realized for $\Omega_H = 0.967$ and $r_H = 0.3$.

and the scalar "cloud" is located at a given distance of the black hole horizon. In the case $q \neq 0$ the scalar field is electrically charged, more precisely every quantum in the "cloud" acquire the same charge q, and then every quantum would electrically try to grow back each other so that the macroscopic scalar field (the position of the maximum) would be located further away compared to the uncharged case. This qualitative argumentation can also explain the existence of q_{max}. This corresponds to the value of the scalar electric charge for which gravity ceases to be strong enough to compensate for the electrostatic repulsion and guarantee a bound state.

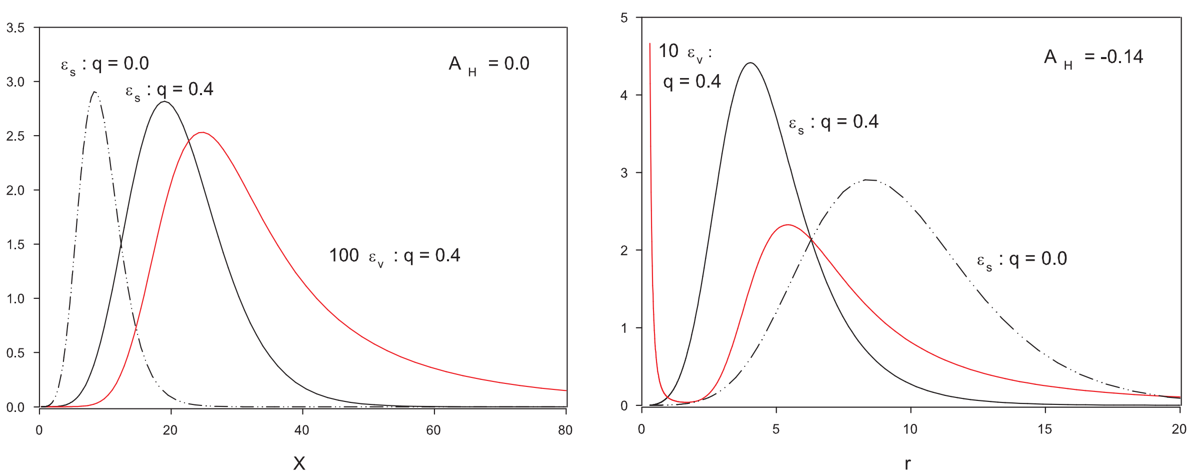


Figure 4: *Left*: The scalar (black) and vector (red) energy density for $q = 0$ (dot-dashed) and for $q = 0.4$ (solid) for $A_H = 0$. *Right:* Idem for $A_H = -0.14$. On both sides the dot-dashed curve is the same allowing the reader to compare the plots. Both plots are realized for $r_H = 0.3$ and $\Omega_H = 0.967$.

5.3 General case : $A_H \neq 0$, $q \neq 0$

In the general case, any uncharged hairy solution (with fixed r_H, Ω_H) lead to a two-parameter family of charged solutions characterized by the value A_H and the gauge coupling constant q. This domain of existence of the solutions is largely limited by the bound state condition (2.17).

Setting for definiteness $A_H < 0$, the numerical results indicate that the solutions stop being localized for $q > q_{\max}$, i.e. when the effective frequency becomes imaginary. These results also strongly suggest that the value $q_{\max}$ is independent of the value A_H. To illustrate this statement, the dependence on q of the quantity appearing in (2.17) is reported on Fig. 3 (left side) for several values of $A_H \leq 0$. The plot clearly suggests that the value $q_{\max}$ is independent of A_H, although the curves are substantially different for the intermediate values of q. The temperature of these families of solutions are reported on the right side of Fig. 3.

For $A_H > 0$, the domain is considerably smaller because, due to the negative sign of V_0, the effective frequency quickly becomes imaginary. We did not study this case in detail.

On Fig. 4 (right side) we report the effective energy density for the scalar part and vector part for $A = -0.14$ and for $q = 0.0$ and $q = 0.4$. Here, the maximum of the scalar energy density is closer to the horizon for $q \neq 0$. This behaviour remains true until q becomes close to $q_{\max}$. While approaching $q_{\max}$ the maximum of the scalar energy density would become more and more distant from the horizon until the bound state condition stop to be satisfied.

The interpretation in this case is similar to the previous one : The case $q = 0$ corresponds to an uncharged scalar field as we already discussed. Here when $q \neq 0$, since $A_H \neq 0$, the system would be subject to electric and magnetic interaction since the black hole as both electric and magnetic charge. For a given value of A_H (and then for a fixed magnetic interaction with the black hole), on most of the allowed parameter space, the magnetic effect is dominant with respect to the electric repulsion. The effect of this magnetic interaction will be to concentrate the scalar field near the horizon. Nevertheless, if one continues to increase q, the electric repulsion will finally become dominant and lead to the dispersion of the scalar field. The interesting thing here is that this breakdown occurs for a value of q independent of the parameter of the model, i.e. not only A_H but also Ω_H and r_H. This "universality" of $q_{\max}$ apears as a surprise and we did not found any physical or analytic explanation for it.

Let us finally mention that, when the value $q_{\max}$ is approached, the mass and angular momentum of the black hole become very large and possibly tend to infinity (see Fig.2, right side). Due to numerical difficulties, this statement can hardly be proven but an inspection of the profiles presented on Fig. 5 show that the fields have a tendency to spread over space for $q \to q_{\max}$. We checked that this behaviour persists for different values of A_H, r_H and Ω_H than the ones presented in the figure.

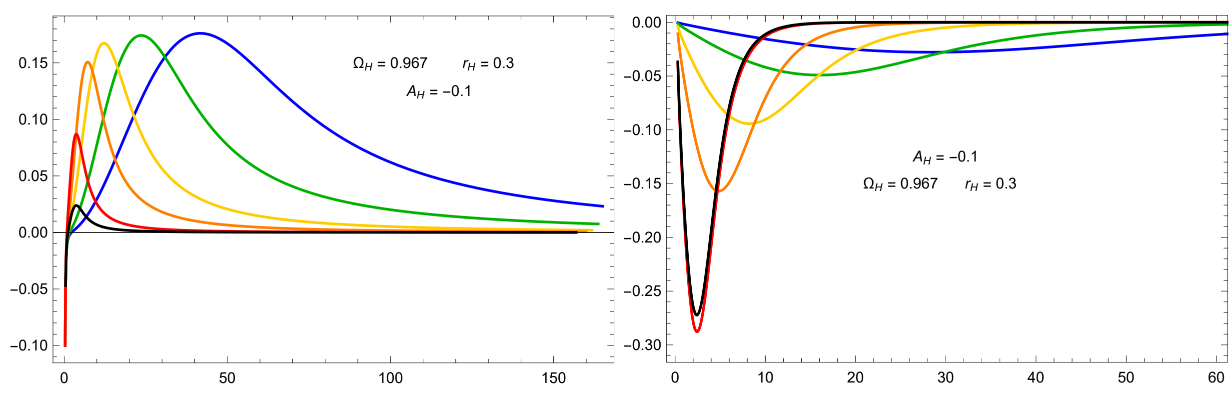


Figure 5: *Left*: Profile of the magnetic potential A with $A_H = -0.1$ for $q = 0.1, 0.3, 0.5, 0.55, 0.57$ and $0.575 \approx q_{\max}$ respectively in black, red, orange, yellow, green and blue. These curves are for $\Omega_H = 0.967$ and $r_H = 0.3$. *Right:* Idem for the scalar function F.

6 Conclusion & outlook

Conclusion

This paper is devoted to the study of higher dimensional rotating charged black holes in Einstein gravity supplemented by a doublet of complex massive scalar fields (see section 2 for details of the model).

This model contains numerous parameters. Within our ansatz, once fixed the units ($8\pi G = 1$ and $\mu = 1$, where μ is the mass of the scalar doublet), the solutions are determined in terms of 4 parameters : the position of the event horizon r_H, the angular velocity of the black hole at the horizon Ω_H, the gauge coupling parameter q and the value of the magnetic potential at the horizon A_H.

In sections 3 and 4 we reviewed the main results already available in the literature for the non-hairy and uncharged hairy subsectors of our model.

In section 5 we finally come with new results for the most general cases *i.e.* charged rotating hairy solutions. We saw that any uncharged hairy solution (described by r_H and Ω_H) gives rise to a two-parameter space of solutions (controlled by q and A_H). As expected, the domain of existence for the solutions is asymmetric with respect to a change of sign of A_H only when $q \neq 0$. We saw that the domain of existence was more restricted for $A_H > 0$ while for $A_H < 0$ solutions might exist for a significantly higher value of $|A_H|$. In both cases, the bound in the domain is controlled by the condition limiting the possible existence of regular solutions with **localized** scalar field. That is $\mu - |\omega - qV_0| > 0$, where ω is the harmonic frequency of the scalar doublet and V_0 the value of the electric potential at infinity. Basically, this condition fix the maximal value of the gauge coupling parameter, say $q_{\max}$, for which the presence of localized scalar field might be supported by the rotating charged black hole. Our numerical results tend to proof that this value $q_{\max}$ is "universal", by this we mean independent of A_H, r_H and Ω_H. In our units, we found $q_{\max} \approx 0.577$.

Finally, let us mention that the link between the harmonic frequency of the scalar doublet ω and the angular velocity of the black hole at the horizon Ω_H was also established and give rise to the condition: $\omega - \Omega_H + qA_H\Omega_H = 0$. This condition generalizes the synchronization condition known for uncharged solutions.

Outlook

Perhaps the most natural question which raises about hairy black holes is the study of their stability – or at least their effective stability – and the estimation of their life time compared to the age of the Universe. Such an analysis is quite involved and out of the scope of this paper in $d = 5$; however we would like to point out some technical differences with respect to the important case $d = 4$ which was addressed recently in [24] and [25].

In the $d = 4$ case it turns out that one unstable mode appears in the sector of the perturbed scalar field [24] and that the instability is more pronounced in the region where the HBH bifurcate from the Kerr solutions. In a sense, the superadiance instability of the Kerr black hole persists by continuity in the region of HBH with small hairs. On the other hand HBH with long hairs have been argued [25] to present a longer life time, i.e. of the order of the age of the Universe.

As discussed in the first section of the present work, the HBH in $d = 5$ are totally disconnected from the Myers-Perry black hole. As a consequence their eventual instability cannot be due to a contamination of instabilities of the background metric. Instabilities of different nature can nevertheless occur. For examples : (i) in the scalar sector which presents four degrees of freedom (instead of two for $d = 4$), (ii) due to the existence of two azimuthal angles of space-time. The quite involved problem of the perturbation analysis of HBH in $d = 5$ could somehow be simplified by specializing into perturbations preserving the symmetries of the metric (2.6) (see e.g. [26]) and of the scalar fields.

References

[1] R. Emparan and H. S. Reall, Living Rev. Rel. **11** (2008) 6.

[2] D. Lovelock, J.Math.Phys. 12 (1971) 498-501.

[3] P. T. Chrusciel, J. Lopes Costa and M. Heusler, Living Rev. Rel. **15** (2012) 7.

[4] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88** (2002) 101101.

[5] R. Ruffini and J. A. Wheeler, Phys. Today **24** (1971) no.1, 30.

[6] H. Luckock and I. Moss, Phys. Lett. B 176, 341 (1986).

[7] J. D. Bekenstein, "Black hole hair: 25 - years after," In *Moscow 1996, 2nd International A.D. Sakharov Conference on physics* 216-219 [gr-qc/9605059].

[8] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112** (2014) 221101.

[9] C. Herdeiro and E. Radu, Phys. Rev. D **89** (2014) no.12, 124018.

[10] C. A. R. Herdeiro, E. Radu and H. F. Runarsson, Int. J. Mod. Phys. D **25** (2016) no.09, 1641014.

[11] R. C. Myers and M. J. Perry, Annals Phys. **172** (1986) 304.

[12] B. Hartmann, B. Kleihaus, J. Kunz and M. List, Phys. Rev. D **82** (2010) 084022.

[13] Y. Brihaye, C. Herdeiro and E. Radu, Phys. Lett. B **739** (2014) 1.

[14] Y. Brihaye, C. Herdeiro and E. Radu, Phys. Lett. B **760** (2016) 279.

[15] O. J. C. Dias, G. T. Horowitz and J. E. Santos, JHEP **1107** (2011) 115.

[16] S. Hod, Phys. Lett. B **755** (2016) 177.

[17] S. Hod, Phys. Lett. B **751** (2015) 177.

[18] S. Hod, Phys. Lett. B **713** (2012) 505.

[19] J. F. M. Delgado, C. A. R. Herdeiro, E. Radu and H. Runarsson, Phys. Lett. B **761** (2016) 234.

[20] N. Grandi and I. Salazar Landea, Phys. Rev. D **97** (2018) no.4, 044042.

[21] Y. Brihaye and B. Hartmann, Class. Quant. Grav. **35** (2018) no.17, 175008.

[22] Y. Brihaye, T. Delplace, C. Herdeiro and E. Radu, Phys. Lett. B **782** (2018) 124.

[23] U. Ascher, J. Christiansen, R. D. Russell, Math. of Comp. **33** (1979) 659;
U. Ascher, J. Christiansen, R. D. Russell, ACM Trans. **7** (1981) 209.

[24] B. Ganchev and J. E. Santos, Phys. Rev. Lett. **120** (2018) no.17, 171101.

[25] J. C. Degollado, C. A. R. Herdeiro and E. Radu, Phys. Lett. B **781** (2018) 651.

[26] H. K. Kunduri, J. Lucietti and H. S. Reall, Phys. Rev. D **74** (2006) 084021.

Appendix

D Hairy Black Holes, Boson Stars and Non-minimal Coupling to Curvature Invariants

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Hairy black holes, boson stars and non-minimal coupling to curvature invariants

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Abstract

The Einstein-Klein-Gordon Lagrangian is supplemented by a non-minimal coupling of the scalar field to specific geometric invariants : the Gauss-Bonnet term and the Chern-Simons term. The non-minimal coupling is chosen as a general quadratic polynomial in the scalar field and allows – depending on the parameters – for large families of hairy black holes to exist. These solutions are characterized, namely, by the number of nodes of the scalar function. The fundamental family encompasses black holes whose scalar hairs appear spontaneously and solutions presenting shift-symmetric hairs. When supplemented by a an appropriate potential, the model possesses both hairy black holes and non-topological solitons : boson stars. These latter exist in the standard Einstein-Klein-Gordon equations; it is shown that the coupling to the Gauss-Bonnet term modifies considerably their domain of classical stability.

1 Introduction

Attempts to escape the rigidity of the minimal Einstein-Hilbert formulation of gravity and the limited number of parameters describing its fundamental solutions – the black holes –, lead naturally physicists to emphasize enlarged models of gravity. Besides their purely Academic interests, these attempts are largely motivated nowadays by intriguing problems such as inflation, dark matter and dark energy.

One of the most popular class of extensions of Einstein gravity consists in the inclusion of scalar fields and appeals for natural interactions between the scalar fields and the geometry through higher curvature terms, leaving a lot of freedom. The general construction of scalar-tensor gravity leading to second order field equations was first obtained in [1]. Recently this theory was revived in the context of Galileon theory [2] and different extensions of it, see e.g. [3].

Apart from their cosmological implications, the extended models of gravity (by scalar or other types of fields) offer possibilities to escape the limitations of the no-hair theorems [4, 5] holding in standard gravity. In the last few years, black holes endowed by scalar hairs have attracted a lot of attention and have been studied in numerous theories. One particularly interesting result is the family of hairy black holes constructed in [6] within the Einstein gravity minimally coupled to a complex scalar field. In this case, the no-hair theorems [4, 5] are bypassed by the rotation of the black hole and the synchronization of the spin of the black hole with the angular frequency of the scalar field. Recent reviews on the topic of black holes with scalar hairs can be found e.g. in [7],[8],[9].

The general theory of scalar-tensor gravity [1], [3] contains a lot of arbitrariness and the study of compact objects such as black holes, neutron stars or boson stars needs to be realized in some particular cases. As an example, the truncation of the Galileon theory to a lagrangian admitting a shift-symmetric scalar field was worked out by Sotiriou and Zhou (SZ in the following) [10] and still leads to a large family of models. Hairy black holes were constructed perturbatively and numerically in the particular case of a scalar field coupled linearly to the Gauss-Bonnet invariant [11].

Abandonning the hypothesis of shift-symmetry, several groups [12], [13], [14] considered during the past years, new types of coupling terms between a scalar field and specific geometric invariants (essentially the Gauss-Bonnet term). In these models the occurrence of hairy black holes results from an unstable mode of the scalar field equation in the background of a vacuum metric (the probe limit). The interacting term of the scalar field with the curvature invariant plays a role of potential and the coupling constant the role of a spectral parameter. By continuity, the hairy black holes then exist as solutions of the full system. It is used to say that the hairy black holes appear through a *spontaneous scalarization* for a sufficiently large value of the coupling constant.

In the present paper we will consider a model of scalar-tensor gravity encompassing the theories presenting a spontaneous scalarization and the shift-symmetry property. Families of classical solutions whose pattern extrapolates smoothly between shift-symmetric hairy black holes and spontaneous scalarized ones will be constructed . The type of structure found holds when coupling the scalar field to the Gauss-Bonnet invariant and to the Einstein-Chern-Simons invariant as well. All black holes solutions found are supported by the non-minimal coupling between the scalar field and the curvature invariant; however the field equations admit other types of solutions: boson stars. These regular solutions exist with a minimal coupling of scalar field to gravity but it will be shown that the non minimal coupling has important consequences on their stability properties.

The paper is organized as follow : in Sect. 2 we present the model to be studied. Namely the Einstein-Klein-Gordon Lagrangian extended by a non-minimal coupling. We discuss the spherically symmetric ansatz and the general form of the field equations. Sect. 3 is devoted to the presentation of the hairy black holes occurring in the model. The boson stars are presented in Sect. 4 with an emphasis on the influence of the non-minimal coupling of the spectrum of the solutions. Conclusions are drawn in Sect. 5. Similar results hold for Einstein-Chern-Simons gravity and are the object of the Appendix; the activation of the Chern-Simons term is realized by means of a NUT charge [15].

2 The model

2.1 The action

We are interested in solutions of the field equations associated with the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi\mathcal{G}}R - \nabla_\mu\phi^*\nabla^\mu\phi - V(\phi) + f(\phi)\mathcal{I}(g)\right], \tag{2.1}$$

which extends the minimal Einstein-Klein-Gordon lagrangian. Here R is the Ricci scalar and ϕ represents a complex scalar field which – in some circumstances – will be chosen real. The usual Klein-Gordon kinetic term is supplemented by a potential $V(\phi)$ which will actually be chosen as a function of the combination $|\phi|^2 \equiv \phi\phi^*$ in order to ensure a $U(1)$ global symmetry for the scalar sector. In the following V will be set in the form

$$V(\phi) = m^2|\phi|^2 + \lambda_4|\phi|^4 + \lambda_6|\phi|^6 \tag{2.2}$$

which is used generically for obtaining Q-balls in the absence of gravity and boson stars when gravity is set in (see e.g. [16], [17] for reviews).

The gravity sector is supplemented by a non-minimal coupling between the scalar field and the geometrical invariant $\mathcal{I}(g)$. For this paper, we will be interested in the case where this invariant is the Gauss-Bonnet-scalar :

$$\mathcal{I}(g) = \mathcal{L}_{GB} \equiv R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}.$$

It is well known that this invariant is a total derivative in four dimensions but it will contribute non trivially to the equations of motion through the non-minimal coupling to the scalar field via $f(\phi)$. For the seek of generality, we have also investigated the case of a coupling to the Chern-Simons invariant, see Appendix A.

In order to preserve the $U(1)$ symmetry of the "usual" scalar sector, we will assume that, just like the potential, $f(\phi)$ is a function of $|\phi|$. In this paper, we will emphasize the effects of a coupling function of the form

$$f(\phi) = \gamma_1|\phi| + \gamma_2|\phi|^2 \tag{2.3}$$

where γ_1, γ_2 are independant coupling constants. Several forms of the function $f(\phi)$ have been emphasized in the literature where the scalar field is usually choosen real. The EGB theory with $\gamma_2 = 0$ and $V = 0$ corresponds to a shift-symmetric theory studied by SZ [10], the case $\gamma_1 = 0$ is considered in [12], [13]. [14]. Several choices of the function $f(\phi)$ have been considered in [18] and very recently in [19],[20]. Solutions with the form of $f(\phi)$ above with two independant constants γ_1, γ_2 was, to our knowledge, not yet investigated.

2.2 Equations of motion

The equations of motion (EOM) for the general action (2.1) read

$$G_{\mu\nu} = 8\pi\mathcal{G}\left(T^{(\phi)}_{\mu\nu} + T^{(\mathcal{I})}_{\mu\nu}\right) \tag{2.4}$$

for the metric function, and

$$-\Box\phi = -\frac{\partial V}{\partial\phi^*} + \frac{\partial f}{\partial\phi^*}\mathcal{I}(g) \tag{2.5}$$

for the scalar field. In these equations, $G_{\mu\nu}$ is the Einstein tensor and $\Box = \nabla_\mu\nabla^\mu$. The energy momentum $T^{(\phi)}_{\mu\nu}$ arise from the variation of the standard Klein-Gordon lagrangian :

$$T^{(\phi)}_{\mu\nu} = \nabla_{(\mu}\phi\nabla_{\nu)}\phi^* - \left(\nabla_\alpha\phi^*\nabla^\alpha\phi + V(\phi)\right)g_{\mu\nu} \quad . \tag{2.6}$$

Finally, $T^{(\mathcal{I})}_{\mu\nu}$ is the energy momentum tensor associated to the non-minimal coupling term[1] $f(\phi)\mathcal{I}(g)$.

From Eq.(2.5), one can see that the invariant $\mathcal{I}(g)$ will act as a source term for the scalar field. Consequently, if one find a space-time solution of the EOM such that $\mathcal{I}(g) \neq 0$, this solution will automatically present a non-trivial scalar field. This mechanism is known as "curvature induced scalarization".

2.3 The ansatz

2.3.1 Metric

We will be interested in spherically symmetric solutions. In this case, it is well known that (in the appropriate coordinate system) the metric can always be set in the form

$$\mathrm{d}s^2 = -N(r)\sigma^2(r)\mathrm{d}t^2 + \frac{1}{N(r)}\mathrm{d}r^2 + g(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \quad , \tag{2.7}$$

where θ and φ are the standard angles parameterising an S^2 with the usual range and r and t are the radial and time coordinates respectivelly.

The usual coordinate choice $g(r) = r^2$ will be used throughout this paper.

2.3.2 Scalar field

Within the same coordinate system, we choose a scalar field of the form

$$\phi(x^\mu) = e^{-i\omega t}\phi(r), \tag{2.8}$$

where ω, the frequency of the scalar field, is a real parameter and $\phi(r)$ a real function. The scalar field will be assumed to be real (*i.e.* $\omega = 0$) in the case of hairy black holes.

This choice above is motivated by the construction of boson stars. Indeed, it is well known [16] that boson stars exist as solutions of the minimal Einstein-Klein-Gordon equations provided the scalar field is chosen complex (typically of the form (2.8)) and supplemented by a mass term (or a more general potential (2.2)) in the equations.

[1] The expression of $T^{(\mathcal{I})}_{\mu\nu}$ is generically quite involved and depends on the explicit form of $\mathcal{I}(g)$. The expression of $T^{(\mathcal{I})}_{\mu\nu}$ for the case considered here can be found in [21] with the same notations.

2.3.3 Reduced equations

With the Ansatz (2.7)–(2.8), the equations (2.4)–(2.5) reduces to a system of three coupled differential equations (plus a constraint) for the radial functions N, σ and ϕ. Using suitable combinations of the equations, the system is amenable to the form

$$N' = F_1(N, \sigma, \phi, \phi') \quad , \quad \sigma' = F_2(N, \sigma, \phi, \phi') \quad , \quad \phi'' = F_3(N, \sigma, \phi, \phi') \tag{2.9}$$

where F_a, with $a = 1, 2, 3$, are involved algebraic expressions whose explicit form is not illuminating enough to be given.

2.3.4 Rescaling and units

In the coming discussion we will set $c = 1$ and $8\pi G = 1$. The equations are then invariant under the rescaling

$$r \to \lambda r \quad , m^2 \to \frac{m^2}{\lambda^2} \quad , \quad \lambda_{4,6} \to \frac{\lambda_{4,6}}{\lambda^2} \quad \gamma_{1,2} \to \lambda^2 \gamma_{1,2} \ , \tag{2.10}$$

where λ has the dimension of *length*$^{-1}$. These rescaled quantities will be used in the following. In the case of black holes we will use it to set the event horizon to unit (*i.e.* $r_h = 1$). In the case of boson stars (which have no horizon) we will set the mass parameter m to one ($m = 1$).

3 Hairy black holes

3.1 Boundary conditions

We now discuss the black holes solutions of the equations. As stated above these solutions exist for a real scalar field, so we set $\omega = 0$ in the equations. Let us first consider the solutions occuring in the absence of potential (*i.e.* setting $m = \lambda_4 = \lambda_6 = 0$ in (2.2)); the influence of a mass term will be emphasized separately, see Sect. 3.2.3.

For black holes, the metric is required to present a regular horizon at $r = r_h$, *i.e.* $N(r_h) = 0$. The occurence of this condition in the equations and the requirement of a regular function $\phi(r)$ at the horizon implies a non trivial relation for the scalar function and its derivative at $r = r_h$. The two conditions at the horizon are summarized as follows

$$N(r_h) = 0 \quad , \quad \phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8 r_h(\gamma_1 + 2\gamma_2 \phi(r_h))} \quad , \quad \Delta = r_h^4 - 96\gamma_1^2 - 384(\gamma_2^2 \phi(r_h)^2 + \gamma_1 \gamma_2 \phi(r_h)) \tag{3.11}$$

Remark that $\Delta \geq 0$ constitutes a necessary condition for solutions to exist. We will see in the next section that it largely determines the domain of the coupling constants γ_1,γ_2 for which solutions exist. The requirement for the solutions to be asymtotically Minkowski further implies

$$\sigma(r \to \infty) = 1 \quad , \quad \phi(r \to \infty) = 0 \ . \tag{3.12}$$

The four conditions (3.11)-(3.12) constitute the boundary values of the field equations. The black holes can be characterized by their mass M and the scalar charge Q_s. These are related respectively to the asymptotic decay of the functions $N(r)$ and $\phi(r)$:

$$N(r) = 1 - \frac{2M}{8\pi r} + O(1/r^2) \quad , \quad \phi(r) = \frac{Q_s}{r} + O(1/r^2) \ . \tag{3.13}$$

The entropy $S = \pi r_h^2$ and temperature $T_H = \sigma(r_h) N'(r_h)/(4\pi)$ characterize the solutions at the horizon. Using the equations the temperature can further be specified :

$$N'(r_h) = \frac{1}{r_h + 4\phi'(r_h)(\gamma_1 + 2\gamma_2 \phi(r_h))} \ . \tag{3.14}$$

Because the equations do not admit closed form solutions, we solved the system by using the numerical routine COLSYS [22] which is well adapted for the problem at hand. It is based on a collocation method for boundary-value differential equations and on damped Newton-Raphson iterations. The equations are solved with a mesh of a few hundred points and relative errors of the order of 10^{-6}. The values M, Q_S, S, T_H can be extracted with such an accuracy from the numerical datas.

3.2 Numerical results

3.2.1 Fundamental branch

We now present the pattern of solutions in the γ_1, γ_2 parameter space. Practically, we start from the hairy black holes constructed in [10], *i.e.* the shift-symmetric theory, corresponding to $\gamma_2 = 0$. A pair of solutions exist for $\gamma_1 \leq \sqrt{1/96} \approx 0.1021$ (with our convention of the non minimal coupling); characterized by the sign $\pm$ appearing in the condition (3.11). We will essentially focus on the family of solutions corresponding to the "+" sign which, in the limit $\gamma_1 \to 0$, smoothly approach the Schwarschild solution. Solutions corresponding to the "–" sign can be constructed as well (see e.g. [23]), forming a second branch whith higher mass. This branch, however is difficult to construct numerically. Moreover no regular solution can be associated to the $\gamma_1 \to 0$ limit for this branch since the value $\phi'(r_h)$ in (3.11) clearly diverge in this case ($\gamma_2 = 0$) for the "$-$" sign. The understanding of this branch is then not aimed in the present paper.

For a fixed value of the parameter γ_1, the SZ solution can be deformed by increasing (or decreasing) gradually the coupling constant γ_2. The pattern of hairy black holes obtained in this way turns out to be quite different for the small values of γ_1 (say for $\gamma_1 \leq 0.005$) and for $0.005 < \gamma_1 < \sqrt{1/96}$. For definiteness let us first discuss the family of black holes corresponding to $\gamma_1 > 0.005$.

(i) Increasing gradually the coupling constant γ_2, it turns out that the value Δ approaches zero at some critical value, say $\gamma_{2,c}$. Accordingly, no solution exist for $\gamma_2 > \gamma_{2,c}$. This is illustrated on Fig. 1 where the quantities Δ (solid lines) and $\phi(r_h)$ (dashed lines) are plotted as functions of γ_2 for two values of γ_1 (see the purple and red lines). The corresponding values of the mass and of $\phi'(r_h)$ is presented on both sides of Fig. 2.

(ii) In the case $\gamma_2 < 0$, a Schwarzschild metric can be approached arbitrarily close, although not exactly. This is due to the fact that the scalar field never reaches $\phi(r) = 0$ due to the presence of the non-homogeneous term in the scalar field equation. Indeed for the Schwarzschild black hole of mass M we have $\mathcal{L}_{GB} = 48M^2/r^6$.

The deformation of the SZ solutions in the region $\gamma_1 \leq 0.005$ for $\gamma_2 \neq 0$ leads to a richer pattern. For a fixed value of $\gamma_1 \leq 0.005$:

(a) Starting from the shift-symmetric solution ($\gamma_2 = 0$) and increasing $\gamma_2 > 0$, we find that the SZ black holes forms a "first branch" of solutions which exists up to a maximal value, say for $\gamma_2 \leq \gamma_{2,\max}$.

(b) Then, decreasing γ_2 from $\gamma_{2,\max}$, a "second branch" of solutions exists for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,\max}]$. As before, the value $\gamma_{2,c}$ coincide with $\Delta = 0$ and the two branches coincide in the limit $\gamma_2 \to \gamma_{2,\max}$. Fig. 1 illustrates this phenomenon for $\gamma_1 = 0.0005$ (see the blue line; in this case we find $\gamma_{2,c} \approx 0.172$ and $\gamma_{2,\max} \approx 0.177$). We note that, on the interval of γ_2 where the two solutions coexist, the solution of the first branch has a lower mass than the corresponding solution on the second branch.

(c) For $\gamma_2 < 0$, while decreasing γ_2, the black holes approach a Schwarzschild metric in the same way as point (ii) above.

To summarize, fixing low enough values of γ_1 and varying $\gamma_2 > 0$, the SZ solution deforms into a family of hairy black holes forming two branches which exist on specific intervals of γ_2. We can now emphasize how this ensemble behave when taking the limit $\gamma_1 \to 0$. It turns out that the solutions of the first branch approach uniformly the Schwarzschild black hole (irrespectively of γ_2). By contrast, the solutions of the second branch have a non trivial limit and approach the set of so called "spontaneously scalarized black holes" for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,\max}]$. These solutions were constructed directly in [12],[13],[14]. The critical values $\gamma_{2,c} \approx 0.1734$ and $\gamma_{2,\max} \approx 0.1814$ found in these papers fit very well with our numerical datas. The occurence of these critical values have different explanations :

(I) In the limit $\gamma_2 \to \gamma_{2,c}$, the parameter Δ (see (3.11)) approaches zero.

(II) In the limit $\gamma_2 \to \gamma_{2,\max}$, the scalar hairs tends uniformly to zero.

The value $\gamma_{2,\max}$ in fact corresponds to an eigenvalue of the scalar field equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 N(r)\frac{\mathrm{d}}{\mathrm{d}r}\phi\right] = \gamma_2 \frac{48M}{r^6}\phi \quad , \quad N(r) = 1 - \frac{2M}{r} \tag{3.15}$$

considered in the background of Schwarzschild solution. This value reflects a tachyonic instability of the Schwarzschild solution in the theory (2.1), opening the way for the vacuum solution to evolve into a hairy black hole. Details about the spectrum of this equation can be found, namely in [13], [21].

The question about the stability of our solutions raises naturally. In the case of the hairy black holes occuring by spontaneous scalarisation with a quadratic coupling to the Gauss-Bonnet term, it was shown in Refs. [19], [20] that the solutions present radial instabilities which can be removed when supplementing a quartic term in the coupling function. The study of the stability is not aimed in this paper; however we would like to argue about the (in)stability in the case where two solutions coexist with different masses on the interval $[\gamma_{2,c}, \gamma_{2,\max}]$ (see Fig. 2). It is likely that the branch with the lower mass which approaches the Schwarzschild metric in the limit $\gamma_1 \to 0$, is linearly stable while the branch with the higher mass, which approaches the spontaneously scalarized solutions, is unstable; this last statement being reinforced by a continuity argument applied to the results of [19], [20].

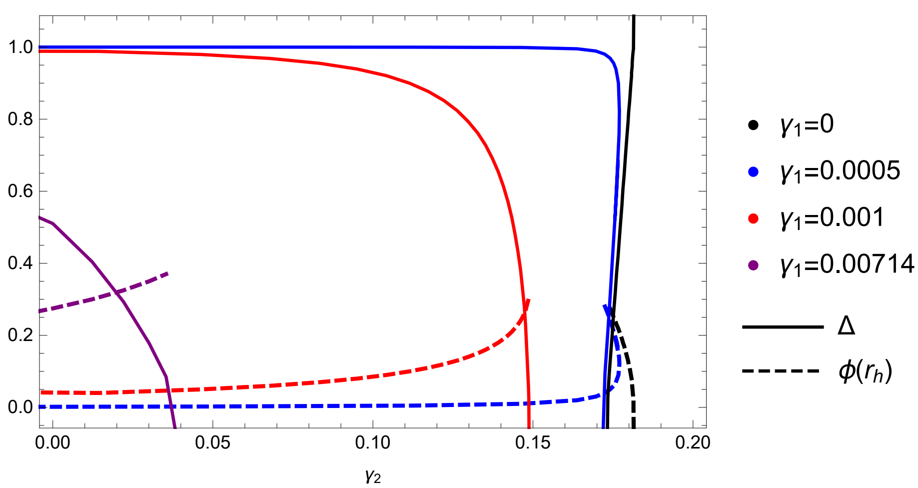

Figure 1: The parameter Δ (solid lines) and the value $\phi(r_h)$ (dashed lines) as functions of γ_2 for several values of γ_1.

3.2.2 Excited solutions

In the shift-symmetric case, *i.e.* with $\gamma_2 = 0$, the condition (3.11) drastically reduces the spectrum of hairy black holes. For each value of $\gamma_1 < 1/\sqrt{96}$ a single solution is allowed whith "+" sign (since $\phi'(r_h)$ does not depend on $\phi(r_h)$ but only on the fixed parameters γ_1 and r_h) and the scalar field is a monotonic function. Consequently, excited solutions (*i.e.* with $\phi(r)$ presenting nodes) do not occur. By contrast, for the spontaneously scalarized black holes (*i.e.* with $\gamma_1 = 0$), the linear equation (3.15) possesses – in

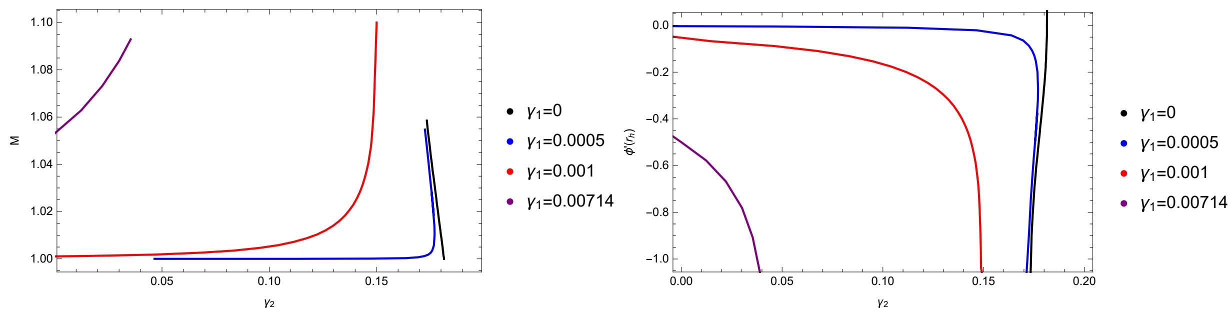

Figure 2: The mass as functions of γ_2 of the solutions of Fig.1. Right : Idem for the value $\phi'(r_h)$.

principle – a series of critical values of γ_2 corresponding to normalizable eigenfunctions $\phi(r)$ presenting one or more nodes. Any of these solutions leads to a branch of excited hairy black holes of the coupled system ($\gamma_1 \neq 0, \gamma_2 \neq 0$). We constructed numerically the branch corresponding to the first excited (or one-node) solution. Values Δ and $\phi(x_h)$ are reported on Fig. 3 as functions of γ_2 for a few values of γ_1 (the red lines correspond to $\gamma_1 = 0$). As for the fundamental (or no-node) solution discussed above, we see that the first excited hairy black holes exists for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,M}]$ where the lower (resp. upper) bound of this interval corresponds to $\Delta = 0$ (resp. to the second eigenvalue of (3.15)).

Switching on the parameter γ_1 leads to a deformation of these excited hairy black holes. The results of Fig. 3 suggest that the excited black holes exist only for $\gamma_2 \geq \gamma_{2,c}$. This contrasts drastically with the spectrum of fundamental solutions (see Fig.1). It is tempting to say that the fundamental solutions are "attracted" by the SZ solutions occurring in the $\gamma_2 = 0$ limit. Having no equivalent, the excited solutions exist only for large values of γ_2.

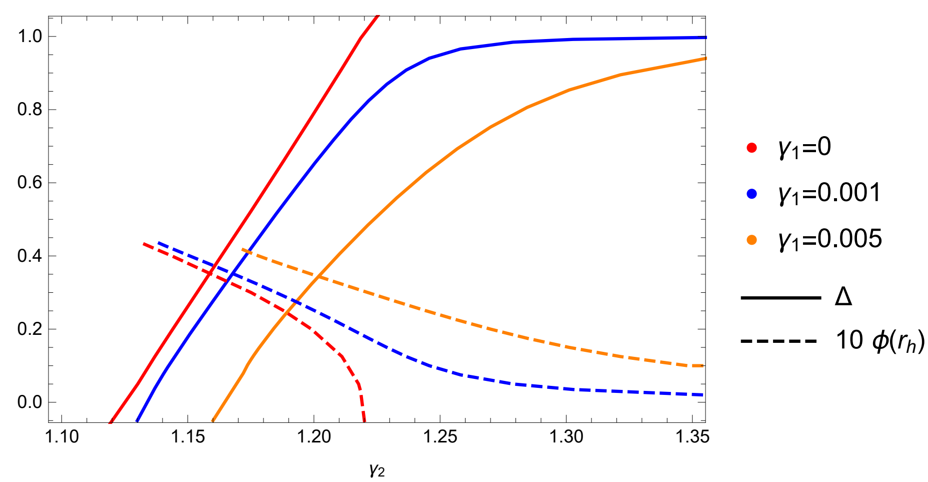

Figure 3: The value $\phi(r_h)$ and the quantity Δ as function of γ_2 for several values of γ_1.

3.2.3 Influence of a mass term

In the previous section, the scalar field ϕ was supposed to be massless. In this section, we discuss the effect of a massive scalar field on the spectrum of hairy black holes. For simplicity we restrict the presentation to the spontaneously scalarized solutions – *i.e.* setting $\gamma_1 = 0$ – and to the mass term only in the potential (2.2) – *i.e.* $\lambda_4 = \lambda_6 = 0$.

In the case of a massive scalar field, the regularity condition (3.11) is more involved :

$$\phi'(r_h) = \frac{-B \pm \sqrt{\Delta}}{2A} \tag{3.16}$$

with

$$A = -\phi_0(12\gamma_2 - m^2 r_h^2(r_h^2 + 8\gamma_2\phi_0^2) + 4\gamma_2 r_h^4 \phi_0^4) \quad , \quad B = 8\gamma_2\phi_0(r_h^2 - \phi_0^2(r_h^4 + 8\gamma_2 r_h^2 - 64\phi_0^2\gamma_2^2)) \quad , \tag{3.17}$$

$$\Delta = (1 - m^2\phi_0^2 r_h^2)^2 \left(r_h^2(r_h^4 - 384\gamma_2^2\phi_0^2) + 256 m^2\gamma_2^2\phi_0^4(r_h^4 + 12\gamma_2 r_h^2 - 96\gamma_2^2\phi_0^2) + 4096 m^4\gamma_2^4\phi_0^8 r_h^2 \right) \tag{3.18}$$

and we posed $\phi(r_h) = \phi_0$. The temperature of the black hole can be evaluated by using :

$$N'(r_h) = \frac{1 - m^2 r_h^2 \phi(r_h)^2}{r_h + 4\phi'(r_h)(\gamma_1 + 2\gamma_2\phi(r_h))} \tag{3.19}$$

instead of (3.14). This suggests that hairy black holes occuring from a massive scalar field can eventually be extremal. However for all values of m that we adressed (see Fig. 4), the parameter $\phi(r_h)$ remains too small for extremal black holes to form.

The numerical results reveals that the inclusion of a massive scalar field results in shifting the interval of existence in γ_2 to larger values, as demonstrated by Fig. 4. The critical phenomena limiting the interval of existence is of the same as discussed above. The shift to larger values of the interval of γ_2 while increasing

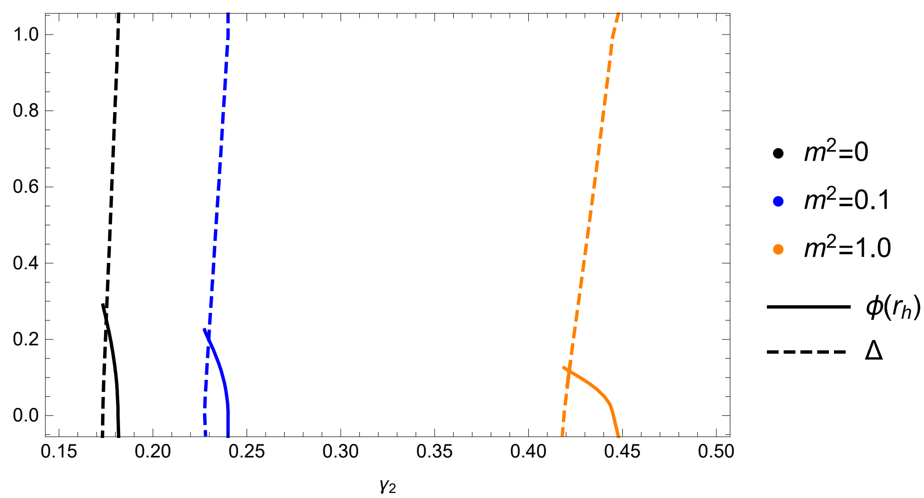

Figure 4: The value $\phi'(r_h)$ and the quantity Δ as function of γ_2 for several values of γ_1 for the solutions with $r_h = 1$.

m can be understood by examining the field equation of the scalar field. With the assumptions made in this section (for instance : a real scalar field, a mass term only and $\gamma_1 = 0$), (2.5) reads

$$-\Box\phi = 2\left(\gamma_2 \mathcal{I}(g) - m^2\right)\phi.$$

One can see that the mass act as a "negative shift constant" on the term $\gamma_2\mathcal{I}(g)$. For $m = 0$ the scalarised solutions appears only when the Gauss-Bonnet term becomes sufficiently important (*i.e.* when γ_2 is large enough to ensure the term $\mathcal{I}(g)$ to trigger the scalar field). It is then intuitive to assert that, since the mass just shift down the trigger term of the scalar field, higher values of γ_2 are needed to allow for spontaneous scalarisation.

4 Boson stars

As we mentionned in Sect. 2, it is well known (see e.g. [16]) that regular solutions – boson stars – exist within a large subclass of the lagrangian (2.1). Let us first specify the conditions :

- The scalar field is complex, of the form (2.8) with $\omega \neq 0$. Accordingly, the Lagrangian possesses a $U(1)$-global symmetry.
- The linear coupling to the Gauss-Bonnet term will be set to zero so $\gamma_1 = 0$ in (2.3). This is because we want to limit ourselves to a polynomial lagrangian in ϕ.
- The potential should contain at least a mass term, so $m > 0$ in (2.2).

Asymptotically, the functions $N(r), \phi(r)$ behave according to

$$N(r) = 1 - \frac{2M}{8\pi r} + O(1/r^2) \quad , \quad \phi(r) \propto \exp(-r\sqrt{m^2 - \omega^2}) \quad , \tag{4.20}$$

contrasting with (3.13). The exponential decay of the scalar field demonstrates the crucial role of frequency parameter ω and of the mass m; in particular boson stars exist for $\omega < m$. Beside the mass M, the solutions are further characterized by the Noether charge associated to the U(1)-symmetry of the Lagrangian. The Noether current and the calculation of the charge Q can be found in numerous papers (see e.g.[24]), for brevity we give the final form of the integral to be computed to evaluate the charge :

$$Q = 8\pi\omega \int \frac{r^2 \phi^2}{N\sigma} \mathrm{d}r. \tag{4.21}$$

This quantity is interpreted as the number of elementary bosons of mass m constituting the star [2].

The construction of boson stars is achieved by solving the field equations of Sect. 2.2 for $r \in [0, \infty]$. The regularity at the origin, the asymptotic flatness and the localization of the scalar field imply the following set of boundary conditions :

$$N(0) = 1 \quad , \quad \phi(0) = F_0 \quad , \quad \phi'(0) = 0 \quad , \quad A(r \to \infty) = 1 \quad , \quad \phi(r \to \infty) = 0 \tag{4.22}$$

which determine the boundary value problem. Practically, the value F_0 at the center is used as control parameter in the numerical resolution; the frequency ω has to be fine-tuned as a function of F_0 for all boundary conditions to be obeyed. The frequency ω, the mass M and the Noether charge Q can then be evaluated as functions of F_0.

4.1 Solutions without self-interaction

Let us first discuss the solutions for a pure mass potential (*i.e.* for $\lambda_4 = \lambda_6 = 0$ in (2.2)). The minimally coupled boson stars (*i.e.* corresponding to $\gamma_2 = 0$) exist on a finite interval of the frequency ω/m, that is to say for $\omega/m \in [\omega_{\min}/m, 1.0]$ with $\omega_{\min}/m \approx 0.76$. The plot of the mass M versus ω presents the form of a spiral as shown by the red line on Fig. 5. From this pattern, it results that two or more solutions can exist with the same frequency on specific sub-intervals of ω. The vacuum (*i.e.* Minkowski space-time) is approached for $F_0 \to 0$ which coincides with the limit $\omega/m \to 1$. The phenomenon limiting the boson stars in the center of the spiral is the following : while increasing F_0 the effects of gravity get stronger at the center of the lump, in particular the value $\sigma(0)$ decreases, finally approaching zero. Correspondingly the value $R(0)$ of the Ricci scalar gets arbitrarily large and a configuration with a singularity at the center is approached.

We now discuss the influence of the non-minimal coupling (*i.e.* with $\gamma_2 > 0$) on this pattern. A look at Fig. 5 reveals that the M versus ω curve has the tendency to unwind for $\gamma_2 > 0$ and that the boson stars exist on a larger interval of ω. The nature of the phenomenon limiting the curves corresponding to $\gamma_2 > 0$ on

[2] One formal analogy can be made with the total charge Q_{EM} of a system of N particles of electric charge q. In such a case the number of components is obtained via the relation $N = Q_{EM}/q$.

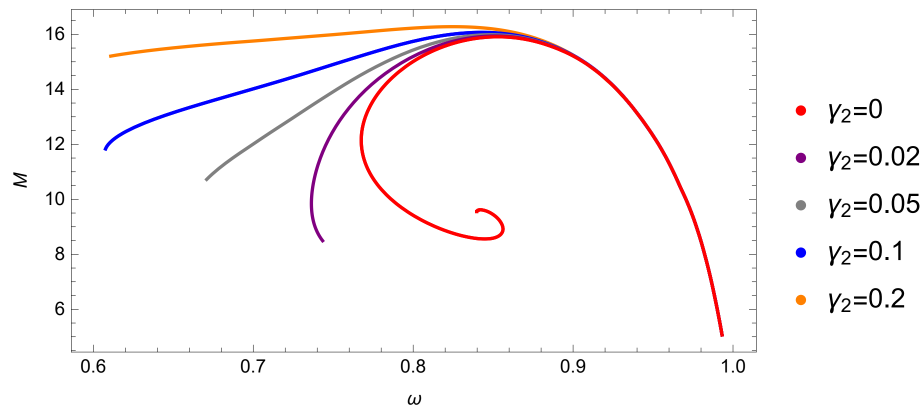

Figure 5: The mass of the boson star as a function ω for no-selfinteraction and for different values of γ_2.

Fig. 5 is different from the case $\gamma_2 = 0$ mentionned above. Denoting $D(r)$ the denominator of the function F_3 in (2.9), it turns out that the values $\sigma(0)$, $D(0)$ both decrease when F_0 increases. However the numerical results strongly indicate that $D(0)$ tends to zero much quicker than $\sigma(0)$ once $\gamma_2 > 0$. This statement is hard to demonstrate because the numerical integration of the equations becomes particularly tricky in this limit. Within the coordinate system used both the numerator and denominator entering in F_3 become quite large in a region of the interval of integration and the accuracy of the numerical solution get lost. The situation is illustrated on Fig. 6 where the pattern of the solutions is shown in the $\omega - \sigma(0)$ plane (left-figure) and in the $\omega - \frac{D(0)}{D(\infty)}10^9$ plane (right-figure). In this plot, the quantity $D(r)$ has been normalized with respect to $D(\infty)$ in order to compare the curves for the different values of γ_2 considered. The logarithmic scale used on the vertical axis of the right plot illustrates the huge variation of $D(r)$ while approaching the critical configuration. The two plots confirm that for, $\gamma_2 \neq 0$, the limit of existence of the boson stars is related to the behaviour of $D(0)$, rather than to $\sigma(0)$ whose values remain finite.

Note that the unwinding phenomenon of the $\omega - M$ relation seems to be closely related to the Gauss-Bonnet term. It was first observed in the construction of boson stars in Einstein-Gauss-Bonnet gravity in five dimensions [25]. In this case, the Gauss-Bonnet term is fully dynamic and does not need coupling to extra field.

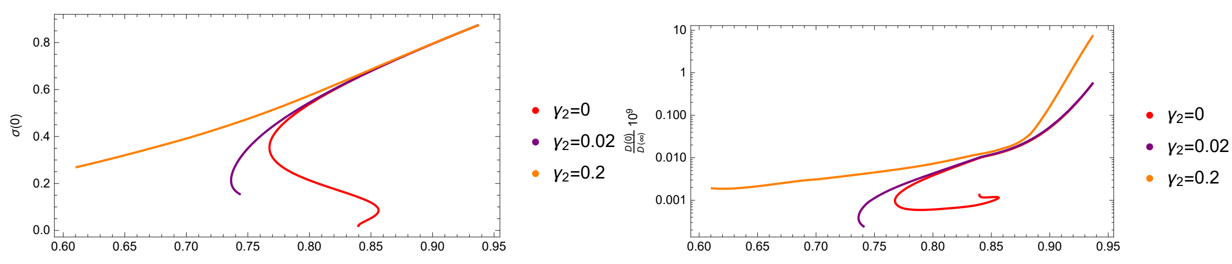

Figure 6: The value $\sigma(0)$ as function of ω for boson stars and three values of γ_2. Right : Idem for the discriminant of the system of equations.

4.2 Effect of a self-interacting term

Because the self-interacting potential depends on two independant parameters (namely λ_4, λ_6), we limited the investigation to the potential of the form : $V = \phi^2(1-\phi^2)^2$. Presenting three degenerate vacua ($\phi = 0, \pm 1$), this potential offers rich possibilities for topological solitons [26]. Recently it was used in [27] for the study of kink-anti-kink collisions in 1+1 dimensions and in [28] to study boson stars in 3+1 dimensions.

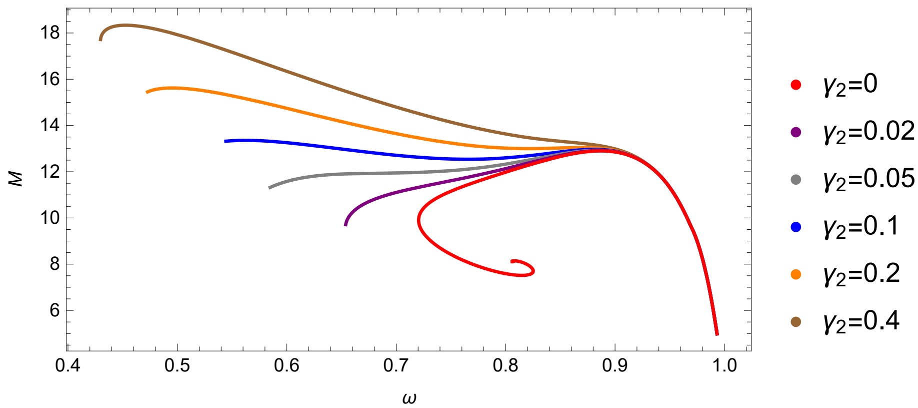

Figure 7: The mass of the boson star as a function ω for a self-interacting potential.

The general effect of the self-interacting potential on the solutions is that the interval of frequencies of the boson stars is significantly larger that for the mass potential. Especially the minimal value $\omega_{\rm min}$ is systematically lower (e.g. $\omega_{\rm min} \approx 0.72$ for $\gamma_2 = 0$). These features are illustrated by Fig. 7, to be compared with Fig. 5. The unwinding feature of the mass-frequency graphic occurring for the mass potential also takes place when the self-interaction is present. The minimal value $\omega_{\rm min}$ is again systematically lower although remaining strictly positive. The combination of self-interacting scalar field and non-minimal coupling to the Gauss-Bonnet term is therefore not suitable to allow for purely real soliton solutions.

One can also note that, in the presence of the self-interaction, an increase of γ_2 can lead to solutions with drastically higher mass compared to solutions with $\gamma_2 = 0$ (compare, for exemple, the difference between the curves corresponding to $\gamma_2 = 0$ (red) and $\gamma_2 = 0.2$ (orange) on Fig. 5 and Fig. 7).

4.3 Classical stability

We now address the stability of the boson stars by invoking a “classical argument”. With the interpretation of Q as the number of bosons of mass m in the lump, it is natural to compare the quantity mQ to the total mass of the solution M. If $M < mQ$, the total mass of the boson star is lower than the sum of its components, *i.e.* the total energy of the system is lower than the energy corresponding to Q “free” bosons. In such a case, as for the mass defect in atoms, we will say that the system is stable, in the sense that the Q bosons can’t exist in a “free” form but have to be bounded within the star. Following the same lines, the case $M > mQ$ will correspond to unstable configurations (remember $m = 1$ as fixing our scale, see section 2.3.4).

The quantity M/Q is reported as a function of ω on Fig. 8 for several values of γ_2. The left part of Fig. 8 characterizes solutions with the mass term only. We see that the solutions emerging from the vacuum limit (*i.e.* $\omega/m = 1$) are classically stable and remain so for sufficiently high values of ω, say for $\omega \geq \omega_s$ where ω_s is such that $M/Q = 1$. For values of ω such that several solutions coexist, the most massive is the most stable.

The most interesting result concern the influence of γ_2 on this pattern. As one can see on the plot, ω_s decreases when γ_2 increases while, for fixed ω, M/Q decreases when γ_2 increases. Consequently, the presence of the interaction between the scalar field and the geometry enhance the stability of the solutions.

This feature remains qualitatively the same for self-interacting solutions as illustrated on the right part of the figure. The presence of the self-interaction reinforces the effects of the curvature and the boson stars are even more stable compared to non-self-interacting ones. For suffcently high values of γ_2 (say for $\gamma_2 > 0.075$), the whole set of solutions is classically stable.

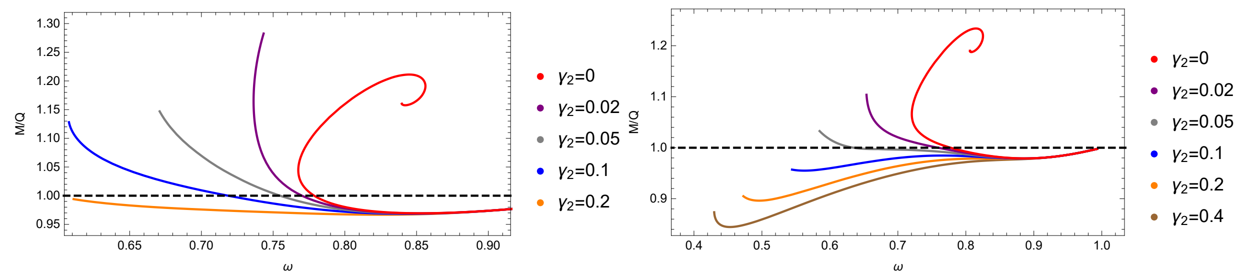

Figure 8: Left : The quantity M/Q as function of ω for several values of γ_2 for solutions without self-interaction. Right : Idem for self-interacting solutions.

5 Conclusion

The investigation for hairy black holes in gravity extended by a Gauss-Bonnet term coupled to a scalar field was a source of huge activity over the past years. In particular the stability of such objects was examined in details in [19],[20]; the construction of such black holes in the presence of a cosmological constant was reported in [31]. The coupling function of the scalar field to the Gauss-Bonnet term is, up to now, left as an arbitrary freedom but its form lead to different patterns for the solutions and turns out to be important for the stability of the hairy black holes.

In this paper we considered as coupling a superposition of the linear and quadratic powers of the scalar field. While spontaneoulsy scalarized black holes – with purely quadratic coupling constant γ_2 – appear on a very limited interval of the coupling constant γ_2, we showed that, when adding a linear part (even with small coupling γ_1), two branches of hairy black holes exist. One of these branches is very close to the spontaneously scalarized black holes while the second extend backward to a solution with shift-symmetric scalar field. This feature is specific for the fundamental solutions and is not repeated for excited solution (*i.e.* with scalar field presenting nodes) .

Extending the scalar sector of scalar-tensor gravity to a massive, complex field, we were also able to construct boson star solutions in the full theory. The qualitative and quantitative effects of the Gauss-Bonnet term have been reported in details in Sect. 4 revealing, for instance, that the presence of the quadratic coupling constant γ_2 can drastically increase the maximal mass of these objects and the range of ω (the frequency of the complex scalar field) for which these solutions exist. In this context, we also show that the critical phenomenon limiting the existence of solutions is different in the minimally and non-minimally coupled case. Interestingly, our results demonstrate that the coupling to the Gauss-Bonnet invariant and/or the inclusion of a self-interacting potential of the scalar field enhances the domain of classical stability of the boson stars.

Finally, in the Appendix we studied the solutions for scalar-tensor gravity extended by the same kind of coupling of the scalar field to the Chern-Simons invariant. Here the space-time is endowed with a NUT charge. The pattern of Nutty-Hairy black holes is qualitatively similar to the Gauss-Bonnet case.

Appendix A : Coupling to the Chern-Simons invariant

In this appendix, we provide an analysis of hairy black holes in the model (2.1) where the curvature invariant is chosen as the Chern-Simons-scalar :

$$\mathcal{I}(g) = \mathcal{L}_{CS} \equiv R\tilde{R} = {}^*R^a{}_b{}^{cd} R^b{}_{acd} \quad , \quad {}^*R^a{}_b{}^{cd} \equiv \frac{1}{2}\eta^{cdef} R^a{}_{bef}$$

where ${}^*R^a{}_b{}^{cd}$ is the Hodge dual of the Riemann-tensor, $\eta^{cdef} = \epsilon^{cdef}/\sqrt{-g}$ the 4-dimensional Levi-Civita tensor and ϵ^{cdef} the Levi-Civita tensor density.

The construction of classical solutions with a non-trivial Chern-Simons term can be performed by enforcing rotations in the metric or by endowing Space-Time with a NUT charge. Nutty-Hairy black holes in Einstein-Chern-Simons gravity were constructed in [29] and [21] for $\gamma_2 = 0$ and $\gamma_1 = 0$ respectively. Similar solutions within Einstein-Gauss-Bonnet (rather than Chern-Simons) gravity were obtained in [23]. The field equations are given by (2.4) and (2.5) with a different expression of $T^{(\mathcal{I})}_{\mu\nu}$ which can be found in [21] with the same notations as in Sect. 2.1.

The ansatz

To construct the solutions we use a metric of the form

$$\mathrm{d}s^2 = -N(r)\sigma^2(r)(\mathrm{d}t + 2n\cos\theta\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{N(r)} + g(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \ ,$$

generalizing the Schwarzschild-NUT solution. Here θ and φ are the standard angles on S^2 with the usual range while r and t are the "radial" and "time" coordinates respectivelly. The NUT parameter n appears as a coefficient in the differential form $\mathrm{d}t + 2n\cos\theta\mathrm{d}\varphi$ (note that $n \geqslant 0$, without any loss of generality). When evaluated with this metric, the Chern-Simons density $\mathcal{L}_{CS}$ is actually proportional to the NUT charge; so it vanishes identically for spherically symmetric solutions ($n = 0$) but becomes non trivial for $n \neq 0$, ensuring a non-trivial behaviour of the scalar field via the curvature induced scalarization only for $n \neq 0$.

In the decoupling limit $\gamma_1 = \gamma_2 = 0$ (implying $\phi = 0$), the functions $N(r), \sigma(r)$ and $g(r)$ are known explicitely :

$$N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2} \ , \qquad \sigma(r) = 1 \ , \qquad g(r) = r^2 + n^2 \ .$$

This metric therefore possesses an horizon at

$$r_h = M + \sqrt{M^2 + n^2} > 0.$$

As in the Schwarzschild limit, $N(r_h) = 0$ is only a coordinate singularity where all curvature invariants are finite. In fact, a nonsingular extension across this null surface can be found [30]. Completing the metric (Appendix A.23), the ansatz for the scalar field is the same as Eq. (2.8).

Numerical results

In the same spirit as in the main part, we have constructed the black hole solutions in the Einstein-Chern-Simons (ECS) model with the mixed coupling (2.1) and using a Nutty space-time in order to make the Chern-Simons term non trivial.

For generic values of γ_1, γ_2, no explicit solution can be found and, again, we relied on a numerical technique. For the construction, we used the gauge $\sigma(r) = 1$. Then the Einstein-Chern-Simons equations can be transformed into a system of three coupled differential equations of the second order for the functions $N(r), g(r)$ and $\phi(r)$. The desired asymptotic form of the solutions require

$$N(r \to \infty) = 1 \ \ , \ \ \sigma(r \to \infty) = 1 \ \ , \ \ \phi(r \to \infty) = \frac{Q_s}{r}$$

where Q_s is the scalar charge. Imposing an horizon $r = r_h$, *i.e.* $N(r_h) = 0$, the conditions of regularity of the solution at the horizon can be determined on the first few coefficients of the Taylor expansion :

$$N(r) = N_1(r-r_h)+O((r-r_h)^2) \ \ , \ \ g(r) = g_0+g_1(r-r_h)+O((r-r_h)^2) \ \ , \ \ \phi(r_h) = \phi_0+\phi_1(r-r_h)+O((r-r_h)^2)$$

Two conditions are finally necessary :

$$g'(r_h) = \frac{1}{N_1}(2 - g_0\phi_0 m^2 - 2nN_1\phi_1(\gamma_1 + 2\gamma_2\phi_0)) \ \ ,$$

$$24\gamma_2\phi_0^2\phi_1(N_1)^3 + N_1(2\gamma_2 n g_0 m^2\phi_0^3 - 12\gamma_2 n\phi_0 - g_0^2\phi_1) + g_0^2\phi_0 m^2 = 0.$$

The pattern of the solutions found for the ECS case is very similar to the case of EGB. In particular, the solutions available for non-zero values of γ_1, γ_2 smoothly extrapolate between the limits $\gamma_1 = 0$ and $\gamma_2 = 0$ found in [21] and [29]. The results are summarized on Fig. 9 for $n = 0.1$ but we have checked that they are qualitatively similar for different values of n.

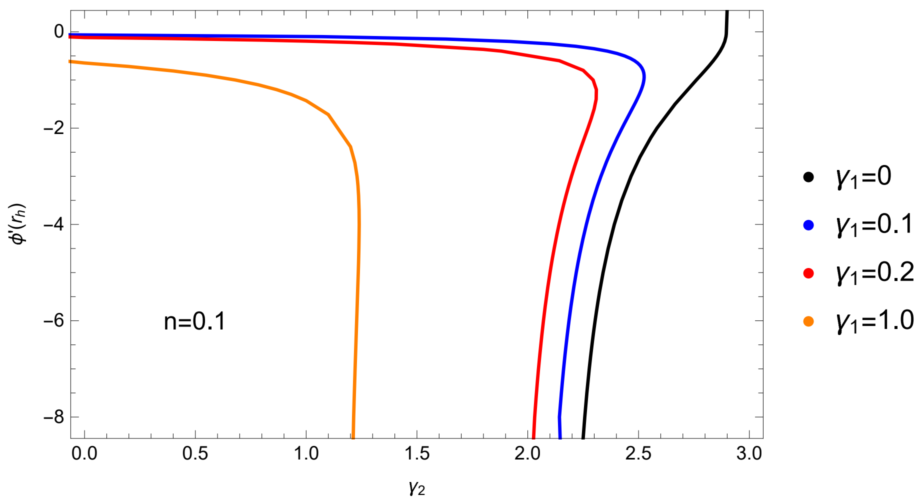

Figure 9: The value $\phi'(r_h)$ as function of γ_2 for several values of γ_1 for the solutions with $r_h = 1$ and $n = 0.1$.

References

[1] G. W. Horndeski, Int. J. Theor. Phys. **10** (1974) 363.

[2] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. **D 79** (2009) 064036.

[3] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D **84** (2011) 064039 [arXiv:1103.3260 [hep-th]].

[4] J. D. Bekenstein, Phys. Rev. Lett. **28** (1972) 452; C. Teitelboim, Lett. Nuovo Cim. **3S2** (1972) 397.

[5] J. D. Bekenstein, Phys. Rev. **D 51** (1995) no.12, R6608.

[6] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112** (2014) 221101. [arXiv:1403.2757 [gr-qc]].

[7] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D **24** (2015) no.09, 1542014 [arXiv:1504.08209 [gr-qc]].

[8] T. P. Sotiriou, Class. Quant. Grav. **32** (2015) no.21, 214002 [arXiv:1505.00248 [gr-qc]].

[9] M. S. Volkov, arXiv:1601.08230 [gr-qc].

[10] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. Lett. **112** (2014) 251102 [arXiv:1312.3622 [gr-qc]].

[11] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. D **90** (2014) 124063 [arXiv:1408.1698 [gr-qc]].

[12] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120** (2018) no.13, 131103 [arXiv:1711.01187 [gr-qc]].

[13] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. **120** (2018) no.13, 131104 [arXiv:1711.02080 [gr-qc]].

[14] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. **120** (2018) no.13, 131102 [arXiv:1711.03390 [hep-th]].

[15] E. T. Newman, L.Tamburino and T. Unti, J. Math. Phys. **4** (1963) 915.

[16] P. Jetzer, Phys. Rep. **220** (1992) 163.

[17] F. E. Schunck and E. W. Mielke, Class. Quant. Grav. **20** (2003) R301. [arXiv:0801.0307 [astro-ph]].

[18] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. D **97** (2018) no.8, 084037. [arXiv:1711.07431 [hep-th]].

[19] M. Minamitsuji and T. Ikeda, Phys. Rev. D **99** (2019) no.4, 044017. [arXiv:1812.03551 [gr-qc]].

[20] H. O. Silva, C. F. B. Macedo, T. P. Sotiriou, L. Gualtieri, J. Sakstein and E. Berti, "On the stability of scalarized black hole solutions in scalar-Gauss-Bonnet gravity," arXiv:1812.05590 [gr-qc].

[21] Y. Brihaye, C. Herdeiro and E. Radu, Phys. Lett. B **788** (2019) 295 arXiv:1810.09560 [gr-qc].

[22] U. Ascher, J. Christiansen, R. D. Russell, Math. Comp. **33** (1979) 659;
U. Ascher, J. Christiansen, R. D. Russell, ACM Trans. **7** (1981) 209.

[23] A. Brandelet, Y. Brihaye, T. Delsate and L. Ducobu, Int. J. Mod. Phys. A **33** (2018) no.32, 1850189. [arXiv:1705.06145 [gr-qc]].

[24] B. Kleihaus, J. Kunz and M. List, Phys. Rev. D **72** (2005) 064002. [gr-qc/0505143].

[25] B. Hartmann, J. Riedel and R. Suciu, Phys. Lett. B **726** (2013) 906. [arXiv:1308.3391 [gr-qc]].

[26] M. A. Lohe, Phys. Rev. D 20 (1979) 3120.

[27] P. Dorey, K. Mersh, T. Romanczukiewicz and Y. Shnir, Phys. Rev. Lett. **107** (2011) 091602. [arXiv:1101.5951 [hep-th]].

[28] Y. Brihaye, A. Cisterna, B. Hartmann and G. Luchini, Phys. Rev. D **92** (2015) no.12, 124061. [arXiv:1511.02757 [hep-th]].

[29] Y. Brihaye and E. Radu, Phys. Lett. B **615** (2005) 1 [gr-qc/0502053].

[30] S. W. Hawking and G. F. R. Ellis, "The Large Scale Structure of Space-Time," doi:10.1017/CBO9780511524646

[31] A. Bakopoulos, G. Antoniou and P. Kanti, "Novel Black-Hole Solutions in Einstein-Scalar-Gauss-Bonnet Theories with a Cosmological Constant," arXiv:1812.06941 [hep-th].

Appendix

E Boson and Neutron Stars with Increased Density

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Boson and neutron stars with increased density

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Abstract

We discuss boson stars and neutron stars, respectively, in a scalar-tensor gravity model with an explicitly time-dependent real scalar field. While the boson stars in our model – in contrast to the neutron stars – do not possess a hard core, we find that the qualitative effects of the formation of scalar hair are similar in both cases : the presence of the gravity scalar allows both type of stars to exist for larger central density as well as larger mass at given radius than their General Relativity counterparts. In particular, we find new types of neutron stars with scalar hair which have radii very close to the corresponding Schwarzschild radius and hence are comparable in density to black holes. This new branch of solutions is stable with respect to the decay into individual baryons.

1 Introduction

With increased interest in astrophysical objects and, in particular, their gravitational properties, compact objects have come to the focus of theoretical research again. These objects are normally defined to have strong gravitational fields and as such are a good testing ground for the momentarily accepted best model of the gravitational interaction – General Relativity (GR) – as well as extensions thereof and even alternative gravity models. Compact objects come in two varieties: either they are star-like with a globally regular space-time or they possess a physical singularity shielded from observation by an event horizons. The former are neutron stars and boson stars, respectively, the latter black holes. While neutron stars and black holes are known to exist and can now be studied with unprecedented precision, boson stars [1] are hypothetical objects made principally of scalar bosonic particles. Evaluating and testing gravity theories is also vital in order to understand two of the great puzzles of current day physics : the nature of dark matter and dark energy. While dark matter is understood to be some kind of matter that interacts only gravitationally and probably has its origin in physics beyond the Standard Model of Particle physics, the nature of dark energy remains elusive. Consequently, suggestions for a modification of GR have been made on the ground of so-called "scalar-tensor" gravity models [2, 3, 4], an idea that relates back to Horndeski [5]. Classes of scalar-tensor gravity models have then been studied thoroughly and a classification, named "Fab Four", was achieved [6, 7]. In this paper, we are interested in a particular model dubbed "John" in this exact classification. As has been shown in [8], static, spherically symmetric black holes can carry scalar hair in this model if the scalar field is explicitly (and linearly) time-dependent. In particular, the Noether current associated to the shift symmetry of the Galileon-type gravity scalar does not diverge on the horizon in this model. In [10], neutron stars have been studied for a specific polytropic equation of state and it has been claimed that the astrophysical objects resulting from the model are viable and not in conflict with constraints from observations. Here, we revisit these results and compare them with those related to another EOS used in [11]. We find that the solutions obtained with the EOS of [11] (a) are in perfect agreement with results obtained in [11] and (b) only this EOS leads to neutron stars possessing the proper mass-radius relation. While neutron stars are matched to the Schwarzschild solution at the exterior radius, we also discuss boson stars in this paper that reach the Schwarzschild solution only asymptotically and hence do not possess a “hard core”.

Our paper is organized as follows: in Section 2 we discuss the scalar-tensor gravity model coupled to an appropriate energy-momentum content. In Section 3, we present our results for boson stars, while Section 4 contains our findings for neutron stars. We summarize and conclude in Section 5.

2 The model

In this paper, we present our results for a scalar-tensor gravity model of Horndeski type coupled minimally to an appropriate matter content with Lagrangian density $\mathcal{L}_{\rm matter}$. The action reads :

$$\mathcal{S} = \int \left(\kappa\mathcal{R} + \frac{\eta}{2}G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \mathcal{L}_{\rm matter}\right)\sqrt{-g}{\rm d}^4x \ , \tag{1}$$

where $\kappa = (8\pi G)^{-1}$. This action contains the standard Einstein-Hilbert term as well as a non-miminal coupling term – first discussed in [6, 7] – that couples a gravity scalar ϕ to the Einstein tensor $G_{\mu\nu}$ via a coupling constant η. For $\eta = 0$, we recover standarad General Relativity (GR).

In the following, we will assume the matter content of the model to be that of (a) a complex valued scalar field and (b) a perfect fluid with a given equation of state, respectively. In the latter case, the model has solutions in the form of neutron stars, while the complex scalar field in curved space-time describes boson stars. The gravity equations then read

$$\kappa G_{\mu\nu} + \eta\left(\partial_\alpha\phi\partial^\alpha\phi G_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\alpha\sigma\rho}R^{\sigma\rho\gamma\delta}\epsilon_{\nu\beta\gamma\delta}\nabla^\alpha\phi\nabla^\beta\phi + g_{\mu\alpha}\delta^{\alpha\rho\sigma}_{\nu\gamma\delta}\nabla^\gamma\nabla_\rho\phi\nabla^\delta\nabla_\sigma\phi\right) = T_{\mu\nu} \ , \tag{2}$$

where $T_{\mu\nu}$ denotes the energy-momentum tensor of the matter content. The model has a shift symmetry $\phi \to \phi + c$, where c is a constant, which leads to the existence of a locally conserved Noether current

$$J^\mu = -\eta G^{\mu\nu}\nabla_\nu\phi \ \ , \ \ \nabla_\mu J^\mu = 0 \ . \tag{3}$$

In the following, we will assume a spherically symmetric Ansatz for our solutions [8]

$${\rm d}s^2 = -b(r){\rm d}t^2 + \frac{{\rm d}r^2}{f(r)} + r^2\left({\rm d}\theta^2 + \sin^2\theta{\rm d}\varphi^2\right) \ , \ \phi(t,r) = qt + F(r) \ , \tag{4}$$

i.e. the tensor part is static, while the gravity scalar has an explicit time-dependence. The non-vanishing components of the Noether current (3) then read

$$J^t = \eta q\frac{f'r + f - 1}{r^2 b} \ \ , \ \ J^r = \eta\phi'\frac{f\left(-b'rf - bf + b\right)}{r^2 b} \ \ , \tag{5}$$

where the prime now and in the following denotes the derivative with respect to r. The norm of the Noether current is

$$J_\mu J^\mu = \eta^2\left[-q^2\frac{(f'r + f - 1)^2}{r^4 b} + \phi'^2\frac{f\left(b'rf + bf - b\right)^2}{r^4 b^2}\right] \ . \tag{6}$$

Since $f(r \ll 1) \sim 1 + f_2 r^2$ and $b(r \ll 1) \sim 1 + b_2 r^2$ with f_2, b_2 constants (see below for explicit expressions), the norm of the Noether current is finite for all $r \in [0 : \infty)$.

We want to consider a non-vanishing energy-momentum tensor that sources the tensor and scalar gravity fields. In the following, we will choose the energy-momentum tensor to be of the form

$$T_\mu^\nu = {\rm diag}(-\rho, P_r, P_t, P_t) \ , \tag{7}$$

where ρ is the energy-density, while P_r and P_t are the radial and tangential pressures, respectively. The gravity equations are then a set of coupled, non-linear ordinary differential equations that have to be solved numerically.

However, we can simplify the analysis by noting that the equation for the gravity scalar ϕ, which comes from the rr-component of (2), can be solved algebraically in terms of the other functions :

$$\eta(\phi')^2 = \frac{2r^2}{f} P_r + \frac{1-f}{bf}\eta q^2 \quad . \tag{8}$$

This allows the elimination of ϕ from the remaining equations and we are left with the equations for the metric functions which read :

$$\mathcal{F}_1 f' + \mathcal{F}_2 = 0 \quad , \quad \frac{b'}{b} = \frac{1-f}{fr} \tag{9}$$

with

$$\mathcal{F}_1 = 4\kappa br + 2br^3 P_r - 3\eta q^2 rf \tag{10}$$

and

$$\mathcal{F}_2 = 3\eta q^2 f(1-f) + 2b\left[\rho r^2(f+1) + 2fr^2 P_r + 4fr^2 P_t + 2\kappa(f-1)\right] \quad . \tag{11}$$

Note that the second equation in (9) ensures that the Noether current J^μ is covariantly conserved, i.e. $\nabla_\mu J^\mu = 0$ and is, in fact, the rt-component of the Einstein equation.

Star-like astrophysical objects are typically characterized in terms of their mass-radius relation. The gravitational mass M_G of this solution is given in terms of the asymptotic behaviour of the metric function $f(r)$:

$$f(r) \xrightarrow[r\to\infty]{} 1 - \frac{M_G}{4\pi\kappa r} + \mathcal{O}(r^{-2}) \ , \tag{12}$$

while the radius will be defined differently in the case of boson stars and neutron stars, see below. Since asymptotically, the metric function $b(r)$ becomes equal to $f(r)$ and we assume in the following that either the pressure P_r tends exponentially to zero asymptotically (in the case of boson stars) or is strictly zero (in the case of neutron stars), we observe that the mass M_G can also be read off from the behaviour of the gravity scalar at infinity. Using (8) we find that

$$(\phi')^2 \xrightarrow[r\to\infty]{} \frac{M_G}{4\pi\kappa r} q^2 \ . \tag{13}$$

In other words: $M_G q^2/(4\pi\kappa)$ constitutes the "charge" associated to the scalar field $(\phi')^2$.

In [9], an action similar to (1) has been discussed, however, with an additional "standard" kinetic term for the scalar field. Hence, the exact argument of loss of hyperbolicity of the metric cannot be translated to our case, but we can modify it accordingly. The question is whether $\phi \equiv 0$ is stable in our model. For that, note that we can interpret the scalar field equation as an equation in an "effective" metric solely given by the Einstein tensor :

$$\tilde{g}_{\mu\nu} = -\eta G_{\mu\nu} \ , \tag{14}$$

which using the Einstein equation and the form of the energy-momentum tensor (7) has determinant

$$\tilde{g} = -g\left(\frac{\eta}{\kappa}\right)^4 \rho P_r P_t^2 \ . \tag{15}$$

where g is the determinant of $g_{\mu\nu}$. This expression demonstrates that the argumentation is independent of the actual sign of η and that – since ρ and P_r are positive – the determinant of $\tilde{g}_{\mu\nu}$ has always opposite sign to the determinant of $g_{\mu\nu}$. This might cause problems if a mass term (or self-interaction terms) would be present for the scalar field, however is irrelevant for the case studied here.

3 Boson stars

In the case of boson stars, the energy-momentum content is that of a complex valued scalar field Ψ, which – in contrast to the neutron star model discussed in Section 4 – is not of perfect fluid type. The energy-momentum tensor reads :

$$T_{\mu\nu} = -g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha\Psi^*\partial_\beta\Psi + \partial_\beta\Psi^*\partial_\alpha\Psi\right) + m^2\Psi\Psi^*\right] + \partial_\mu\Psi^*\partial_\nu\Psi + \partial_\nu\Psi^*\partial_\mu\Psi \ , \tag{16}$$

where m denotes the scalar boson mass. This model contains an additional conserved Noether current due to the internal global U(1) symmetry $\Psi \to \exp(i\chi)\psi$, where χ is a constant. This reads

$$j^\mu = -\frac{i}{2}\left(\Psi^*\nabla^\mu\Psi - \Psi\nabla^\mu\Psi^*\right) \quad , \quad \nabla_\mu j^\mu = 0 \ . \tag{17}$$

With the standard spherically symmetric Ansatz for boson stars

$$\Psi(r,t) = \exp(i\omega t)H(r) \ , \tag{18}$$

where $\omega > 0$ is a constant, the non-vanishing components of the energy-momentum tensor read

$$\rho = f(H')^2 \quad + \quad \left(m^2 + \frac{\omega^2}{b}\right)H^2 \ ,$$
$$P_r = f(H')^2 - \left(m^2 - \frac{\omega^2}{b}\right)H^2 \quad , \quad P_t = -f(H')^2 - \left(m^2 - \frac{\omega^2}{b}\right)H^2 \ . \tag{19}$$

The locally conserved current and associated globally conserved Noether charge are :

$$j^t = -\frac{\omega H^2}{b} \quad , \quad Q = -\int \mathrm{d}^3x\ \sqrt{-g}\ j^t = 4\pi\omega\int \mathrm{d}r\ r^2\ \frac{H^2}{\sqrt{bf}} \tag{20}$$

Note that in the model with ungauged U(1) symmetry, the Noether charge Q is frequently interpreted as the number of bosonic particles of mass m that make up the boson star. Finally the field equation for Ψ reads :

$$H'' + \frac{1}{2}\left(\frac{4}{r} + \frac{f'}{f} + \frac{b'}{b}\right)H' + \frac{1}{f}\left(\frac{\omega^2}{b} - m^2\right)H = 0 \ . \tag{21}$$

The asymptotic behaviour of $H(r)$ that can be read of from (21) is :

$$H(r) \xrightarrow[r\to\infty]{} \frac{1}{r}\exp\left(-\sqrt{m^2-\omega^2}r\right) \ , \tag{22}$$

i.e. although the scalar field making up the boson star decays fast, the star does not have a "hard surface" like the neutron star discussed below. Rather, its energy density ρ and pressures P_r and P_t, respectively, tend to zero only asymptotically. We can, however, use an estimate of the radius R of the boson star which is given as follows :

$$\langle R\rangle = \frac{1}{Q}\int \mathrm{d}^3x\sqrt{-g}\ r\ j^t \ = \frac{4\pi\omega}{Q}\int \mathrm{d}r\ r^3\frac{H^2}{\sqrt{bf}} \ . \tag{23}$$

The equations (9) and (21) have to be solved with boundary conditions that guarantee the regularity of the solution at the origin and its finiteness of energy. The appropriate conditions read :

$$b'(0) = 0 \ , \ H'(0) = 0 \ , \ b(\infty) = 1 \ , \ H(\infty) = 0 \tag{24}$$

where the constant $H(0) \equiv H_0$ is an *a priori* free parameter that determines the value of ω as well as the central density of the boson star, see (19), via $\rho(0) = (m^2 + \omega^2/b(0))H_0^2$. As is well known from boson stars in GR, the parameter $H(0)$ can be increased arbitrarily such that a succession of branches of boson stars exist that end only for $H(0) \to \infty$ and $b(0) \to 0$ in this limit. This will be different for the scalar-tensor boson stars studied here. The expansion of the fields around the origin already gives hints that this should be the case. We find :

$$b(r) = b_0\left[1 + \frac{4H_0^2(2\omega^2 - b_0m^2)}{3(4\kappa b_0 - 3\eta q^2)}r^2 + \mathcal{O}(r^4)\right] \quad , \quad f(r) = f_0\left[1 + \frac{1}{6}\left(m^2 - \frac{\omega^2}{b_0}\right)r^2 + \mathcal{O}(r^4)\right] \ , \tag{25}$$

where $b_0 = b(0)$ and $f_0 = f(0)$. This implies that we have to require $4\kappa b_0 - 3\eta q^2 \neq 0$. As we will demonstrate in the following, this condition is crucial in the limitation of the domain of existence of the solutions for $\eta > 0$. Note that for $\eta < 0$ another limitation exists, related to the requirement of positivity of the right hand side of

(8).
The system of equations is unchanged under the following rescalings

$$r \to \frac{r}{m} \quad , \quad \omega \to m\omega \quad , \quad H \to \sqrt{\kappa} H \quad , \quad \eta \to \kappa\eta \quad , \quad \phi \to \frac{\phi}{m} \ , \tag{26}$$

which rescales the radius, mass and Noether charge of the boson star as follows :

$$\langle R \rangle \to \frac{\langle R \rangle}{m} \quad , \quad M_G \to \frac{M_G}{m} \quad , \quad Q \to \frac{\kappa}{m^2} Q \ . \tag{27}$$

In the following we will choose $\kappa = 1$, $m = 1$, $\eta = \pm 1$ without loss of generality.

3.1 Numerical results

We have solved the equations numerically using a collocation method for boundary-value differential equations using damped Newton-Raphson iterations [12]. The relative errors of the solutions are on the order of $10^{-6} - 10^{-10}$. The constants to be varied are the combination ηq^2 as well as ω (or equivalently $H(0)$). From (22) we know that with the rescalings (26) the angular frequency is restricted by: $\omega^2 \leq 1$.

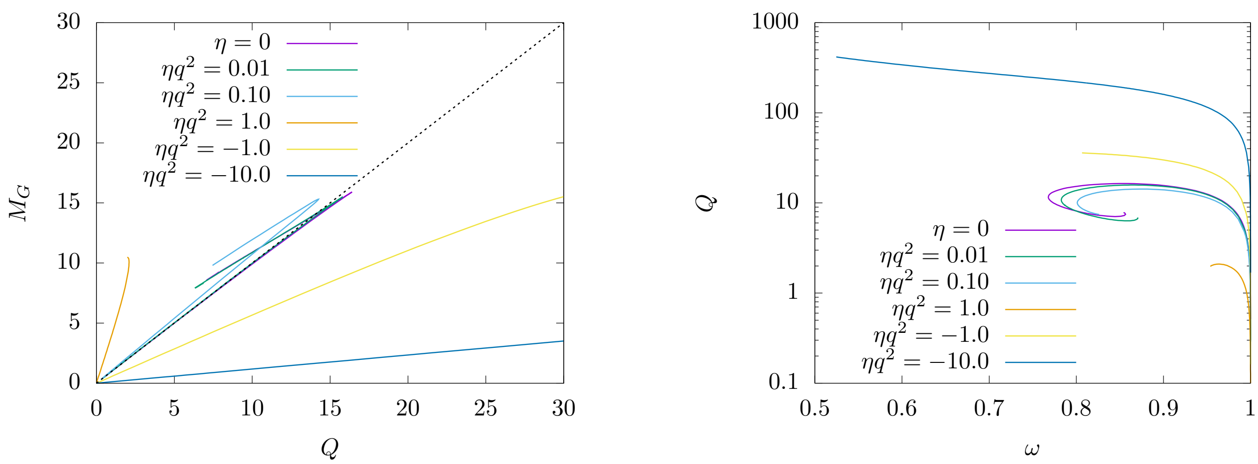

Figure 1: We show the gravitational mass M_G as function of the Noether charge Q (left) as well as the Noether charge Q as function of ω (right) for GR boson stars ($\eta = 0$) and boson stars with time-dependent scalar hair for several values of ηq^2.

In Fig. 1 we show the relation between Noether charge Q and gravitational mass M_G (left) and the dependence of the Noether charge on ω (right), respectively, for several values of ηq^2 including the GR case $\eta = 0$. While for $\eta = 0$, we can increase the value of $H(0)$ arbitrarily, this is no longer the case in the scalar-tensor gravity model studied here. For $\eta q^2 > 0$, the curves shown in Fig. 1 are limited by the requirement discussed above which, with our choice of constant, reads : $4b_0 - 3\eta q^2 > 0$. We find that the branches of solutions stop at $4b_0 - 3\eta q^2 = 0$. For the GR case and $H(0) \to \infty$ the value of the metric function $b(r)$ at $r = 0$, b_0, tends to zero. This is obviously no longer true and hence boson stars with time-dependent scalar hair are limited in their central density of the star. For ηq^2 sufficiently large, see the curves for $\eta q^2 = 1.0$, this also leads to the observation that the Noether charge Q is strongly limited and much smaller than in the GR case. On the other hand, the mass M_G is of the same order of magnitude. Hence, scalar-tensor boson stars with time-dependent

ηq^2	$M_{G,\max}$	$Q_{\max}$	$\langle R\rangle^*$	ω^*	$\rho(0)^*$	$P(0)^*$
0	15.91	16.40	3.10	0.85	0.19	0.04
0.01	15.50	15.81	3.22	0.86	0.15	0.03
0.1	15.32	14.27	3.54	0.87	0.12	0.02
1.0	10.46	2.10	6.44	0.96	$3 \cdot 10^{-3}$	$2 \cdot 10^{-4}$
−1.0	16.68	35.93	2.84	0.83	0.47	0.11
−10.0	17.90	416.81	2.50	0.79	4.13	1.00

Table 1: We give the maximal values of the mass $M_{G,\max}$ as well as the maximal value of the Noether charge $Q_{\max}$ for different values of ηq^2. Also given is the mean radius $\langle R\rangle^*$, the angular frequency ω^*, the central density $\rho(0)^*$ as well as the central pressure $P_r(0)^* = P_t(0)^* \equiv P(0)^*$ at the maximal value of the mass, i.e. at $M_{G,\max}$.

scalar fields and $\eta q^2 > 0$ are comparable in mass, but consist of an order of magnitude smaller number of scalar bosonic particles as compared to their GR counterparts. Moreover, their central density $\rho(0)$ and central pressure $P_r(0) = P_t(0) \equiv P(0)$ is comparable to the GR case, see Table 1 as long as ηq^2 is not too large. For $\eta q^2 = 1.0$, we find that both the central density as well as the central pressure are very small.

For $\eta q^2 < 0$, we observe the exact opposite: the boson stars can contain many more scalar particles. The Noether charge increases strongly, while the mass remains of the same order of magnitude. We present some numerical values of our results in Table 1. As can be clearly seen here in combination with the data presented in Fig. 2, the mass M_G varies only slightly with ηq^2 and decreases when increasing ηq^2. The Noether charge Q on the other hand varies strongly with ηq^2. Moreover, as can be seen from Fig. 2 a gap in ηq^2 exist for which scalarized boson stars are not possible. This gap depends on the value of the frequency ω and increases when ω decreases, i.e. when ω decreases.

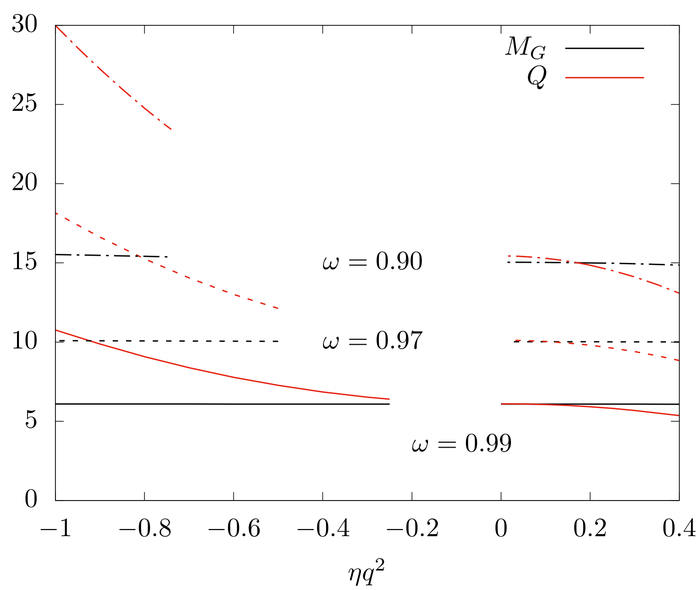

Figure 2: We show the Noether charge Q (red) and the gravitational mass M_G (black), in dependence of ηq^2 for the boson star solutions with $\omega = 0.99$ (solid), $\omega = 0.97$ (dashed) and $\omega = 0.90$ (dotted-dashed) respectively.

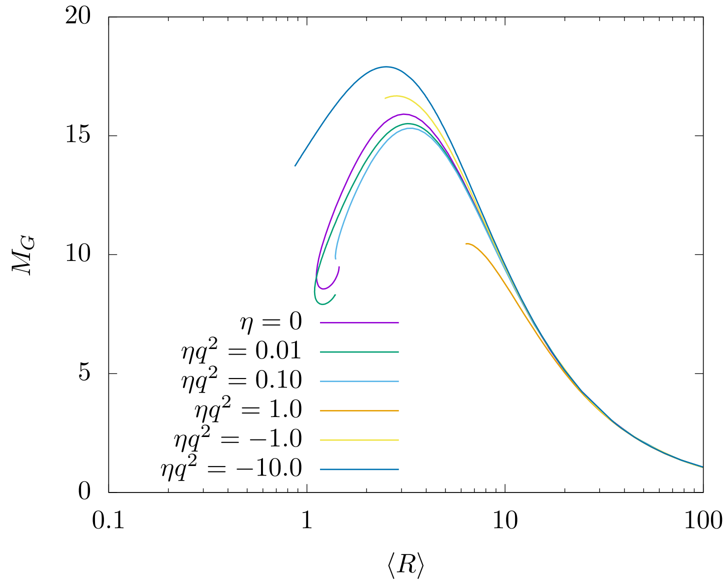

Figure 3: We show the gravitational mass M_G in function of the mean radius $\langle R \rangle$ of the boson star with time-dependent scalar hair for several values of ηq^2. For comparison we also show the mass-radius relation for the GR limit ($\eta = 0$).

Finally, and since we want to compare neutron stars with scalar hair with boson stars with scalar hair in this paper, we show the mass-radius relation for the boson stars in Fig. 3 for several values of ηq^2, see also Table 1 for some values. We find that boson stars with large radius are practically not influenced by the scalar-tensor coupling, but very compact boson stars are. The radius of the boson star at maximal mass, $\langle R \rangle^*$ (see Table 1) is larger for all positive ηq^2 that we have studied, however smaller for all negative values of ηq^2.

If we use the standard argument that a boson star can be thought of as a system of a number Q of scalar particles of mass m, we can compare the actual mass M_G of the boson star and the mass of Q scalar bosons which is mQ. For $M_G < mQ$, we expect the boson star to form a bound system of these individual bosons and hence be stable with respect to the decay into those particles. Note that with our rescalings, the scalar boson mass $m \equiv 1$. Inspection of Fig. 1 demonstrates that decreasing ηq^2 from zero, the binding between the scalar particles increases, suggesting that for $\eta < 0$ the non-minimal coupling has effectively an attractive nature. On the other hand, for $\eta q^2 > 0$, we find that $M_G > Q$ for a part of the second branch of solutions (see $\eta q^2 = 0.01$) or that – for sufficiently large ηq^2 – all boson star solutions are unstable to decay into Q individual bosons (see curves for $\eta q^2 \geq 0.01$.)

4 Neutron stars

The energy-momentum tensor for a neutron star is typically assumed to be that of a perfect fluid with $P_r = P_t \equiv P$ and an equation of state (EOS) relating ρ and P. In addition to the gravity equation (2), we then also have to solve the Tolman-Oppenheimer-Volkoff (TOV) equation which reads :

$$P' = -\frac{b'}{2b}(P + \rho) \ . \tag{28}$$

In the following, we will use different equations of state to study the properties of neutron stars. It will be of the general polytropic form

$$\rho = CP + KP^{1/\Gamma} \tag{29}$$

where C and K are constants and Γ is the so-called adiabatic index. We have restricted our analysis to some specific cases :

- the first equation of state ("EOSI" in the following) has been used in [10] in the exact same context as in our work and has $C = 1$ and $\Gamma = 3/2$,
- the second equation of state ("EOSII") has $C = 1$ and $\Gamma = 5/3$,
- the third equation of state ("EOSIII") has $1/C = \Gamma - 1$ and $\Gamma = 2.34$. This has been used in [16] as a good fit to a realistic equation of state.

Note that although we use the letter K in (29) for all three equations of state, this coupling has different mass dimensions in the individual cases. Remembering that ρ and P have mass dimension -2 in natural units, the mass dimension of K is $-2 + 2/\Gamma$.

In Fig.4, we show a realistic equation of state [17] - the so-called BSk20 EOS, which is based on Hartree-Fock-Bogoliubov mass models, in comparison to the polytropic EOSI, EOSII and EOSIII, respectively. We have chosen the respective values of K corresponding to the maximal value of the gravitational mass (for details see our numerical results below). This figure demonstrates that especially the polytropic equation of state EOSIII fits the realistic BSk20 equation of state very well at high density and high pressure.

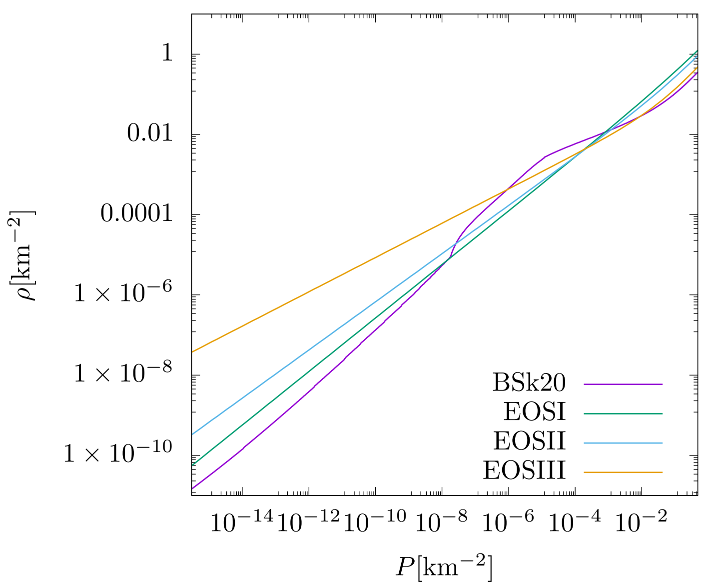

Figure 4: We show the energy density ρ as function of the pressure P in natural units ($8\pi G = c = \hbar = 1$) for the BSk20 equation of state. In comparison we show the three polytropic equations of state used in this paper, see (29). For EOSI, EOSII and EOSIII we have chosen $K = 1.5$, $K = 0.67$ and $K = 0.16$, respectively, corresponding to the maximal mass value.

The radius R of the neutron star is defined differently than that of the boson star. Here, the star has a "hard core", i.e. a surface outside of which the space-time is given by the Schwarzschild solution. The relevant conditions to impose in this case are :

$$P(R) = 0 \quad , \quad b(R) = f(R) \ . \tag{30}$$

To connect the results to physically realistic values for the mass and radius of the neutron stars, K has to be chosen accordingly. However for the purpose of our study, we note that the equations of motion are invariant under the following rescaling :

$$r \to \lambda r \quad , \quad M_G \to \lambda M_G \quad , \quad P \to \lambda^{-2} P \quad , \quad \rho \to \lambda^{-2}\rho \quad , \quad K \to \lambda^{-2+2/\Gamma} K \ . \tag{31}$$

Then, a dimensionless radius $\tilde{R}$ and a dimensionless mass $\tilde{M}_G$ of the configuration can be defined according to

$$\tilde{R} = R K^{\Gamma/(2\Gamma-2)} \quad , \quad \tilde{M}_G = M_G K^{\Gamma/(2\Gamma-2)} \quad . \tag{32}$$

Note that we are using natural units here with $\hbar = c = 8\pi G \equiv 1$. Reinstalling the natural constants, we find that the mass M_G and R given in Fig. 7 are related to the dimensionful mass $M_{G,\text{phys}}$ and dimensionful radius R_{phys} as follows

$$\frac{M_{G,\text{phys}}\,[M_\odot]}{R_{\text{phys}}\,[\text{km}]} \approx 0.68 \frac{M_G}{R} = 0.68 \frac{\tilde{M}_G}{\tilde{R}} \ . \tag{33}$$

In the following, it will also be useful to define the number N_B of baryons of mass m_B that make up the neutron star. This is the equivalent of the Noether charge Q for the boson star and can be used to estimate whether the constructed solutions are stable to decay into a number of individual baryons or if they form a bound system. We follow the discussion in [15]. As argued in this latter paper, the particle number conservation follows from the energy-momentum conversation when

$$\frac{n'}{n} = \frac{\rho'}{\rho + p} \ , \tag{34}$$

where n is the particle number denisty $n = n(r)$. Combining this with the TOV equation (28) gives

$$n(r) = \frac{\sqrt{b}(\rho + p)}{m_B \sqrt{1 - \frac{2M_G}{R}}} \ , \tag{35}$$

where the integration constant has been fixed by assuming the conditions on the surface of the star to be $p(r = R) = 0$, $\rho(r = R) = \rho_0$, $n(r = R) = n_0$ and using the relation between ρ and n: $\rho_0 = m_B n_0$. The globally conserved quantity associated to the particle density current nu^μ, u^μ the 4-velocity of a particle, is the total baryon number

$$N_B = \int \sqrt{-g} n u^0 \mathrm{d}^3 x = \frac{4\pi}{m_B \sqrt{1 - \frac{2M_G}{R}}} \int_0^R (\rho + p) \sqrt{\frac{b}{f}} r^2 \mathrm{d}r \tag{36}$$

where we have used that for a particle at rest $u^0 = 1/\sqrt{b}$. Comparing $m_B N_B$ with M_G will tells us whether the neutron star is stable ($m_B N_B > M_G$) or unstable ($m_B N_B < M_G$), respectively, to decay into N_B individual baryons of mass m_B.

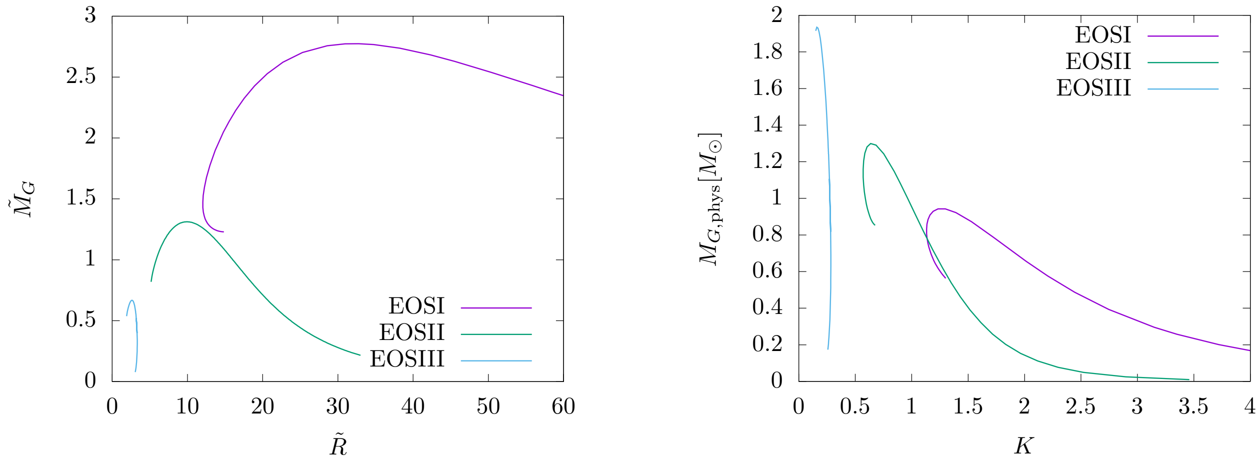

Figure 5: We show the mass $\tilde{M}_G$ as function of the radius $\tilde{R}$ of the neutron star solutions for $\eta = 0$ and three different EOS, see (29), respectively (left). We also show $M_{G,\rm phys}$ in units of solar masses $M_\odot$ as function of K for the same EOS and $R = 10$ (right).

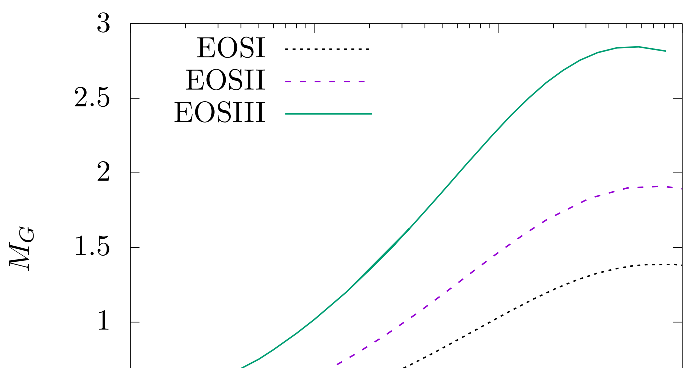

Figure 6: We show the mass M_G in function of the central pressure $P(0)$ of the neutron star solutions for $\eta = 0$ and three different EOS, see (29), respectively.

4.1 Numerical results

In this first part, we will discuss and review already existing results to clarify our construction and compare the three different EOS discussed above in the GR limit. We will then turn to new scalar-tensor neutron stars using EOSII and EOSIII, respectively.

4.1.1 Neutron stars in GR

In Fig. 5 (left) we show the dimensionless quantity $\tilde{M}_G$ in function of the radius $\tilde{R}$ of the neutron star in the GR limit and for the three different equations of state. Note that using (31), the axes in this plot have to be rescaled by the same factor K in order to find the physical values of mass and radius of the neutron star. Contrary to what is presented in [10], we find that for a typical neutron star of radius $R_{\rm phys} = 10$km (corresponding to the maximum of the curve) the ratio $M/M_\odot \approx 0.6$, and not $M/M_\odot \approx 1.2$ as stated in [10]. Moreover, the qualitative relation between mass and radius is different to that in Fig. 2 of [10].

Comparing e.g. with the gravitational wave detections GW170817 from a binary neutron star merger [18] which suggests that the two neutron stars in the merger had masses between $0.86M_\odot$ and $2.26M_\odot$ and radii between 10.7 km and 11.9 km [19] (compare also very new results in [20]), we find that EOSI seems to have neutron stars of too low mass. We have hence considered EOSII and EOSIII, respectively. In Fig.5 we show the mass $\tilde{M}_G$ in function of the radius $\tilde{R}$ (left) and $M_{G,\rm phys}$ in solar mass units $M_\odot$ in function of K for $R = 10$ (right) for EOSI, EOSII and EOSII. The combination of the data shown in this figure gives a maximal mass of a $R = 10$ neutron star of $M_{G,\rm phys} \approx 0.95M_\odot$ at $K = 1.23$ for EOSI, of $M_{G,\rm phys} \approx 1.17M_\odot$ at $K = 0.67$ for EOSII and $M_{G,\rm phys} \approx 1.94M_\odot$ at $K = 0.16$ for EOSIII, respectively. Note that these are also the values of K used in Fig. 4. We conclude that EOSIII seems to be a good approximation to the realistic BSk20 equation of state for high pressure and high density neutron stars, but that we have also computed stars with low density and found consistency. In order to make sure that non-uniqueness does not exist for the neutron stars using the EOSI, EOSII and EOSII for the given parameters, we have plotted M_G in function of the central density $P(0)$ of the star in Fig. 6. This demonstrates clearly that there is one solution for a given value of $P(0)$ for all equations of state that we have studied in this paper.

4.1.2 Scalar-tensor neutron stars

We now turn to the description of the influence of the non-minimal scalar-tensor coupling on the neutron star solutions constructed with EOSII and EOSIII

We find that the existence of neutron stars – very similar to that of boson stars – is limited by the requirement of positivity of the denominator in the expansion (25) for $\eta q^2 > 0$ and by the requirement of positivity of ϕ'^2 (see (8)) for $\eta q^2 < 0$, respectively. Our results for the mass-radius relation of neutron stars for different values of ηq^2 are shown in Fig.7. The maximal mass $M_{G,\max}$ of the scalarized neutron stars is reached at roughly the same value of $R \approx 10$, however, when increasing ηq^2, the value of the maximal mass decreases as compared to the GR limit. When decreasing ηq^2 from zero, we find an interesting new phenomenon which appears for both EOSII and EOSIII. Let us choose the value $\eta q^2 = -0.25$ for EOSII to explain this in more detail : when increasing the central pressure of the star, $P(0)$, we find a branch of solutions for $P(0) \leq 0.009$ (in our units) corresponding to $R > 11.4$. The solutions constructed for larger $P(0)$ (and $R \leq 11.4$) have $(\phi')^2 < 0$ in some region and are therefore not acceptable, i.e. we find an interval of $P(0)$ for which no scalarized neutron stars exist. Interestingly, we observe that when increasing $P(0)$ sufficiently (in fact, $P(0) > 0.12$) a new, second branch of scalarized neutron stars for which $(\phi')^2 > 0$, exists. The reason for the existence of this new branch can be understood when considering (8) and the plot of the energy density ρ, pressure P, the metric functions $f(r)$ and $b(r)$ as well as ϕ'^2 given in Fig. 9 for neutron star corresponding to the second branch of solutions. This neutron star has $R = 10$ and $P(0) = 0.66$. Clearly, all functions are well behaved in particular $\phi'^2 \geq 0$ inside the star. The reason for the existence of these solutions then also becomes clear : since $b(r)$ is very small everywhere inside the star by inspection of (8) the value of ϕ'^2 can become positive again. The crucial point is hence the presence of the explicit time-dependence of the scalar field, i.e. the fact that $q \neq 0$. Not surprisingly, these neutron stars are very dense : as Fig. 7 demonstrates (see also Fig. 8) they are very close to the branch of Schwarzschild black holes. Note that for $\Gamma = 2.34$ and $\eta q^2 = -0.0625$, the second branch of solutions exists for

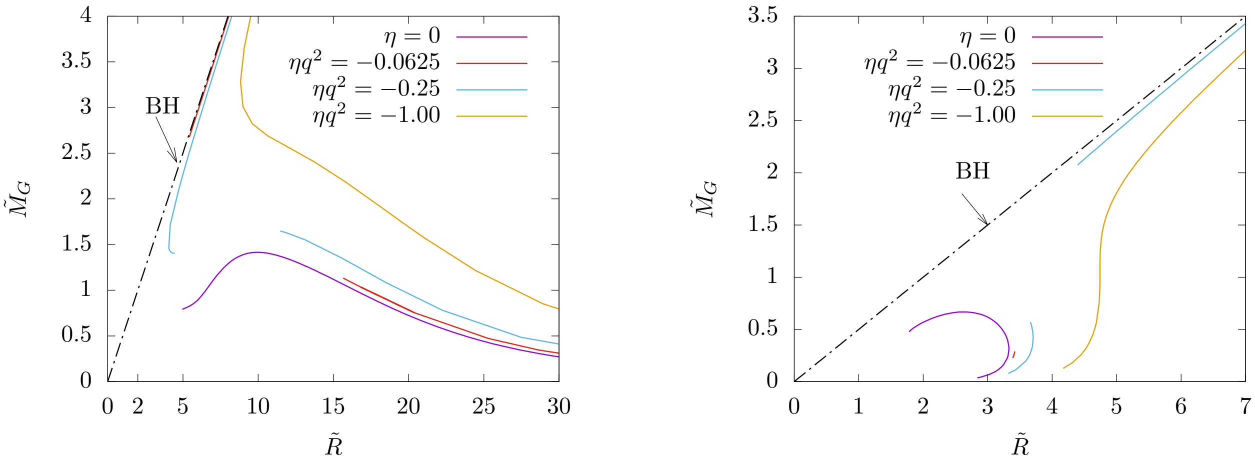

Figure 7: We show the mass $\tilde{M}_G$ as function of the radius $\tilde{R}$ of the neutron star solutions for $\Gamma = 3/5$ (left) and $\Gamma = 2.34$ (right) for several values of ηq^2. The mass-radius relation of the corresponding Schwarzschild black hole is indicated by "BH".

$R \geq 12.15$, i.e. is not visible in the figure. At $R = 12.15$, the mass of these solutions is $M_G \approx 6.07$, i.e. is very close to the black hole limit. When decreasing ηq^2 further, see the curve for $\eta q^2 = -1.0$ in Fig. 7, we find that there exists a continuous branch of solutions along which the central pressure $P(0)$ increases and $(\phi')^2$ stays always positive. Hence, we find neutron stars that through a continuous deformation of the central pressure can reach mass densities that are very close to that of black holes. In order to get an idea of the astrophysical scales of these objects, we have plotted the physical mass $M_{G,\mathrm{phys}}$ in solar mass units versus the radius of the stars in km in Fig. 8.

We have also studied the stability of the neutron stars with respect to the decay into N_B individual baryons with mass m_B. For all solutions obtained, we observe that increasing ηq^2 from zero leads to a decrease in the binding energy and that for sufficiently large ηq^2 the neutron star becomes unstable to decay into individual baryons. This is, however, different when decreasing ηq^2 from zero. Remember that the existence of solutions in linked to the requirement of the positivity of the quantity $(\phi')^2$ and hence the domain of the parameter K for which solutions exist depends strongly on ηq^2. This is shown in Fig. 10 (left) for some values of $\eta q^2 < 0$, where we give the ratio between M_G and $m_B N_B$. A ratio smaller than one indicates stability of the neutron star with respect to this specific decay. As is apparent from Fig. 10, the inclusion of a gravity scalar with ηq^2 negative increases the binding between the individual baryons. Moreover, when two separate branches of solutions are present, the branch closer to the BH limit has much stronger binding. The typical cusp-like structure for neutron stars in GR when plotting $m_B N_B - M_G$ versus $m_B N_B$ is something we do not observe for neutron stars with $\eta q^2 < 0$, see Fig. 10 (right). Rather, the difference between the energy of N_B individual baryons of mass m_B and the mass of the neutron star M_G is a monotonically increasing function of $m_B N_B$.

5 Conclusions

In this paper, we have studied the properties of boson and neutron stars in a scalar-tensor gravity models which contains an explicitly time-dependent real scalar field. The norm of the Noether current associated to the shift symmetry of the gravity scalar is finite everywhere in the space-time. We find that the explicit time-dependence does allow non-trivial scalar fields to exist in both the space-time of a boson star and neutron star, respectively.

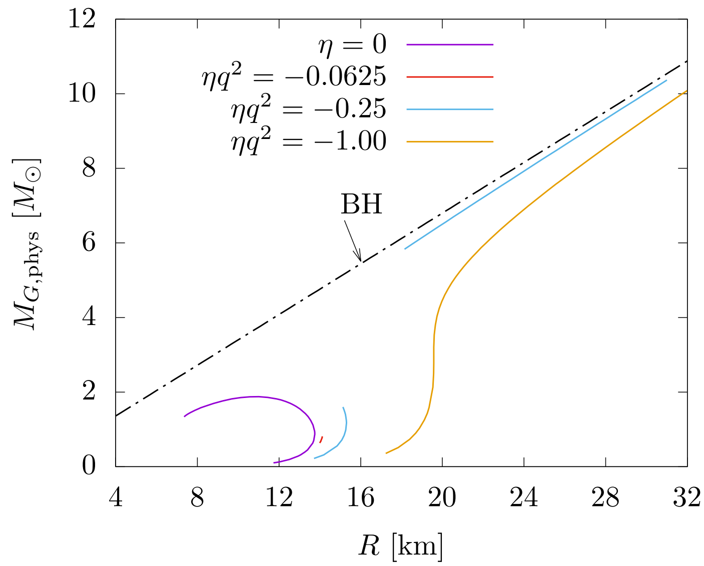

Figure 8: We show the physical mass $M_{G,\mathrm{phys}}$ in solar mass units as function of the radius R in km of the neutron star solutions for $\Gamma = 2.34$ for several values of ηq^2. The mass-radius relation of the corresponding Schwarzschild black hole is indicated by "BH".

Moreover, the presence of the gravity scalar has interesting consequence for the properties of these objects. While the boson star's mass does not vary strongly when increasing or decreasing the scalar-tensor coupling from zero, it has a large effect on the number of scalar bosonic particles making up the boson star, the mean radius and central density and pressure. This means that while in the GR limit, boson stars of the type studied here, so-called "mini boson stars", have radius of a few Schwarzschild radii (see e.g. [1]), the radius of the scalar-tensor counterparts could, in fact, be much closer to the Schwarzschild radius.

For neutron stars, we have investigated two polytropic equation of states out of which one seems to be a very good fit to realistic equations of state. While neutron stars have a "hard core" outside which the pressure is strictly zero, the change of properties is comparable to that of boson stars. In particular, for negative scalar-tensor coupling and the gravity scalar changing slowly in time, we find that new branches of solutions of neutron stars exist that have a mass-radius relation very close to that of Schwarzschild black holes. Increasing the time change of the gravity scalar, we find that we can continuously deform "standard" mass neutron stars to these objects with large central pressure $P(0)$. We observe that this phenomenon arises for both equations of state that we have investigated.

In summary, our results indicate that the presence of a gravity scalar in the case of globally regular, compact objects prevents these objects from collapsing to a black hole at the values known in GR due to an increased central pressure allowed inside the stars. Our results furthermore indicate that the scalar-tensor objects studied here are stable to decay into their individual constituents. Surely this does not mean that they are generally stable under perturbations - a study outside the scope of this paper - but it indicates that when considering extensions of GR, the formation of star-like objects that are in density close to that of black holes is a viable possibility.

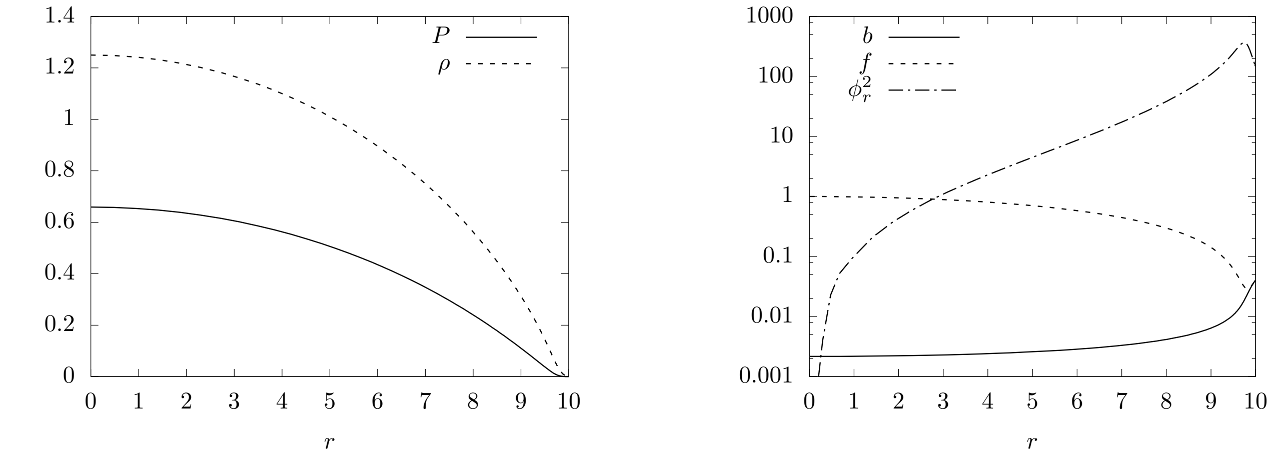

Figure 9: *Left* : We show the profiles of the pressure P (solid) and the energy density ρ (dashed) of a scalar-tensor neutron star with radius $R = 10$ for $\eta q^2 = -0.25$, $P(0) = 0.66$ and EOSII. *Right* : We also show the metric functions b (solid) and f (dashed) as well as $\phi_r^2 \equiv \phi'^2$ (dotted-dashed) for the same solution (right).

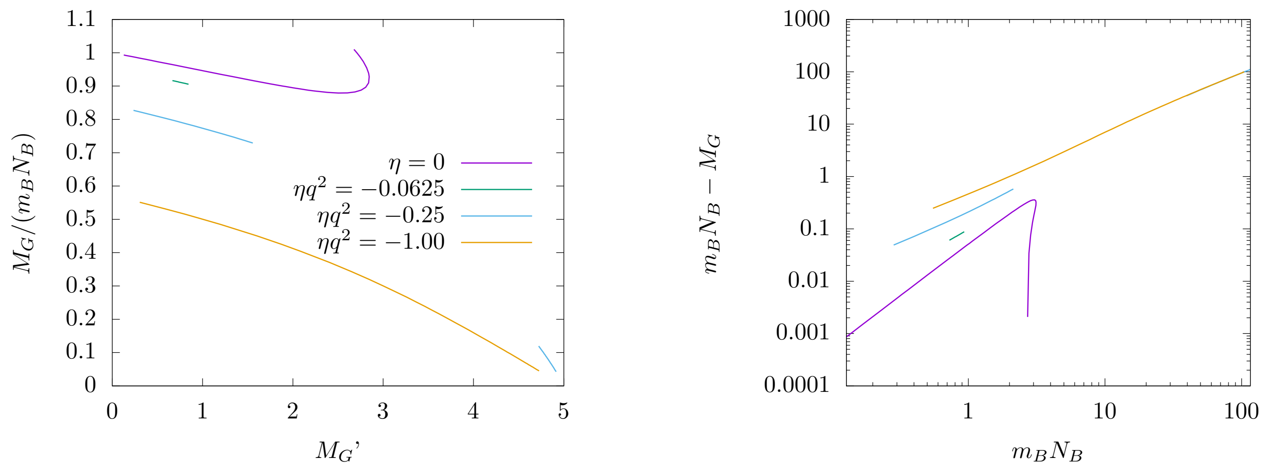

Figure 10: *Left* : We show the ratio $M_G/(m_B N_B)$ between the mass M_G of the neutron star and the mass of N_B individual baryons of mass m_B in function of M_G for EOSIII and several values of ηq^2. *Right*: We show the difference in energy between N_B baryons of mass m_B and the mass M_G of the neutron star in function of $m_B N_B$.

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References

[1] F. E. Schunck and E. W. Mielke, *General relativistic boson stars*, Class. Quant. Grav. **20** (2003), R301-R356

[2] A. Nicolis, R. Rattazzi and E. Trincherini, *The Galileon as a local modification of gravity*, Phys. Rev. D **79** (2009) 064036

[3] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, *From k-essence to generalised Galileons*, Phys. Rev. D **84** (2011) 064039.

[4] C. Deffayet and D. A. Steer, *A formal introduction to Horndeski and Galileon theories and their generalizations*, Class. Quant. Grav. **30** (2013) 214006.

[5] G. W. Horndeski, *Second order scalar-tensor field equations in a four dimensional space*, Int. J. Theor. Phys. **10**, 363 (1974).

[6] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, *Self-tuning and the derivation of a class of scalar-tensor theories*, Phys. Rev. D **85** (2012), 104040.

[7] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, *General second order scalar-tensor theory, self tuning, and the Fab Four*, Phys. Rev. Lett. **108** (2012), 051101.

[8] E. Babichev and C. Charmousis, *Dressing a black hole with a time-dependent Galileon*, JHEP **1408** (2014) 106.

[9] N. Andreou, N. Franchini, G. Ventagli and T. P. Sotiriou, *Spontaneous scalarization in generalised scalar-tensor theory* , Phys. Rev. D **99** (2019) 12, 124022.

[10] A. Cisterna, T. Delsate and M. Rinaldi, *Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling*, Phys. Rev. D **92** (2015) no.4, 044050.

[11] S. Ray, M. Malheiro, J. P. S. Lemos and V. T. Zanchin, *Charged polytropic compact stars*, Braz. J. Phys. **34** (2004) 310.

[12] U. Ascher, J. Christiansen, R. D. Russell, *A collocation solver for mixed order systems of boundary value problems*, Math. of Comp. **33** (1979) 659;
U. Ascher, J. Christiansen, R. D. Russell, *Collocation software for boundary-value ODEs*, ACM Trans. **7** (1981) 209.

[13] V. R. Pandharipande and D.G. Ravenhall, *Hot Nuclear Matter*, in Nuclear Matter and Heavy Ion Collisions, NATO ADS Ser., Vol. B205, ed. M. Soyeur, H. Flocard, B. Tamain, and M. Porneuf (Dordrecht: Reidel), 103 (1989).

[14] F. Douchin and P. Haensel, *A unified equation of state of dense matter and neutron star structure*, Astron. Astrophys. **380** (2001), 151.

[15] I. Goldman and N. Rosen, *Binding energy and stability of a cold neutron star*, The Astrophysical Journal **225** (1978), 708.

[16] T. Damour and G. Esposito-Farese, *Nonperturbative strong field effects in tensor - scalar theories of gravitation*, Phys. Rev. Lett. **70** (1993), 2220.

[17] S. Goriely, N. Chamel and J. M. Pearson, *Further explorations of Skyrme-Hartree-Fock-Bogoliubov mass formulas. XII: Stiffness and stability of neutron-star matter*, Phys. Rev. C **82** (2010), 035804.

[18] B. Abbott *et al.* [LIGO Scientific and Virgo], *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, Phys. Rev. Lett. **119** (2017) no.16, 161101.

[19] B. Abbott *et al.* [LIGO Scientific and Virgo], *GW170817: Measurements of neutron star radii and equation of state*, Phys. Rev. Lett. **121** (2018) no.16, 161101.

[20] B. Abbott *et al.* [LIGO Scientific and Virgo], *GW190425: Observation of a Compact Binary Coalescence with Total Mass* $\sim 3.4 M_{\odot}$, [arXiv:2001.01761 [astro-ph.HE]].

Appendix

F Scalarized Black Holes in Teleparallel Gravity

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Scalarized Black Holes in Teleparallel Gravity

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Black holes play a crucial role in the understanding of the gravitational interaction. Through the direct observation of the shadow of a black hole by the event horizon telescope and the detection of gravitational waves of merging black holes we now start to have direct access to their properties and behaviour, which means the properties and behaviour of gravity. This further raised the demand for models to compare with those observations. In this respect, an important question regarding black holes properties is to know if they can support "hairs". While this is famously forbidden in general relativity, in particular for scalar fields, by the so-called no-hair theorems, hairy black holes have been shown to exist in several class of scalar-tensor theories of gravity. In this article we investigate the existence of scalarized black holes in scalar-torsion theories of gravity. On one hand, we find exact solutions for certain choices of couplings between a scalar field and the torsion tensor of a teleparallel connection and certain scalar field potentials, and thus proof the existence of scalarized black holes in these theories. On the other hand, we show that it is possible to establish no-scalar-hair theorems similar to what is known in general relativity for other choices of these functions.

I. INTRODUCTION

Black holes (BHs) play a crucial role to describe astrophysical objects. According to General Relativity (GR) they are only characterised by three parameters: their mass M, angular momentum J and charge Q [1–3]. This fact is established by the so-called no-hair theorem [4, 5] which gave rise to the conjecture that in the presence of any kind of matter, the result of a gravitational collapse would give rise to a Kerr-Newmann black hole which is fully described by these three physical quantities [6]. For realistic astrophysical systems, the charge is thought to be very small so it is usually neglected when one is studying black hole configurations. Furthermore, by assuming that the black hole is a non-rotating one, GR predicts that the unique vacuum spherically symmetric solution is the Schwarzschild solution which characterises a black hole only by its mass [7, 8].

Different studies have been trying to find black holes solutions violating the no-hair theorem. This is of course not possible in vacuum for GR, but it is possible to evade the assumptions on which the theorem is based by adding a scalar field minimally or non-minimally coupled to the gravitational sector. Several studies have found that, depending on how the coupling is implemented, the no-hair theorem is still valid or not. In [9], it was shown that the simplest massless minimally coupled scalar field with a Lagrangian of the form $\mathcal{L} = \frac{1}{4}\mathring{R} - \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi)$ still respects the no-hair theorem. Further, adding a mass term $-\frac{1}{2}\mu^2\psi^2$ still does not change this fact [10]. However, by replacing the mass term by a generic potential $\mathcal{V}(\psi)$, i.e. for a general Einstein-Klein-Gordon Lagrangian, one may circumvent the no-hair theorem depending on the choice of $\mathcal{V}(\psi)$ [3, 11–15]. Another way to construct scalarized black hole solutions can be found by allowing the scalar field to be complex [16–18] or considering non-minimal couplings between the scalar field and the Ricci scalar, for example in Lagrangians like $\mathcal{L} = \frac{1}{4}F(\psi)\mathring{R} - \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) - \mathcal{V}(\psi)$ with some specific potentials and coupling functions [19–23]. Other attempts to construct scalarized black hole solutions consider a scalar field coupled to modifications and extensions of GR, motivated by the search for dark matter, dark energy and quantum gravity (see [24–29] for reviews). These new theories of gravity contain additional terms to the Ricci

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scalar (or its reformulations in terms of torsion or non-metricity) in the gravitational action. One of the most popular theories exhibiting scalarized black holes is the one where a Gauss-Bonnet invariant $\overset{\circ}{G}$ is coupled to the scalar field, namely where the Lagrangian is of the form $\mathcal{L} = \frac{1}{4}\overset{\circ}{R} - \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) + F(\psi)\overset{\circ}{G}$. In these theories, it was shown that scalarized black holes exists for several choices of the non-minimal coupling function $F(\psi)$ [30–36]. The first example was provided in [30, 31] assuming a linear coupling ($F(\psi) \propto \psi$). The Gauss-Bonnet invariant being a total divergence in $4D$, such a coupling ensure that the model present a symmetry under $\psi \to \psi + c$ for a given constant c, dubbed as the shift-symmetry. Later on, under the assumption of a quadratic coupling ($F(\psi) \propto \psi^2$) hairy black holes were constructed in [33] as the result of the so-called spontaneous scalarization process. One should also see [26] for the construction of hairy black holes extrapolating between the shift-symmetric and spontaneously scalarized types of solutions ($F(\psi) = \gamma_1\psi + \gamma_2\psi^2$). Some of these black holes are constructed in such a way that one has an asymptotically flat spacetime described by the Schwarzschild metric when the scalar field is vanishing. Finally, hairy black holes have also be found in theories including a non-minimal coupling between the first derivative of the scalar field and the Einstein tensor, namely $\mathcal{L} = \overset{\circ}{R} - (\eta\, g^{\mu\nu} - \beta\, \overset{\circ}{G}{}^{\mu\nu})\partial_\mu\psi\partial_\nu\psi$, see [37]. In this case, the scalar field is allowed to be time dependent even tough the spacetime is assumed to be spherically symmetric. It should be noted that all these theories can evade the no-hair theorem even without potential.

To the best of our knowledge, all the scalarised black holes found so far are described by extensions of GR based on Riemannian geometry, i.e. on the Levi-Civita connection with vanishing torsion and non-metricity. In this paper, we will study theories constructed in the framework of teleparallel gravity where the geometry is described by a connection with vanishing curvature and non-metricity, but non-vanishing torsion [38–40]. It is well known that in this framework, it is possible to formulate a theory which is dynamically equivalent to GR, that is usually called the teleparallel equivalent of general relativity (TEGR) [38]. The TEGR action, defined by the so called torsion scalar T, and the Einstein-Hilbert action differ only by a boundary term B as the Ricci scalar for the Levi-Civita connection $\overset{\circ}{R}$ can be expressed as $\overset{\circ}{R} = -T + B$. Since B is a boundary term in the action it does not influence the field equations of the theory as long as it appears linearly in the action. However, as in the standard case of Gauss-Bonnet scalar-tensor theory, the boundary term would contribute to the dynamics when coupled with the scalar field in the action.

In this work, we extend the TEGR action by considering a scalar field which is non-minimally coupled to B and T. These theories are the so-called scalar-torsion theories (or teleparallel scalar-tensor theories), considered in the context of cosmology only with the torsion scalar already in [41], later with the boundary term in [42], and constructed in all generality in the series of articles [43–45] as well as in the review [39] (see Sec. 5.8). Exact wormholes solutions, induced by the existence of a non-trivial scalar field, were already found in [46, 47]. Furthermore, it has been obtained that the PPN parameters of this theory only differs from the GR PPN parameters in α and β when there is a coupling between the boundary term and the scalar field [48]. Our main aim is to find teleparallel scalarised black holes and to investigate the existence of no-hair theorems within these theories.

This paper is organised as follows: In Sec. II we give a brief overview of teleparallel theories of gravity and we present the scalar-torsion theory considered with its corresponding field equations. Sec. III gives the most important results of the paper: we demonstrate the existence of new scalarised black hole solutions. The section starts by presenting the field equations in spherical symmetry in Sec. III A for the two possible tetrads which solve the antisymmetric field equations of the theory. Then, we analyse two main theories, namely, one which assumes only a coupling between the boundary term with the scalar field (Sec. III B) and the other which only considers a coupling between the torsion scalar and the scalar field (Sec. III C). In these two theories we analyse the possible two tetrads and we provide new exact spherically symmetric solutions. In Sec. IV we discuss no-hair theorems for certain classes of theories. Finally, we provide the main conclusions of our findings in Sec. V. Throughout this paper we assume the units where $c = 1$ and the metric signature is $(+---)$. Objects labeled with a $^\circ$ are constructed with help of the Levi-Civita connection of the metric defined by the tetrad. Spacetime indices are raised in lowered with the spacetime metric, Latin indices refer to the tangent spacetime indices and Greek ones to the spacetime ones.

II. TELEPARALLEL GRAVITY WITH NON-MINIMALLY COUPLED SCALAR FIELDS

We briefly recall the setup of covariant teleparallel gravity as well as the action of the scalar-torsion gravity theory and its field equations, which we will solve in the next section to demonstrate the existence of scalarized black holes in scalar-torsion theories.

Standard references for teleparallel gravity are [38, 39, 49, 50]. Scalar-tensor theories in teleparallel gravity have been discussed very detailed in the series of articles [43–45].

A. Covariant teleparallel gravity

The fundamental variables in teleparallel gravity are a tetrad coframe $\theta^a = \theta^a{}_\mu dx^\mu$, resp. its dual $e_a = e_a{}^\mu \partial_\mu$ satisfying

$$\theta^a{}_\mu e_a{}^\nu = \delta^\nu_\mu, \quad \theta^a{}_\mu e_b{}^\mu = \delta^a_b, \quad g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu \,, \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric, and a flat, metric compatible Lorentz spin connection with coefficients $\omega^a{}_{b\mu}$ which is generated by Lorentz transformation matrices $\Lambda^a{}_b$

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b \,. \tag{2}$$

The torsion tensor of the spin connection is given by

$$T^\rho{}_{\mu\nu} = e_a{}^\rho \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu\right) \,. \tag{3}$$

The spin connection is constructed such that under the action of local Lorentz transformations on the tetrads and the spin connection, the torsion tensor with only spacetime indices, as we just displayed, is invariant.

In the formulation of teleparallel geometry, i.e. on a manifold whose geometry is defined by tuple $(\theta^a, \omega^a{}_{b\mu})$ consisting of a tetrad and a flat, metric compatible spin connection, it is well known that one can work in the so called Weitzenböck gauge. This means with a tetrad (the Weitzenböck tetrad) and vanishing spin connection, i.e. with the tuple $(\theta^a, 0)$. Throughout this article we will work with Weitzenböck tetrads.

For the scalar-torsion theories of gravity we study in Sec. II B we will need further ingredients. First of all, one can show that the Ricci scalar $\mathring{R}$ computed with the Levi-Civita connection can be related to the so-called torsion scalar T and the boundary term B as follows

$$\mathring{R} = -T + B \,, \quad T = \frac{1}{2} T^{\rho\mu\nu} S_{\rho\mu\nu} \,, \quad B = 2\mathring{\nabla}_\nu T_\mu{}^{\mu\nu} \,, \tag{4}$$

where we have defined the superpotential as

$$S_{\rho\mu\nu} = \frac{1}{2}\left(T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho}\right) - g_{\rho\mu} T^\sigma{}_{\sigma\nu} + g_{\rho\nu} T^\sigma{}_{\sigma\mu} \,. \tag{5}$$

Eq. (4) says that the Ricci scalar differs by a boundary term with respect to the torsion scalar. Thus, a Lagrangian constructed by T would provide the same equations of motions that the Einstein-Hilbert action constructed from $\mathring{R}$. That particular theory is called the "Teleparallel equivalent of GR" (TEGR) since it provides the same equations as the Einstein's field equations, expressed in terms of the teleparallel quantities.

B. Scalar-torsion theories of gravity

To demonstrate that the teleparallel boundary term induces scalarized black holes, similarly to what the Gauss-Bonnet boundary term does, we consider scalar-torsion theories of gravity as they have been introduced in [43]. Let $\theta = \det \theta^a{}_\mu$ and let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{V}$ be functions of the scalar field ψ. Then,

$$S_{\mathrm{g}}\left[\theta^a, \omega^a{}_b, \psi\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T + 2\mathcal{B}(\psi)X + 2\mathcal{C}(\psi)Y - 2\kappa^2\mathcal{V}(\psi)\right]\theta\, \mathrm{d}^4x \,, \tag{6}$$

defines a scalar-torsion theory of gravity with X being the kinetic term of the scalar field and Y a derivative coupling term defined by

$$X = -\frac{1}{2} g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \,, \quad Y = g^{\mu\nu} T^\rho{}_{\rho\mu}\partial_\nu\psi \,. \tag{7}$$

It has been demonstrated that this action is equivalent to the action

$$S_{\mathrm{g}}\left[\theta^a, \omega^a{}_b, \psi\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T + 2\mathcal{B}(\psi)X - \tilde{\mathcal{C}}(\psi)B - 2\kappa^2\mathcal{V}(\psi)\right]\theta\, \mathrm{d}^4x \,, \tag{8}$$

up to a boundary term, when one chooses the coupling function $\mathcal{C}(\psi) = \partial_\psi \tilde{\mathcal{C}}(\psi)$. In the following we always assume this equivalence.

This gravitational part of the action is coupled to matter actions S_m, which we assume to depend only on the tetrad via the metric, not on the spin connection or the torsion. Variation of the total action $S = S_\mathrm{g} + S_\mathrm{m}$ with respect to the tetrad, then leads to gravitational field equations of the form

$$E_a{}^\mu = \kappa^2 \Theta_a{}^\mu \,. \tag{9}$$

Multiplying with the components of tetrad and lowering the indices with the metric yields equations of the type $E_{\mu\nu} = \kappa^2\Theta_{\mu\nu}$, which decay into symmetric $E_{(\mu\nu)} = \kappa^2\Theta_{(\mu\nu)}$ and antisymmetric $E_{[\mu\nu]} = 0$ part. It can be shown that the atisymmetric equations of the variation of the action with respect to the tetrad are identical to the equations one obtains when one varies the action with respect to the spin connection [39, 49]. Since we are working with the Weitzenböck tetrad, the tetrad alone must satisfy both, the symmetric and the antisymmetric field equations.

The general scalar-torsion theories of gravity (6) and (8) are dynamically equivalent to general relativity with a minimally coupled scalar field by choosing either ($\mathcal{A} = \tilde{\mathcal{C}} = \alpha, \mathcal{B} = \beta$) or ($\mathcal{A} = \alpha, \mathcal{B} = \beta, \tilde{\mathcal{C}} = 0$), for constants α and β. The first case yields the standard Einstein-Klein-Gordon action, while the second differs from that by a boundary term, and is the standard TEGR action supplemented by a minimally coupled scalar field.

The field equations of the general theory (6) have been derived in [43]. Hereafter we will assume that $\mathcal{B}(\psi) = \beta$. By taking variations with respect to the tetrads, the symmetric part is given by

$$\begin{aligned}\left(\mathcal{A}'(\psi) + \tilde{\mathcal{C}}'(\psi)\right) S_{(\mu\nu)}{}^\rho \psi_{,\rho} + \mathcal{A}(\psi)\left(\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}\right) + \left(\frac{1}{2}\beta - \tilde{\mathcal{C}}''(\psi)\right)\psi_{,\rho}\psi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} \\ - (\beta - \tilde{\mathcal{C}}''(\psi))\psi_{,\mu}\psi_{,\nu} + \tilde{\mathcal{C}}'(\psi)\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\psi - \mathring{\Box}\psi g_{\mu\nu}\right) + \kappa^2\mathcal{V}(\psi)g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}\,,\end{aligned} \tag{10}$$

where here and in the following a $'$ denote the derivative of a function with respect to its argument, while the antisymmetric part of the field equation becomes

$$\left(\mathcal{A}'(\psi) + \tilde{\mathcal{C}}'(\psi)\right) T^\rho{}_{[\mu\nu}\psi_{,\rho]} = 0\,. \tag{11}$$

Variations with respect to the scalar field provides us the modified Klein-Gordon equation:

$$\frac{1}{2}\mathcal{A}'(\psi)T + \frac{1}{2}\tilde{\mathcal{C}}'(\psi)B - \beta\mathring{\Box}\psi + \kappa^2\mathcal{V}'(\psi) = \kappa^2\epsilon\Theta\,, \tag{12}$$

where ϵ provides a non-minimal coupling between the scalar field and the matter content.

To demonstrate that scalarized black holes exist in the scalar-torsion theories of gravity we just displayed, we will focus on the equations in vacuum, i.e. for $\Theta_{\mu\nu} = 0$. Observe that the symmetric field equation (10) can also be understood as Einstein equations with effective energy momentum tensor $\Theta^{(\psi)}{}_{\mu\nu}$ sourced by the superpotential of the torsion and the scalar field, by writing (10) as

$$\begin{aligned}\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = -\frac{1}{\mathcal{A}(\psi)}\Big[\left(\mathcal{A}'(\psi) + \tilde{\mathcal{C}}'(\psi)\right) S_{(\mu\nu)}{}^\rho\psi_{,\rho} + \left(\frac{1}{2}\beta - \tilde{\mathcal{C}}''(\psi)\right)\psi_{,\rho}\psi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} \\ - (\beta - \tilde{\mathcal{C}}''(\psi))\psi_{,\mu}\psi_{,\nu} + \tilde{\mathcal{C}}'(\psi)\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\psi - \mathring{\Box}\psi g_{\mu\nu}\right) + \kappa^2\mathcal{V}(\psi)g_{\mu\nu}\Big] =: \Theta^{(\psi)}{}_{\mu\nu}\,.\end{aligned} \tag{13}$$

This interpretation of the field equations helps us to identify an effective energy $\mathcal{E}_\psi$ of the scalar field to characterise its behaviour. Assuming the existence of a timelike killing vector field $\xi = \xi^\mu\partial_\mu$, $\mathcal{E}_\psi$ is given by [51, 52]

$$\mathcal{E}_\psi \coloneqq \int_\mathcal{H} \Theta^{(\psi)\mu}{}_\nu\, \xi^\nu n_\mu \sqrt{-\det(h)}\mathrm{d}^3x\,, \tag{14}$$

where $\mathcal{H}$ is a spacelike hypersurface with induced negative definite metric h and timelike conormal field $n = n_\mu \mathrm{d}x^\mu$.

From Sec. III on we will consider spacetime geometries with a metric of the form

$$\mathrm{d}s^2 = A^2\mathrm{d}t^2 - \frac{C^2}{A^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{15}$$

where $\mathrm{d}\Omega^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$. The slightly unusual parametrization in terms of the functions $A = A(r)$ and $C = C(r)$ here is chosen to ensure non-degeneracy of the metric at the black hole horizon. For the analysis of the solutions we present in the following, we remark that such metrics are asymptotically flat if and only if $A(r) \stackrel{r\to\infty}{\longrightarrow} c < \infty$ and $C(r) \stackrel{r\to\infty}{\longrightarrow} c$, where c is a constant, so that $CA^{-1} \stackrel{r\to\infty}{\longrightarrow} 1$.

The metric clearly possesses a timelike Killing vector field $\xi = \partial_t$ and we can express the energy of the scalar field in the black hole exterior region, i.e. on a spacelike hypersurface $\mathcal{H}_{\rm ext}$ that is defined by $x^0 = t = \text{const.}$ and $x^1 = r \geq r_h$, r_h being the radius of the event horizon, with conormal $n = A\mathrm{d}t$ and induced metric $h = -C^2 A^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$, as

$$\mathcal{E}_\psi = \int_{r_h}^{\infty}\int_{S^2} \Theta^{(\psi)t}{}_t C r^2 \mathrm{d}r\,\mathrm{d}\Omega = 4\pi\int_{r_h}^{\infty} \rho_\psi(r) C r^2 \mathrm{d}r\,, \tag{16}$$

where $\rho_\psi(r) \coloneqq \Theta^{(\psi)t}{}_t(r)$ will be interpreted as the (effective) scalar field energy density.

III. SCALARIZED TELEPARALLEL SPHERICALLY SYMMETRIC STATIC BLACK HOLES

The field equations in spherical symmetry can be derived for the general Lagrangian (8). Based on what is know from teleparallel theories of gravity in spherical symmetry [53–55], it is straightforward to find two classes of tetrads which solve the antisymmetric fields equations, a real and a complex one. For these tetrads, one can display the remaining symmetric field equations, which is what we do first in Sec. III A.

Solving them in general, for arbitrary coupling functions $\mathcal{A}(\psi)$ and $\tilde{\mathcal{C}}(\psi)$ is not possible. However, for specific choices of theories, with certain fixed values for these coupling functions we find scalarized black hole solutions. In Sec. III B we consider only a coupling between the scalar field and the boundary term ($\mathcal{A}(\psi) = \alpha$), while in Sec. III C only a coupling between the torsion scalar and the scalar field ($\tilde{\mathcal{C}}(\psi) = 0$) is considered.

In both classes of scalar-torsion extensions of general relativity, scalarized black holes exist for suitable choices of the coupling functions.

In the next section IV we will discuss the existence and non-existence of no-hair theorems for scalar-torsion theories.

A. The field equations for the real and complex tetrad

In [54] it was found that for a generic $f(T, B, \psi, X)$ gravity theory, the antisymmetric field equations (11) are solved for only two possible classes of tetrads (in the Weitzenböck gauge). This statement assumes that both the metric and teleparallel connection respect spherical symmetry. The first tetrad is real and it is described by [55, 56]

$$\theta^{(1)a}{}_\mu = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & CA^{-1}\cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ 0 & CA^{-1}\sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ 0 & CA^{-1}\cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{17}$$

whereas the second one is complex and it is given by [54]

$$\theta^{(2)a}{}_\mu = \begin{pmatrix} 0 & \frac{iC}{A} & 0 & 0 \\ iA\sin\theta\cos\phi & 0 & -r\sin\phi & -r\sin\theta\cos\theta\cos\phi \\ iA\sin\theta\sin\phi & 0 & r\cos\phi & -r\sin\theta\cos\theta\sin\phi \\ iA\cos\theta & 0 & 0 & r\sin^2\theta \end{pmatrix}. \tag{18}$$

The remaining equations, which need to be integrated, are the symmetric ones (10) and the scalar field equation (12). There will be two different sets of equations depending on the choice of the tetrad which satisfy the antisymmetric field equations. In addition, we require the scalar field to only depend on the radial coordinate to also respect spherical symmetry $\psi = \psi(r)$.

For the real tetrad (17) the torsion scalar, vector part of torsion and boundary term become

$$T = \frac{2(A-C)(2rA' + A - C)}{r^2C^2}\,, \tag{19}$$

$$T_\mu = \left(0, -\frac{A'}{A} - \frac{2}{r} + \frac{2C}{rA}, 0, 0\right), \tag{20}$$

$$B = \frac{2\left(rCA'(rA' - 2C) + A\left(-r^2A'C' + rC(rA'' + 6A') - 2C^2\right) + 2A^2(C - rC')\right)}{r^2C^3}\,, \tag{21}$$

while for the complex tetrad (18) these quantities are

$$T = \frac{2\left(2rAA' + A^2 + C^2\right)}{r^2C^2}\,, \tag{22}$$

$$T_\mu = \left(0, -\frac{A'}{A} - \frac{2}{r}, 0, 0\right), \tag{23}$$

$$B = \frac{2\left(r^2CA'^2 + rA\left(C\left(rA'' + 6A'\right) - rA'C'\right) + 2A^2\left(C - rC'\right)\right)}{r^2C^3}. \tag{24}$$

Notice that as expected, both tetrads give the same form of the Ricci scalar $\overset{\circ}{R}$

$$\overset{\circ}{R} = -T + B = -\frac{2A\left(rA' + 2A\right)C'}{rC^3} + \frac{2\left(r^2A'^2 + rA\left(rA'' + 4A'\right) + A^2\right)}{r^2C^2} - \frac{2}{r^2}. \tag{25}$$

One can immediately notice some differences between the real and complex tetrad. First of all, the torsion scalar and boundary term only vanish for the real tetrad when one considers a Minkowski metric. Further, the vectorial part of torsion T_μ in the complex tetrad does not depend on C whereas in the real tetrad it does. Another interesting point to remark is that in the Schwarzschild case, the torsion scalar and boundary term become $4/r^2$ for the complex tetrad and therefore they are regular at the horizon $r = 2M$, while, for the real tetrad, these scalars diverge at $r = 2M$.

1. *The field equations for the real tetrad*

Employing the first tetrad (17) in the symmetric vacuum field equations, (10) and (12) with $\Theta_{\mu\nu} = 0$, leads to

$$\begin{aligned} E^{(1)}_{tt} = 0 \;=\;& \mathcal{A}(\psi)\frac{A^2\left(-2rACA' - A^2\left(C - 2rC'\right) + C^3\right)}{r^2C^3} + \tilde{\mathcal{C}}'(\psi)\left(\frac{A^4\psi''}{C^2} - \frac{A^3\psi'\left(-rCA' + rAC' - 2C^2\right)}{rC^3}\right) \\ & - \frac{2A^3(A - C)\psi'\mathcal{A}'(\psi)}{rC^2} + \frac{A^4\psi'^2\tilde{\mathcal{C}}''(\psi)}{C^2} - \frac{A^4\psi'^2\beta}{2C^2} + \kappa^2A^2\mathcal{V}(\psi), \end{aligned} \tag{26a}$$

$$E^{(1)}_{rr} = 0 = -\frac{\left(2rAA' + A^2 - C^2\right)\mathcal{A}(\psi)}{r^2A^2} + \left(\frac{A'}{A} + \frac{2}{r}\right)\psi'\tilde{\mathcal{C}}'(\psi) + \frac{\kappa^2C^2\mathcal{V}(\psi)}{A^2} + \frac{1}{2}\beta\psi'^2, \tag{26b}$$

$$\begin{aligned} E^{(1)}_{\theta\theta} = 0 \;=\;& -\frac{A\psi'\left(rA' + A - C\right)\mathcal{A}'(\psi)}{rC^2} + \tilde{\mathcal{C}}'(\psi)\left(\frac{A\psi'\left(rCA' - rAC' + C^2\right)}{rC^3} + \frac{A^2\psi''}{C^2}\right) \\ & + \frac{\mathcal{A}(\psi)\left(-rCA'^2 + A\left(rA'C' - C\left(rA'' + 2A'\right)\right) + A^2C'\right)}{rC^3} + \frac{A^2\psi'^2\tilde{\mathcal{C}}''(\psi)}{C^2} - \frac{A^2\psi'^2\beta}{2C^2} + \kappa^2\mathcal{V}(\psi), \end{aligned} \tag{26c}$$

$$\begin{aligned} E^{(1)}_{\psi} = 0 \;=\;& \frac{(A - C)\left(2rA' + A - C\right)\mathcal{A}'(\psi)}{r^2C^2} + \frac{\beta A\psi'\left(2rCA' + A\left(2C - rC'\right)\right)}{rC^3} + \frac{\beta A^2\psi''}{C^2} + \kappa^2\mathcal{V}'(\psi) \\ & + \frac{\tilde{\mathcal{C}}'(\psi)\left(rCA'\left(rA' - 2C\right) + A\left(-A'r^2C' + rC\left(rA'' + 6A'\right) - 2C^2\right) + 2A^2\left(C - rC'\right)\right)}{r^2C^3}. \end{aligned} \tag{26d}$$

2. *The field equations for the complex tetrad*

For the complex tetrad (18), the Eqs. (10) and (12) in vacuum become

$$\begin{aligned} E^{(2)}_{tt} = 0 \;=\;& \mathcal{A}(\psi)\frac{A^2\left(-2rACA' - A^2\left(C - 2rC'\right) + C^3\right)}{r^2C^3} + \tilde{\mathcal{C}}'(\psi)\frac{A^3\left(C\left(A'\psi' + A\psi''\right) - AC'\psi'\right)}{C^3} - \frac{2A^4\psi'\mathcal{A}'(\psi)}{rC^2} \\ & + \frac{A^4\psi'^2\tilde{\mathcal{C}}''(\psi)}{C^2} - \frac{\beta A^4\psi'^2}{2C^2} + \kappa^2A^2\mathcal{V}(\psi), \end{aligned} \tag{27a}$$

$$E^{(2)}_{rr} = 0 = \frac{\mathcal{A}(\psi)\left(-2rAA' - A^2 + C^2\right)}{r^2A^2} + \left(\frac{A'}{A} + \frac{2}{r}\right)\psi'\tilde{\mathcal{C}}'(\psi) + \frac{\kappa^2C^2\mathcal{V}(\psi)}{A^2} + \frac{1}{2}\beta\psi'^2, \tag{27b}$$

$$\begin{aligned} E^{(2)}_{\theta\theta} = 0 \;=\;& -\mathcal{A}'(\psi)\frac{A\left(rA' + A\right)\psi'}{rC^2} + \tilde{\mathcal{C}}'(\psi)\frac{A\left(C\left(A'\psi' + A\psi''\right) - AC'\psi'\right)}{C^3} \\ & + \frac{\mathcal{A}(\psi)\left(-rCA'^2 + A\left(rA'C' - C\left(rA'' + 2A'\right)\right) + A^2C'\right)}{rC^3} + \frac{A^2\psi'^2\tilde{\mathcal{C}}''(\psi)}{C^2} - \frac{\beta A^2\psi'^2}{2C^2} + \kappa^2\mathcal{V}(\psi), \end{aligned} \tag{27c}$$

$$E^{(2)}_{\psi} = 0 = \frac{\left(2rAA' + A^2 + C^2\right)\mathcal{A}'(\psi)}{r^2C^2} + \frac{\beta A\left(2rCA'\psi' + A\left(C\left(r\psi'' + 2\psi'\right) - rC'\psi'\right)\right)}{rC^3}$$

$$+\frac{\tilde{\mathcal{C}}'(\psi)\left(r^2CA'^2+rA\left(C\left(rA''+6A'\right)-rA'C'\right)+2A^2\left(C-rC'\right)\right)}{r^2C^3}+\kappa^2\mathcal{V}'(\psi)\,. \tag{27d}$$

3. Remarks on the field equations

A first observation is that the rr-component and the modified Klein-Gordon equation for the two tetrads differ only by

$$E_{rr}^{(1)}-E_{rr}^{(2)} = -\frac{2\left(rA'+A\right)}{r^2C}\left(\mathcal{A}'(\psi)+\tilde{\mathcal{C}}'(\psi)\right), \tag{28}$$

$$E_{\psi}^{(1)}-E_{\psi}^{(2)} = -\frac{(C-1)C}{r^2A^2}\left(\mathcal{A}(\psi)+\kappa^2r^2\mathcal{V}(\psi)\right), \tag{29}$$

and the difference between the other components contain much more terms. This means that when $C=1$ ($g_{rr}=-1/g_{tt}$), the modified Klein-Gordon equations become the same for the two tetrads.

Moreover ,the systems (26a)-(26d) and (27a)-(27d) there are only four equations since we did not display $E_{\phi\phi}$, which is directly related to $E_{\theta\theta}$ via $E_{\phi\phi}=\sin^2\theta E_{\theta\theta}$, due to the spherical symmetry we are considering. We like to remark that these four equations for the three variables A, C, and ψ are not independent. For example, for the first tetrad, one can notice that Eq. (26c) is solved, by employing A' (and its derivative) from (26b), C' from (26a) and ψ'' from (26d).

Thus generically, we have three equations for three unknowns A, C and ψ as function of r. Hence, in principle, there should exist solutions for sufficiently regular choices of the couplings $\mathcal{A}$, $\tilde{\mathcal{C}}$ and the potential $\mathcal{V}$. In this paper, we will focus on explicit solutions that admit a closed form in terms of either usual or special functions. It is worth noting that surely the spectrum of solutions in general is way larger. Indeed, the systems (26) or (27) are highly non-linear systems of differential equations. In general it will not be possible (i.e. for any choice of $\mathcal{A}(\psi)$, $\mathcal{V}(\psi)$ and $\tilde{\mathcal{C}}(\psi)$) to find solutions in closed form. The existence of such solutions is a remarkable property achieved for some specific choices of the functions $\mathcal{A}(\psi)$, $\mathcal{V}(\psi)$ and $\tilde{\mathcal{C}}(\psi)$. The analysis in this paper could then be completed by a numerical investigation of the spectrum of solutions for some specific choices of these functions. We keep such an analysis for later work.

In the next sections we find non-trivial analytic solutions for the field equations for specific scalar-torsion extensions of general relativity.

B. $\mathcal{A}(\psi)=\alpha$: only non-minimal coupling between the scalar field and the boundary term

It is well known that non-minimally coupled scalar to boundary terms built from the metric and the curvature of the Levi-Civita connections, such as for example most famously the Gauss-Bonnet boundary term, lead to scalar-tensor theories of gravity which have scalarized black hole solutions [30, 33, 35, 36].

In this section we consider a coupling between the scalar field and the teleparallel boundary term B, which is the difference between the torsion scalar and the Ricci scalar. We find that such theories also allow for several classes of scalarized solutions. Among them is the Schwarzschild-(anti)-de-Sitter (S-(A)dS) spacetime equipped with a non-trivial scalar field.

1. Analytical analysis of the field equation for the real tetrad

For the real tetrad we discuss several types of analytical solutions to boundary-term non-minimal coupling scalar-torsion gravity.

Let us first explore the case $\alpha=0$ which is a limiting case when the theory does not have a GR limit. When we solve Eqs. (26a)-(26b) for the coupling functions $\tilde{\mathcal{C}}'$ and $\tilde{\mathcal{C}}''$ and then we replace those expressions into (26c) we get $\mathcal{V}(\psi)=-\frac{\beta A^2\psi'^2}{2\kappa^2C^2}$. By replacing this form of the potential back in (26b) we immediately get that $\tilde{\mathcal{C}}'\psi'=0$ which is only true for the trivial case when either the scalar field is a constant or the contribution from the boundary term disappears in the equations ($\tilde{\mathcal{C}}=$ const). Thus we found that, as expected since the torsion scalar is missing in the action, this case gives only trivial solutions of a non-dynamical scalar field. Another limiting case is when we set $A(r)=A_0/r^2$ which eliminates the coupling function $\tilde{\mathcal{C}}'(\psi)$ in (26b). For this case, again there are only trivial solutions to the system.

Non-trivial solutions can be found by setting without loss of generality $\alpha = 1 \neq 0$, and $A \neq A_0/r^2$. From now on, we will assume those conditions. We can solve (26a)-(26c) for $\tilde{\mathcal{C}}'(\psi), \tilde{\mathcal{C}}''(\psi)$ and the potential $\mathcal{V}(\psi)$ and then replace them in (26d) yielding

$$\begin{aligned} 0 \;=\; & -r^3A^2A'''C^2 + r^2A^2C\,(rA'-A)\,C'' + 3r^2A^2\,(A-rA')\,C'^2 + r^2ACC'\,(2A'\,(rA'+C) + A\,(3rA''-2A')) \\ & +C^2\Big[\left(-2r^2CA'^2 + rC^2A' + r^3A'^3 - 2C^3\right) + A^2\left(r\,(rA''+A') + C\left(2-\beta r^2\psi'^2\right)\right) \\ & +A\left(-2r^2CA'' + r^2A'\,(A'-2rA'') + 3C^2\right) - 3A^3\Big]\,. \end{aligned} \tag{30}$$

This equation is thus a general necessary relationship that must be always true for any form of the potential and any form of the coupling function. The above equation admits several interesting solutions, but is involved to be solved in all generality. We discuss two classes of solutions.

- A first class of solutions can be found in the case $C(r) = 1$ and $\mathcal{V}(\psi) = 0$. Solving (26b)-(26c) for $\tilde{\mathcal{C}}', \tilde{\mathcal{C}}''$ and replacing them in (26a), gives a remaining equation

$$\begin{aligned} 0 \;=\; & 2\left(r^3A'^3 + rA' - 1\right) + A^2\left(2 + 4r^2A'' - 2rA' - \beta r^2\psi'^2\right) - 4A^3 \\ & +2A\left(2r^2A'^2 + rA'\left(r^2A''+2\right)+2\right)\,. \end{aligned} \tag{31}$$

An exact solution of this equation is

$$\mathrm{d}s^2 = \Big(1-Kr\Big)^2\mathrm{d}t^2 - \Big(1-Kr\Big)^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{32}$$

$$\psi(r) = \sqrt{-\frac{6}{\beta}}\log(1-Kr)\,, \quad \tilde{\mathcal{C}}(\psi) = \sqrt{\frac{-2\beta}{3}}\psi\,, \quad \mathcal{V}(\psi) = 0\,. \tag{33}$$

To obtain this solution we have first found the scalar field and then we have inverted it from $\psi = \psi(r)$ to $r = r(\psi)$. After doing this, we can write down the remaining field equations depending on ψ (not r). Then, the coupling function and potential can be easily found by solving the remaining field equations. This procedure will be also used in the next sections to find the form of the potential and the coupling function. This solution is non-asymptotically flat and contains a horizon at $r_h = 1/K$. Since we assumed that the scalar field and the coupling function are real, the kinetic parameter must be $\beta < 0$.

- A second class of solutions we like to display has $C(r) = -\frac{1}{2\sqrt{1-\Lambda r^2}}$ and the metric behaves as

$$\mathrm{d}s^2 \;=\; \Big(1-\Lambda r^2\Big)\mathrm{d}t^2 - \frac{1}{4}\Big(1-\Lambda r^2\Big)^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{34}$$

$$\psi(r) \;=\; \sqrt{\frac{6}{\beta}}\log r\,. \tag{35}$$

Note that $C(r)^2$ appears in the metric and $C(r)$ in the tetrad so that its sign could only affects the tetrad. Nevertheless, $C(r) = +\frac{1}{2\sqrt{1-\Lambda r^2}}$ does not have the same solution as above. The reason comes from the fact that $C(r)$ appears linearly in some of the equations and thus, its sign affects the form of the solution even though the metric is unaffected. By replacing the above solution in the field equations we find that in order to obtain the potential and the coupling function one needs to invert $\psi = \psi(r)$ to $r = r(\psi)$ using (35). Then, it is easy to solve the remaining part of the system (26a)-(26d) leading to the following form of the potential and coupling function

$$\kappa^2\mathcal{V}(\psi) \;=\; 3\left(16\Lambda^2 e^{\sqrt{\frac{2\beta}{3}}\psi} + e^{-\sqrt{\frac{2\beta}{3}}\psi} - 14\Lambda\right)\,, \tag{36}$$

$$\tilde{\mathcal{C}}(\psi) \;=\; -\frac{1}{4}\left(\sqrt{6\beta}\psi + 5\log\left(1-\Lambda e^{\sqrt{\frac{2\beta}{3}}\psi}\right)\right)\,. \tag{37}$$

This solution represents again a non-asymptotically flat spherically symmetric solution, which behaves similarly as a S-(A)dS spacetime without a mass or at large r and $g_{rr} \neq -1/g_{tt}$. Contrary to the solution (33), the above solution requires $\beta > 0$ to ensure that both the coupling function and the scalar field are real. We like to point out that for $\Lambda > 0$ this solution cannot be interpreted as a black hole since the determinant of the metric, proportional to $C(r)$, diverges at the horizon. For $\Lambda < 0$ there is no horizon at all.

Thus we found two classes of non-asymptotically flat scalarized solutions for a scalar field non-minimally coupled to the teleparallel boundary term in the case of a real tetrad.

2. Analytical analysis of the field equation for the complex tetrad

In this section we will find exact solutions for the complex tetrad when one assumes that there is only a coupling between the boundary term and the scalar field. If we replace Eqs. (27a) and (27b) into (27c) we find that the metric functions must obey the following differential equation

$$C'(r) = \frac{C\left(r^2AA''+r^2A'^2-A^2+C^2\right)}{rA\left(rA'-A\right)}\,. \tag{38}$$

This equation is, analogously what we found for the real tetrad, a necessary condition that has to hold, independently of the theory considered. It cannot easily be solved without making further assumptions.

We investigate two different main cases.

- First we set again $C=1$, which implies from (38) that the metric becomes

$$\mathrm{d}s^2=\Big(1-\frac{2M}{r}-\Lambda r^2\Big)\mathrm{d}t^2-\Big(1-\frac{2M}{r}-\Lambda r^2\Big)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,. \tag{39}$$

  This means that for any form of the potential or coupling function this case yields the unique solution for the metric, which is given by a S-(A)dS metric. This result goes in a similar directions as what was found for $f(T)$-gravity in [54], where it was shown that for any $f(T)$ gravity theory, the condition $g_{tt}=-1/g_{rr}$ imposes the metric to be Schwarzchild-de-Sitter.

  Having found the metric which solves the necessary condition (38) we still need to solve the remaining field equations. There are different ways of solving them since one can assume either a form of $\mathcal{V}(\psi)$, $\tilde{\mathcal{C}}(\psi)$ or even the form of the scalar field ψ.

  To demonstrate the existence of scalarized S-(A)dS spacetimes we choose the coupling function to be $\tilde{\mathcal{C}}(\psi)=(\tilde{\mathcal{C}}_0/p)\,\psi^p$, which makes Eqs. (27a)-(27b)

$$2\kappa^2r^2\psi\,\mathcal{V}(\psi) = 2\tilde{\mathcal{C}}_0\left(3M+3\Lambda r^3-2r\right)\psi^p\psi'+r\psi\left(\beta\left(2M+\Lambda r^3-r\right)\psi'^2-6\Lambda r\right)\,, \tag{40}$$

$$0 = \tilde{\mathcal{C}}_0(p-1)r\psi^p\psi'^2+\tilde{\mathcal{C}}_0\psi^{p+1}\left(r\psi''-2\psi'\right)-\beta r\psi^2\psi'^2\,. \tag{41}$$

  When we now fix p, we only need to solve equation (41) for ψ to obtain complete solutions of the field equations with a potential given by (40), where the difficulty lies in expressing the potential as function of ψ, instead of as a function of r. We display solutions for several choices:

  – $p=2$, $\tilde{\mathcal{C}}_0\neq\beta/2$:

$$\psi(r) = \left(u\,r^3+v\right)^{\frac{\tilde{\mathcal{C}}_0}{u}}\,,\quad \tilde{\mathcal{C}}(\psi)=\frac{1}{2}\tilde{\mathcal{C}}_0\psi^2\,,\quad u=2\tilde{\mathcal{C}}_0-\beta\,,\quad v=3\tilde{\mathcal{C}}_0\psi_1\,, \tag{42}$$

$$\begin{aligned}\kappa^2\mathcal{V}(\psi) &= -3\alpha\Lambda+\frac{18\tilde{\mathcal{C}}_0^3\psi^{2-\frac{u}{\tilde{\mathcal{C}}_0}}(Mu-\Lambda v)}{u^2}-\frac{9\tilde{\mathcal{C}}_0^2v\beta\psi^{2-\frac{2u}{\tilde{\mathcal{C}}_0}}(2Mu-\Lambda v)}{2u^2}+\frac{9\tilde{\mathcal{C}}_0^2\Lambda\psi^2(2\tilde{\mathcal{C}}_0+u)}{2u^2}\\ &\quad+\frac{3\tilde{\mathcal{C}}_0^2\left(3v\beta\psi^{2-\frac{2u}{\tilde{\mathcal{C}}_0}}-(6\tilde{\mathcal{C}}_0+u)\psi^{2-\frac{u}{\tilde{\mathcal{C}}_0}}\right)\sqrt[3]{\psi^{u/\tilde{\mathcal{C}}_0}-v}}{2u^{4/3}}\,.\end{aligned} \tag{43}$$

  – $p=2$, $\tilde{\mathcal{C}}_0=\beta/2$:

$$\psi(r) = e^{\frac{r^3\psi_1}{3}}\,,\quad \tilde{\mathcal{C}}(\psi)=\frac{\beta}{4}\psi^2\,, \tag{44}$$

$$\kappa^2\mathcal{V}(\psi) = -3\alpha\Lambda+\frac{1}{2}\beta\psi^2\left(9\Lambda(\log(\psi)+1)\log(\psi)+3M\psi_1(2\log(\psi)+1)-\sqrt[3]{3}\psi_1^{2/3}(3\log(\psi)+2)\sqrt[3]{\log(\psi)}\right)\,. \tag{45}$$

  – $p=1$:

$$\psi(r) = -\frac{\tilde{\mathcal{C}}_0}{\beta}\log\left(3\tilde{\mathcal{C}}_0\psi_1-\beta r^3\right)\,,\quad \tilde{\mathcal{C}}(\psi)=\tilde{\mathcal{C}}_0\psi\,, \tag{46}$$

$$\kappa^2\mathcal{V}(\psi) = -3\alpha\Lambda-\frac{9p\tilde{\mathcal{C}}_0^2\Lambda}{2\beta}+\frac{27\tilde{\mathcal{C}}_0^3\psi_1e^{\frac{2\beta\psi}{\tilde{\mathcal{C}}_0}}(3\tilde{\mathcal{C}}_0\Lambda\psi_1+2\beta M)}{2\beta}-\frac{3}{2}\tilde{\mathcal{C}}_0^2e^{\frac{\beta\psi}{\tilde{\mathcal{C}}_0}}\sqrt[3]{\frac{3\tilde{\mathcal{C}}_0\psi_1-e^{-\frac{\beta\psi}{\tilde{\mathcal{C}}_0}}}{\beta}}\left(9\tilde{\mathcal{C}}_0\psi_1e^{\frac{\beta\psi}{\tilde{\mathcal{C}}_0}}+1\right)\,. \tag{47}$$

Here, ψ_1 is an integration constant. All these solutions are scalar fields on S-(A)dS spacetimes, solving the scalar-torsion field equations. As mentioned before, instead of choosing the coupling function $\tilde{\mathcal{C}}(\psi)$ one can also choose either ψ or $\mathcal{V}(\psi)$ and obtain new solutions.

Even for $\Lambda \to 0$, i.e. when the metric just becomes the Schwarzschild metric, we see that these theories allow for non-trivial scalar field solutions. However, these do not have any influence on the geometry since the integrand in the expression for the energy of the scalar field (16) is zero. This is expected since we know from (13) that the effective energy density of the scalar field must be proportional to $\Theta^{(\psi)t}{}_t(r)$ which is equated to the tt component of the Einstein tensor. The latter vanishes for a S-(A)dS spacetime.

- Second we consider $C \neq 1$ and a vanishing potential ($\mathcal{V}(\psi) = 0$).

  Following a similar approach as before, it is possible to manipulate the field equations to get an equation that does not depend on the coupling function. To do this, we first solve (27b) for $\tilde{\mathcal{C}}(\psi)$ and (27c) for $\mathcal{V}(\psi)$ and replace those expressions in the modified Klein-Gordon equation (27d). Then, we assume that $\mathcal{V}(\psi) = 0$ and rearrange the equation yielding

$$
\begin{aligned}
0 \;=\; & r^2 A A'^2 \left(A^2\left(6 - 2\beta r^3 \psi'\psi'' - \beta r^2 \psi'^2\right) + 10C^2\right) + rA'\Big(A^2C^2\left(4 + \beta r^2\psi'^2\right) \\
& + A^4\left(6 - 2\beta r^3\psi'\psi'' - \beta r^2\psi'^2\right) - 2C^4\Big) - 2r^3A^2A'^3\left(6 + \beta r^2\psi'^2\right) + A''\Big[2r^3A^3A'\left(6 + \beta r^2\psi'^2\right) \\
& + r^2A^2\left(A^2\left(6 + \beta r^2\psi'^2\right) - 6C^2\right)\Big] + 2A^3C^2\left(2 + \beta r^2\psi'^2\right) - 4AC^4 + 4\beta r^2 A^5\psi'\left(r\psi'' + \psi'\right) .
\end{aligned} \tag{48}
$$

To find solutions to this equations, we will further assume that the theory is $\tilde{\mathcal{C}}(\psi) = \tilde{\mathcal{C}}_0\psi$ and also that the scalar field behaves as $\psi = \psi_0 \log(r)$. By replacing these assumptions in (27b) and in the above equation we find

$$
C(r)^2 \;=\; \frac{A}{2}\Big(2rA'(2 - \tilde{\mathcal{C}}_0\psi_0) + A(2 - \psi_0(\beta\psi_0 + 4\tilde{\mathcal{C}}_0))\Big) , \tag{49}
$$

$$
A(r) \;=\; r^{-\frac{\beta\psi_0}{\tilde{\mathcal{C}}_0} - 2}\left(A_0 + r^{\frac{4\beta\psi_0 + 6\tilde{\mathcal{C}}_0 - \beta\tilde{\mathcal{C}}_0\psi_0^2}{2\beta\psi_0 + 6\tilde{\mathcal{C}}_0}}\right)^{\frac{\beta\psi_0 + 3\tilde{\mathcal{C}}_0}{2\tilde{\mathcal{C}}_0 - \tilde{\mathcal{C}}_0^2\psi_0}} , \tag{50}
$$

with A_0 being an integration constant. Finally, by replacing all these expressions in the remaining field equation we find that $\tilde{\mathcal{C}}_0 = (-2 \pm \sqrt{4 - 2\beta\psi_0^2})/\psi_0$. Thus, the final solution gives us the following form of the metric

$$
\mathrm{d}s^2 = r^{-2}\Big(A_0 r^{\pm\sqrt{2w}} + r^{\pm\sqrt{w/2}}\Big)\mathrm{d}t^2 - \frac{A_0\sqrt{w}}{2} r^{\pm\sqrt{w/2}}\Big(A_0 r^{\pm\sqrt{w/2}} + 1\Big)^{-1}(\pm 2\sqrt{2} - \sqrt{w})\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 , \tag{51}
$$

with $w = 2 - \beta\psi_0^2$, corresponding to the coupling function, potential and scalar field being equal to

$$
\tilde{\mathcal{C}}(\psi) = \Big(\frac{-2 \pm \sqrt{2w}}{\psi_0}\Big)\psi , \quad \mathcal{V}(\psi) = 0 , \quad \psi = \psi_0 \log(r) . \tag{52}
$$

The metric (51) does not obey $g_{tt} = -1/g_{rr}$ and it is always non-asymptotically flat. Note that $w \geq 0$ to ensure that the physical quantities are real.

Thus we demonstrated explicitly the existence of non-trivial scalar fields in scalar-torsion theories of gravity based on a non-minimal coupling to the teleparallel boundary term B, for the real and the complex tetrad in spherical symmetry. For a non-trivial scalar field all the solutions we found in this section were not asymptotically flat. In Sec. IV, we find no hair theorems for certain classes of scalar-torsion theories with non-minimal coupling to B, which help to narrow down the class of theories in which one should search for asymptotically flat scalarized solutions.

Next, we will see that further interesting solutions exist for a non-minimal coupling to the torsion scalar.

C. $\tilde{\mathcal{C}}(\psi) = 0$: only non-minimal coupling between the scalar field and the torsion scalar

Another scalar tensor theory in teleparallel gravity is to introduce a non-minimal coupling between the torsion scalar and the scalar field, for which boson stars have been found [57]. Here we present several scalarized solutions for such theories for the real and the complex tetrad. In contrast to the previous section, in this section we find asymptotically flat scalarized black holes.

1. Analytical analysis of the field equation for the real tetrad

We follow a similar strategy as the previous section. By solving the equations (26a)-(26c) for $\mathcal{A}(\psi), \mathcal{A}'(\psi)$ and $\mathcal{V}(\psi)$ and then replacing these expressions (and their derivatives) into (26d) one finds a cumbersome equation which is generically independent of the potential $\mathcal{V}(\psi)$ and the coupling function $\mathcal{A}(\psi)$ and thus only depends on $A(r)$, $C(r)$, $\psi(r)$. In this case the equation is much more involved than (30). For completeness, this equation is displayed in the appendix (see Eq. (A1)).

Again, by making some assumptions on the solution we are looking for, we can solve the equation. Imposing that $C = 1$ we can discuss two classes of solutions.

- The first one is given by the ansatz $A(r) = 1 - \frac{K}{r^p}$ which implies the following metric

$$\mathrm{d}s^2 = \Big(1 - \frac{K}{r^p}\Big)^2 \mathrm{d}t^2 - \Big(1 - \frac{K}{r^p}\Big)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,. \tag{53}$$

By replacing this ansatz in the necessary constraint equation (A1) that does not depend on the coupling function and the potential, we find that ψ is represented by Appell hypergeometric functions for any power-law p. In particular, three particular special cases can have less involved form for the scalar field, namely

$$\psi(r) = -\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}}\,, \quad p = 1\,, \tag{54}$$

$$\psi(r) = \frac{2\psi_0}{5K\left(1 - Kr^2\right)^{5/4}}\,, \quad p = -2\,, \tag{55}$$

$$\psi(r) = \frac{4\psi_0\sqrt[4]{r}\left(1 - \frac{\sqrt{r}}{K}\right)^{5/6} {}_2F_1\left(\frac{1}{2}, \frac{11}{6}; \frac{3}{2}; \frac{\sqrt{r}}{K}\right)}{K\left(K - \sqrt{r}\right)^{5/6}}\,, \quad p = 1/2\,, \tag{56}$$

where ${}_2F_1$ is the hypergeometric function. Interestingly, the first solution, with $p = 1$, reproduces the metric called the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution found in Riemannian conformal scalar-vacuum theory [19, 58, 59]. This solution behaves as a extremal Reissner-Nordström black hole for $K \neq 0$ and reduces to the Minkowski metric for $K = 0$. Clearly, the scalar field diverges at the horizon $r = r_h = K$. However, the energy of the scalar field (16) for this solution is $\mathcal{E}_\psi = 4\pi K$ which is always positive and finite. This solution was shown to be unstable against linear perturbations [60] but those studies were performed by considering the Riemannian conformal scalar-vacuum theory. In our case, this solution could have a different behaviour since the field equations are different. There are other claims about the BBMB solution related to the energy-momentum tensor which emphasises that it is ill-defined at the horizon [61].

For the cases $p = 1$ and $p = -2$, we can solve the remaining field equations to get the following exact solutions

$$\mathcal{A}(\psi) = -\frac{1}{8}\beta\psi^2\,, \quad \mathcal{V}(\psi) = 0\,, \quad p = 1\,, \tag{57}$$

$$\mathcal{A}(\psi) = \frac{5\beta}{8}\psi^2\,, \quad \kappa^2\mathcal{V}(\psi) = -\frac{15}{4}K\beta\psi^2\,, \quad p = -2\,. \tag{58}$$

Even though the case $p = 1/2$ is an analytical solution to the field equations, it is not easy to invert $\psi = \psi(r)$ to $r = r(\psi)$ to then find an explicit form of the coupling function and the potential.

- The second class of exact solutions can be obtained by adding the assumption $\mathcal{V}(\psi) = 0$ to $C = 1$. Assuming furthermore that $rA' + A - 1 \neq 0$ (which would reduce to the solution (53) with $p = 1$ in case of an equality) and $2rAA' + A^2 - 1 \neq 0$ (which would require $\psi =$ const. in case of an equality), one can solve (26b) and (26c) for $\mathcal{A}, \mathcal{A}'$ and then replace those expressions in (26a) to find the following differential equation

$$0 = rA(A-1)A'' + (A-1)^2 A' - r(A+1)A'^2\,, \tag{59}$$

which has the following solution for the metric

$$\mathrm{d}s^2 = \Big(2 - \frac{r}{2K} + \frac{\sqrt{r(r-4K)}}{2K}\Big)^2 \mathrm{d}t^2 - \Big(2 - \frac{r}{2K} + \frac{\sqrt{r(r-4K)}}{2K}\Big)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2\,, \tag{60}$$

where K is an integration constant. By replacing the above equation in the modified Klein-Gordon equation (26d) we find that the scalar field is

$$\psi(r) = \frac{\psi_0\left(\sqrt{r(r-4K)} - 4K + r\right)\sqrt[4]{\sqrt{r(r-4K)} - 2K + r}}{3Kr^{3/4}\sqrt{r-4K}}\,. \tag{61}$$

Finally, we find that the coupling function must be of the form

$$\mathcal{A}(\psi) = \frac{3\beta}{8}\psi^2\,. \tag{62}$$

The solution (60) is well defined for $K \to 0$ and has one horizon at $r_h = 4K$. Moreover it is asymptotically flat. The energy of the scalar field (16) is $\mathcal{E}_\psi = -8\pi K$. For $K > 0$ there exists a well defined horizon, but the energy of the scalar field is negative, while for $K < 0$ the energy of the scalar field is positive, but the spacetime does not possess a horizon, since there does not exist any $r > 0$ such that $g_{tt}(r) = 0$.

A series expansion for $K \ll 1$ gives us the following metric and scalar field up to second order in K

$$\mathrm{d}s^2 \;=\; \Big(1 - \frac{2K}{r} - \frac{3K^2}{r^2}\Big)\mathrm{d}t^2 - \Big(1 - \frac{2K}{r} - \frac{3K^2}{r^2}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 + \mathcal{O}(K^3)\,, \tag{63}$$

$$\psi(r) \;=\; \frac{2\sqrt[4]{2}\psi_0}{3K} - \frac{\sqrt[4]{2}\psi_0}{r} - \frac{3K\psi_0}{2\sqrt[4]{8}r^2} - \frac{35K^2\psi_0}{12\sqrt[4]{8}r^3} + \mathcal{O}(K^3)\,, \tag{64}$$

which behaves similarly as a Reisser-Nordstrdöm (RN) metric with $K = M$ acting as a mass but the charge appearing with the opposite sign behaving as an imaginary charge ($Q^2 = -3M^2$).

To summarize this section: we demonstrated that there exist several non-trivial scalarized solutions to scalar-torsion gravity with a non-minimal coupling to the torsion scalar, which goes with a coupling function $\mathcal{A} \sim \psi^2$ and $\tilde{\mathcal{C}} = 0$ for the real tetrad. In particular we also found asymptotically flat solutions. As we will see in Sec. IV, the classes of theories we considered here consistently evade the no-hair theorems since the potentials do not satisfy a condition which is necessary for the no-hair theorems to hold.

2. *Analytical analysis of the field equation for the complex tetrad*

We will now find exact solutions for the complex tetrad (18) for the class of theories defined by $\tilde{\mathcal{C}}(\psi) = 0$.

Due to the structure of the field equations (27a)-(27d), it is convenient to separate the study into the case with vanishing and non-vanishing potential. The latter case turns out to be very complicated and it is not possible to find exact solutions, which is why we assume $\mathcal{V}(\psi) = 0$ throughout this section.

When we solve (27a) and (27b) for $\mathcal{A}'(\psi)$ and $\mathcal{A}(\psi)$ and replace these expressions into (27c), we find, as in the previous cases, a model independent necessary equation that the metric components have to satisfy. For any non-trivial coupling function $\mathcal{A}(\psi)$ the following must hold

$$C(r)^2 = A\left[rA' - A\left(\frac{rA''}{A'} + 1\right)\right]\,. \tag{65}$$

- Let us again first consider the case $C(r) = 1$. Then, it is straightforward to solve (65) and then the remaining equations for the system (27a)-(27d) can be manipulated in the following way. First, we can solve the equation (27b) for the coupling function and replace this expression into (27a). The corresponding equation can be easily solved for ψ. Lastly, we invert $\psi = \psi(r)$ to $r = r(\psi)$ and solve the remaining equation for the coupling function. After doing all of this procedure, we find the following solution

$$\mathrm{d}s^2 \;=\; \Big(\frac{1}{2p+1} - 2Mr^{-2p-1}\Big)\mathrm{d}t^2 - \Big(\frac{1}{2p+1} - 2Mr^{-2p-1}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{66}$$

$$\mathcal{A}(\psi) \;=\; -\frac{1}{4}p\beta\,\psi^2\,, \quad \psi(r) = \psi_0 r^p\,, \quad \mathcal{V}(\psi) = 0\,, \quad p \neq 0\,, \tag{67}$$

with p, ψ_0 and M being integration constants. In this solution, we have assumed that $p \neq 0$ and thus $\psi = \psi(r)$. Again this solution is non-asymptotically flat.

- As second case we assume $C \neq 1$. We can solve (27a) and (27b) for $\tilde{\mathcal{C}}'(\psi)$ and $\tilde{\mathcal{C}}''(\psi)$ and replace them into into the modified Klein-Gordon equation (27d). If one further replaces (65) and assumes $\mathcal{V}(\psi) = 0$, one finds that the scalar field $\psi(r)$ and the metric function $A(r)$ must satisfy the following differential equation

$$\begin{aligned} 0 \;=\; & 5r^3A^3A''^3\psi' - 3r^3A'^6\psi' - r^2AA'^5\left(4r\psi''+5\psi'\right) + A'^4\left(2r^3AA''\psi' - 4rA^2\left(r\psi''+2\psi'\right)\right) \\ & + A'^3\left(r^3A^2A'''\psi' - 2r^2A^2A''\psi' + 8A^3\left(r\psi''+2\psi'\right)\right) + A'^2\Big(-4r^2A^3A'''\psi' - 2r^3A^2A''^2\psi' \\ & + 12rA^3A''\left(r\psi''+2\psi'\right)\Big) + A'\left(r^2A^3A''^2\left(4r\psi''+15\psi'\right) - 3r^3A^3A'''A''\psi'\right) . \end{aligned} \tag{68}$$

To be able to solve this equation we consider two sub cases:

- A way to find a solution for (68) is by imposing that the terms multiplying A''' in (68) vanish. After doing this, we can find the form of the scalar field by solving the equation (68). Then, to find a solution we just need to use the remaining field equation to get the coupling function. All this procedure gives the following exact solution

$$\mathrm{d}s^2 \;=\; \frac{1}{r}\left(2+Kr^{1/3}\right)^3\mathrm{d}t^2 - \frac{1}{3}\Big(1+\frac{2}{Kr^{1/3}}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{69}$$

$$\mathcal{A}(\psi) \;=\; \frac{3\beta e^{-\frac{\psi}{3}}\left(Ke^{\psi/3}-1\right)^2}{4K}\,, \quad \psi(r) = \log\left(\frac{r}{\left(Kr^{1/3}+2\right)^3}\right)\,, \quad \mathcal{V}(\psi) = 0\,. \tag{70}$$

This solution is non-asymptotically flat.

- We now fix the coupling constant $\mathcal{A}(\psi)$ and then solve the field equations. Choosing $\mathcal{A}(\psi) = \mathcal{A}_0\psi^2$ and also by imposing $\psi(r) = \psi_0 r^p$, we can easily solve the system (27a)-(27d) which gives a more general solution than (66). This solution has the same form as (66) for the scalar field $\psi(r)$ and the coupling function $\mathcal{A}(\psi) = -\frac{1}{4}p\beta\psi^2$, meaning that the field equations are solved only when the coupling constant is $\mathcal{A}_0 = -\frac{p\beta}{4}$. We consider the non-trivial case $p \neq 0$. For this solution the metric does not obey $C \neq 1$. Thus, the solution has the following form

$$\mathrm{d}s^2 \;=\; \Big(A_0 - 2Mr^{-2p-1}\Big)\mathrm{d}t^2 - \Big(\frac{A_0(2p+1)}{A_0 - 2Mr^{-2p-1}}\Big)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,. \tag{71}$$

Here, M, p and A_0 are integration constants. Again, this solution is non-asymptotically flat. By choosing $A_0 = \frac{1}{1+2p}$, one recovers the metric (66). It should be noted that the solution (66) was found by only assuming $C(r) = 1$ while the above solution was obtained by assuming the coupling function and also the form of the scalar field. This means that the solution (66) is the most general one satisfying $C(r) = 1$ for our theory while the above solution might not be the most general solution for the squared coupling theory. Observe that the metric (71) is a generalisation of the metric (66) which we found earlier.

Let us now study (71) by supposing that $p \ll 1$ and expanding the terms up its first order corrections. By doing this, we find that the metric and the scalar field quantities becomes Schwarzschild plus a modified correction related to the scalar field:

$$\mathrm{d}s^2 = \Big[1 - \frac{2M}{r} + \frac{4Mp}{r}\log(r)\Big]\mathrm{d}t^2 - \Big(1-\frac{2M}{r}\Big)^{-1}\Big[1+2p-\Big(1-\frac{2M}{r}\Big)^{-1}\frac{4Mp}{r}\log(r)\Big]\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 + \mathcal{O}(p^2)\,, \tag{72}$$

$$\mathcal{A}(\psi) = -\frac{1}{4}p\beta\psi^2\,, \quad \psi(r) = \psi_0 + p\phi_0\log(r) + \mathcal{O}(p^2)\,, \quad \mathcal{V}(\psi) = 0\,, \quad p \neq 0\,. \tag{73}$$

Here, we have also set $A_0 = 1$ to recover Schwarzschild at the background level. Notice that since we assumed that, the above expanded metric would have a different behaviour as (66). This solution then corresponds to a scalarised black hole solution with p being the parameter that controls the scalar field. It is worth mentioning that this expanded form of the metric contains a logarithmic term similarly as it was previously obtained in $f(T)$ and $f(T,B)$ gravity where perturbed solutions around Schwarzschild were found [54, 62–65].

Thus, also for the complex tetrad we could find scalarized solutions for torsion scalar non-minimally coupled scalar fields, however none of them is asymptotically flat.

IV. ON NO-HAIR THEOREMS IN SCALAR-TORSION THEORIES OF GRAVITY

In the context of scalar-tensor theories of gravity several no-hair theorems have been proven, which state that, under certain assumptions, there exist no non-trivial scalar fields on black hole spacetimes [3].

In the previous section we demonstrated the existence of static spherically symmetric solutions in scalar-torsion theories for certain types of non-minimal coupling between the scalar field and the torsion of the teleparallel connection. Hence, a general no-hair theorem cannot be expected.

However, when we study the field equations carefully we can derive some necessary constraints that have to be satisifed by the coupling functions $\mathcal{A}$ and $\tilde{\mathcal{C}}$ as well as by the potential $\mathcal{V}$ so that non-trivial scalar fields can exist.

On the one hand, taking the trace of the symmetric vacuum field equations (10), we find

$$-\mathcal{A}\mathring{R} - 2(\mathcal{A}' + \tilde{\mathcal{C}}')T^\mu\partial_\mu\psi + (\beta - 3\tilde{\mathcal{C}}'')\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - 3\tilde{\mathcal{C}}'\mathring{\Box}\psi + 4\kappa^2\mathcal{V} = 0\,. \tag{74}$$

A little further manipulation of the $\mathring{\Box}\psi$ term yields the convenient form

$$-\mathcal{A}\mathring{R} - 2(\mathcal{A}' + \tilde{\mathcal{C}}')T^\mu\partial_\mu\psi + \beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi + 4\kappa^2\mathcal{V} = 3\mathring{\nabla}_\mu(\tilde{\mathcal{C}}'\mathring{\nabla}_\mu\psi)\,. \tag{75}$$

On the other hand the vacuum scalar field equation (12) has to hold

$$\frac{1}{2}\mathcal{A}'(\psi)T + \frac{1}{2}\tilde{\mathcal{C}}'(\psi)B - \beta\mathring{\Box}\psi + \kappa^2\mathcal{V}' = 0\,, \tag{76}$$

which can be multiplied with ψ to obtain

$$\mathring{\nabla}_\mu\left(\left(\mathcal{A}' + \tilde{\mathcal{C}}'\right)\psi T^\mu - \beta\psi\mathring{\nabla}^\mu\psi\right) + \beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - \frac{1}{2}\psi\mathcal{A}'\mathring{R} - \left(\mathcal{A}' + \tilde{\mathcal{C}}' + \psi\mathcal{A}'' + \psi\tilde{\mathcal{C}}''\right)T^\mu\partial_\mu\psi + \kappa^2\psi\mathcal{V}' = 0\,. \tag{77}$$

We can integrate equation (75) and (77) over any volume $V \subset M$ to obtain the constraints

$$\int_V \mathrm{d}^4x\;\theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - \mathcal{A}\mathring{R} - 2(\mathcal{A}' + \tilde{\mathcal{C}}')T^\mu\partial_\mu\psi + 4\kappa^2\mathcal{V}\right) = \int_{\partial V} 3(\tilde{\mathcal{C}}'\mathring{\nabla}^\mu\psi)n_\mu\mathrm{d}\sigma \tag{78}$$

$$\int_V \mathrm{d}^4x\;\theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - \frac{1}{2}\psi\mathcal{A}'\mathring{R} - (\mathcal{A}' + \tilde{\mathcal{C}}' + \psi\mathcal{A}'' + \psi\tilde{\mathcal{C}}'')T^\mu\partial_\mu\psi + \kappa^2\psi\mathcal{V}'\right) = -\int_{\partial V}((\mathcal{A}' + \tilde{\mathcal{C}}')\psi T^\mu - \beta\psi\mathring{\nabla}^\mu\psi)n_\mu\mathrm{d}\sigma\,, \tag{79}$$

where n^μ is the normal of the boundary of the volume V and $\mathrm{d}\sigma = \sqrt{h}\,\mathrm{d}^3u$ is the pull-back of the volume element $\mathrm{d}^4x\;\theta$ to ∂V equipped with coordinates u.

From now on, we will assume that M is a spherically symmetric asymptotically flat black hole spacetime and V its exterior region. The boundary of this volume ∂V is thus given by the event horizon $\mathcal{H}$ and the asymptotic flat region $\mathcal{H}_\infty$ $(A(r) \to 1, C(r) \to 1)$ at infinity. Regarding the behaviour of the scalar field, we will assume that ψ only depends on r (to respect the spherical symmetry) and that approaching $\mathcal{H}_\infty$ we have $\partial_\mu\psi \to 0$.

We know that, at the horizon $\mathcal{H}$, the non-vanishing components of the normal are given by n^t and n^ϕ only (since $\mathcal{H}$ is a Killing horizon). In addition, for the spherically symmetric tetrads which we employ, we found that the vector torsion has only a non-vanishing T^r component, see (20) and (23). Consequently, the boundary term in (78) vanishes, while the boundary term in (79) reduces to

$$\int_{\mathcal{H}_\infty}\left(\left(\mathcal{A}' + \tilde{\mathcal{C}}'\right)\psi T^\mu\right)n_\mu\mathrm{d}\sigma\,. \tag{80}$$

This boundary term vanishes for both the real and complex tetrad, since in both cases, the vector torsion vanishes where $A(r) \to 1, A'(r) \to 0$ and $r \to \infty$ (in the asymptotic flat regions), see again (20) and (23).

Let us continue with the analysis of the equations (78) and (79). Under the above assumptions, these equations reduce to

$$\int_V \mathrm{d}^4x\;\theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - \mathcal{A}\mathring{R} - 2\left(\mathcal{A}' + \tilde{\mathcal{C}}'\right)T^\mu\partial_\mu\psi + 4\kappa^2\mathcal{V}\right) = 0\,, \tag{81}$$

$$\int_V \mathrm{d}^4x\;\theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi - \frac{1}{2}\psi\mathcal{A}'\mathring{R} - \left(\mathcal{A}' + \tilde{\mathcal{C}}' + \psi\mathcal{A}'' + \psi\tilde{\mathcal{C}}''\right)T^\mu\partial_\mu\psi + \kappa^2\psi\mathcal{V}'\right) = 0\,. \tag{82}$$

Substracting the two equations yields

$$\int_V \mathrm{d}^4x\ \theta\left(\left(\mathcal{A}-\tfrac{1}{2}\psi\mathcal{A}'\right)\mathring{R}+\left(\mathcal{A}'+\tilde{\mathcal{C}}'-\psi\mathcal{A}''-\psi\tilde{\mathcal{C}}''\right)T^\mu\partial_\mu\psi+\kappa^2\left(\psi\mathcal{V}'-4\mathcal{V}\right)\right)=0\,. \tag{83}$$

Thus, in total we find the necessary constraints (81), (82) and (83), which need to be satisfied in order for a non-trivial scalar field to exist. They impose restrictions on the kinetic term of the scalar field, the coupling functions and the potential. The third constraint (83) is of particular importance, since it allows us to replace the integral over the curvature scalar or the the vector torsion in (81) and (82) for several models. These then yield a constraint on the potential for the existence of non-trivial scalar fields in classes of scalar-torsion theories of gravity.

We explicitly discuss four cases:

1. A pure polynomial coupling to the torsion scalar by choosing $\mathcal{A}=\alpha\psi^m$ and $\tilde{\mathcal{C}}=0$. In this case equation (83) can be solved for the integral over the Ricci scalar

$$-\int_V \mathrm{d}^4x\ \theta\ \alpha\psi^m\mathring{R}=\int_V \mathrm{d}^4x\ \theta\left(2m\alpha\psi^{m-1}T^\mu\partial_\mu\psi+\frac{2\kappa^2}{2-m}(\psi\mathcal{V}'-4\mathcal{V})\right). \tag{84}$$

   Using this result in equations (81) and (82) gives from both equations the same constraint

$$\int_V \mathrm{d}^4x\ \theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi+\frac{2\kappa^2}{m-2}\left(2m\mathcal{V}-\psi\mathcal{V}'\right)\right)=0\,. \tag{85}$$

   Consequently, for all potentials $\mathcal{V}$ satisfying

$$\frac{2}{\beta(m-2)}\left(2m\mathcal{V}(\psi)-\psi\mathcal{V}'(\psi)\right)\leq 0\,,\quad \forall\,\psi(r)\,, \tag{86}$$

   non-trivial scalar field solutions cannot exist, since $\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi=-\frac{C^2}{A^2}\psi'^2\leq 0$. If the inequality (86) holds strictly, there cannot exist any scalar-field solution, since it would immediately contradict (85), while if the equality in (86) holds, then (85) implies that the scalar field must be constant.

2. Choosing $\mathcal{A}=\alpha\psi^2$ and $\psi\tilde{\mathcal{C}}''=\tilde{\mathcal{C}}'$ (i.e. $\tilde{\mathcal{C}}=\frac{c_1}{2}\psi^2+c_2$, in particular also for $\tilde{\mathcal{C}}=0$), gives from (83) directly

$$\int_V \mathrm{d}^4x\ \theta\ \kappa^2(\psi\mathcal{V}'-4\mathcal{V})=0\,. \tag{87}$$

   Hence for potentials $\mathcal{V}$ for which either $\psi\mathcal{V}'>4\mathcal{V}$ or $\psi\mathcal{V}'<4\mathcal{V}$ $(\forall\,\psi)$ holds, non-trivial scalar field cannot exist. The solutions we found in (57), (58) and around (62) evade this constraint for the potential. In particular for polynomial potential $\mathcal{V}=\psi^n$ we find that $\int_V\kappa^2(n-4)\psi^n=0$ which cannot be satisfied for a non-trivial scalar field and n being even, except for $n=4$.

3. Choosing $\mathcal{A}=\alpha$ and $\tilde{\mathcal{C}}'+\psi\tilde{\mathcal{C}}''=0$ (i.e. $\tilde{\mathcal{C}}=c_1\ln(\psi)+c_2$), turns the constraint (82) into

$$\int_V \mathrm{d}^4x\ \theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi+\kappa^2\psi\mathcal{V}'\right)=0\,. \tag{88}$$

   Hence, for this logarithmic non-minimal coupling to the boundary term, $\psi=\psi(r)$, we obtain a class of scalar-torsion theories of gravity for which a scalar no-hair Theorem holds similar to the minimally coupled general relativity, namely, there cannot be a scalarized static spherically symmetric black hole for any potential $\mathcal{V}$ which satisfies $\frac{\psi\mathcal{V}'}{\beta}\leq 0$.

4. Choosing a polynomial coupling to the boundary term $\tilde{\mathcal{C}}=\frac{\gamma}{m+1}\psi^{m+1}$ and $\mathcal{A}=\alpha$ gives from (83)

$$-\int_V \mathrm{d}^4x\ \theta\ \alpha\mathring{R}=\int_V \mathrm{d}^4x\ \theta\left((1-m)\gamma\psi^mT^\mu\partial_\mu\psi+\kappa^2(\psi\mathcal{V}'-4\mathcal{V})\right)\,, \tag{89}$$

   Using this in equation (81), as well as evaluating (82) which is already independent of the Ricci scalar, both gives

$$\int_V \mathrm{d}^4x\ \theta\left(\beta\mathring{\nabla}_\mu\psi\mathring{\nabla}^\mu\psi-(m+1)\gamma\psi^mT^\mu\partial_\mu\psi+\kappa^2\psi\mathcal{V}'\right)=0\,. \tag{90}$$

From this derived constraint we find that there cannot exist non-trivial scalar field profile if the potential $\mathcal{V}$ under consideration satisfies either

$$\frac{1}{\beta}\left(\psi\mathcal{V}' - (m+1)\gamma\psi^m T^r\psi'\right) \leq 0\,, \quad \forall \quad r > r_h\,, \tag{91}$$

or, using (90) in (89)

$$\frac{(m+1)}{m-1}\frac{1}{\beta}\left(\alpha\mathring{R} + \kappa^2(\psi\mathcal{V}' - 4\mathcal{V})\right) \leq 0\,, \quad \forall \quad r > r_h\,. \tag{92}$$

Since these inequalities depend on the spacetime torsion, resp. the spacetime curvature, we cannot formulate these conditions in terms of ψ alone. The interplay between the potential and the properties of spacetime needs to be taken into account.

Let us summarize our findings in a first no-hair theorem for scalar-torsion theories:

Theorem 1. *Consider a scalar-torsion theory of gravity defined by the action* (8)*, the tetrads* (17) *or* (18) *and a scalar field* $\psi = \psi(r)$*. There exist no spherically symmetric asymptotically flat scalarized black holes for the following couplings and potentials satisfying the corresponding displayed inequalities:*

1. $\mathcal{A} = \alpha\psi^m$, $\tilde{\mathcal{C}} = 0$ *and* $\frac{2}{\beta(m-2)}\left(2m\mathcal{V} - \psi\mathcal{V}'\right) \leq 0$;
2. $\mathcal{A} = \alpha\psi^2$, $\tilde{\mathcal{C}} = \frac{c_1}{2}\psi^2 + c_2$ *and either* $\psi\mathcal{V}' > 4\mathcal{V}$ *or* $\psi\mathcal{V}' < 4\mathcal{V}$;
3. $\mathcal{A} = \alpha$, $\tilde{\mathcal{C}} = c_1\ln(\psi) + c_2$ *and* $\frac{\psi\mathcal{V}'}{\beta} \leq 0$;
4. $\mathcal{A} = \alpha$, $\tilde{\mathcal{C}} = \frac{\gamma}{m+1}\psi^{m+1}$ *and* $\frac{1}{\beta}\left(\psi\mathcal{V}' - (m+1)\gamma\psi^m T^r\psi'\right) \leq 0$ *or* $\frac{(m+1)}{m-1}\frac{1}{\beta}\left(\alpha\mathring{R} + \kappa^2(\psi\mathcal{V}' - 4\mathcal{V})\right) \leq 0$.

Observe that a vanishing potential $\mathcal{V} = 0$ (and thus $\mathcal{V}'$=0) evades the no-hair constraints in Case 2. For the Cases 1 and 3, a vanishing potential implies that there are no asymptotically flat spherically symmetric scalarized black hole solutions. Finally, in Case 4, the existence of scalarized black holes is connected to the sign of the vector torsion, or equivalently, the Ricci scalar.

No hair theorems of this type can easily be obtained for further scalar-torsion theories of gravity, i.e. different choices of $\mathcal{A}$ and $\tilde{\mathcal{C}}$ as we presented here, by employing the algorithm we outlined below (83).

V. CONCLUSIONS

In this paper, we studied spherical symmetry in scalar-torsion theories of gravity by considering a scalar field non-minimally coupled to both the torsion scalar and the teleparallel boundary term. This theory contains different subclasses that have been studied in both teleparallel gravity and the standard Riemannian scalar-tensor gravity. For instance, when the non-minimal coupling is reduced to the minimal case (setting $\mathcal{A}(\psi) = \tilde{\mathcal{C}}(\psi) = 1/2$ and $\mathcal{B}(\psi) = \kappa^2$in (8)), the theory obtained is the standard Riemannian scalar-tensor theory minimally coupled with the Ricci scalar with the Einstein-Klein-Gordon Lagrangian $\mathcal{L} = \frac{1}{4\kappa^2}\mathring{R} - \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) - \mathcal{V}(\psi)$. Further, by choosing instead $\tilde{\mathcal{C}}(\psi) = -\mathcal{A}(\psi)$ in (8), the theory is extended to have a non-minimal coupling between the scalar field and the Riemannian Ricci scalar of the form $\mathcal{A}(\psi)\mathring{R}$. In all the other non-trivial cases, the theory would have couplings that cannot be obtained in the Riemannian case and they can be considered to be related only to teleparallel gravity.

As usual in teleparallel theories of gravity the field equations can be decomposed into a symmetric (10) and antisymmetric part (11). By imposing spherical symmetry, one can solve the antisymmetric field equation in two different ways : one the one hand with a real tetard expressed in (17) and, on the other hand with a complex one given by (18). Consequently, the symmetric field equations in spherical symmetry have two branches which were presented in Secs. III A 1-III A 2. We studied these equations in two main teleparallel scalar-torsion theories, one with only a non-minimal coupling between the boundary term and the scalar field (see Sec. III B) and another one with a only a non-minimal coupling between the torsion scalar and the scalar field (see Sec. III C). For these two theories, we split the study for the two possible tetrads and we presented exact solutions to scalar-torsion theories of gravity for the first time and thus demonstrated the existence of scalarized black holes in these theories.

For the boundary term coupled theory (see Sec. III B), we found two exact spherically symmetric solutions for the real tetrad, see Eqs. (32)-(33) and (34)-(37), respectively. These solutions are non-asymptotically flat and both metrics have one horizon. For the complex tetrad, we found S-(A)dS as exact solutions (39) for different non-minimal couplings (see (42)-(47)). One notices that even for the $\Lambda = 0$ case, the scalar field can have a non-trivial profile leading to a Schwarzschild geometry endowed with a non-trivial scalar field. Moreover, for the complex tetrad we also found a non-asymptotically flat solution described by a non-trivial scalar field (52) and a power-law type form of the metric (51).

In the theory which is defined by a coupling between the torsion scalar and the scalar field (see Sec. III C), we found three exact solutions for the real tetrad. The first one has the same metric as (53) with $p = 1$ which is the so-called BBMB black hole. The form of the scalar field and the coupling functions were displayed in (54) and (57). This solution is asymptotically flat and it has been found before in a Riemannian scalar-tensor theory which is conformally flat. Another solution that we found that leads to a non-asymptotically flat metric is described by (53) with $p = -2$ and the scalar field, potential and couplings are given by (55) and (58). The last solution for the real tetrad we found has an asymptotically flat metric (60) with a non-trivial scalar field (61) having a zero potential with a coupling of the form $\mathcal{A} = \frac{3\beta}{8}\psi^2$. When the constant related to the scalar field is assumed to be small, the metric becomes a RN-like metric with the charge being equal to the mass (extremal RN) but having an opposite sign in the charge-type term as in the RN metric. For the complex tetrad (see Sec. III C 2), we obtained three exact solutions and all of them are non-asymptotically flat. For the solution (71), one notices that when the scalar field contribution becomes small, the metric (72) can be written approximately as a Schwarzschild modified metric with the scalar field acting as an extra term that modifies the spacetime to be non-asymptotically flat.

To summarize, we found several non-asymptotically flat solutions and, most noteworthy, two asymptotically flat scalarized solutions (53) and (60) which emerge for a non-minimal coupling between the scalar field ψ and the torsion scalar T which is proportional to ψ^2.

As a natural complement to this analysis, we have also investigated no-scalar-hair arguments limiting the sectors in which spherically symmetric scalarized black holes can be found in these theories. These results are summarized in Theorem 1.

This paper is a first step in the systematic analysis of the existence of hairy black-holes in scalar-torsion theories of gravity. It will be continued by extending the investigations on teleparallel Gauss-Bonnet scalar-torsion theory, such as the theories discussed in [66–68]. In the Riemannian case, it is well known that these theories have asymptotically flat scalarized black holes with spontaneous scalarization [32, 33, 36]. Since the Gauss-Bonnet scalar-torsion theories contains the standard Riemannian case in a certain limit, it is obvious to mention that those theories also will contain those solutions. However, the nature of the pure teleparallel part is unknown and it would be interesting to explore those theories to find what kind of new scalarized black hole solutions can appear. Further the study of pseudo scalar/axion couplings [69] will also be extended to spherical symmetry.

Finally, a next step is to extend the results found here to rotating teleparallel black holes in axial symmetry [70, 71].

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Appendix A: Non-minimally coupling between scalar field and torsion scalar - real tetrad

For the real tetrad, one can use the field equations (26a)-(26d) to derive the following equation that does not depend on the form of the potential nor the coupling function:

$$
\begin{aligned}
0 \;=\;& \left(A^2+2rA'A-C^2\right)\Big[\left(\left(3C-rC'\right)\psi'+2rC\psi''\right)A^5+A^4\Big\{C'\psi'\left(A'+rA''\right)r^2+C\Big(C'\psi'-A'\left(3\psi'+2r\psi''\right)\\
&+r\left(r\psi'A'''-A''\left(3\psi'+2r\psi''\right)\right)\Big)r-C^2\left(5\psi'+4r\psi''\right)\Big\}+A^3\Big\{A'C'\psi'\left(A'+C'-rA''\right)r^3-C\Big(\left(\psi'+2r\psi''\right)A'^2\\
&+\left(C'\left(3\psi'+2r\psi''\right)+r\left(-2rA''\psi''-\psi'\left(2A''+C''-rA'''\right)\right)\right)A'-2r^2\psi'A''^2\Big)r^2+C^2\Big\{3C'\psi'+A'\left(3\psi'+2r\psi''\right)\\
&+r\left(A''\left(3\psi'+4r\psi''\right)-2r\psi'A'''\right)\Big\}r-2C^3\psi'\Big\}+A^2\Big\{CA'\Big(\psi'\left(-A'^2+\left(r\left(A''-C''\right)-2C'\right)A'+C'\left(C'+3rA''\right)\right)
\end{aligned}
$$

$$+2rA'(A'+C')\psi''\Big)r^3+C^2r^2\Big\{2(\psi'+2r\psi'')A'^2-(C'(3\psi'-2r\psi'')+r(2rA''\psi''+\psi'(3A''+C''-rA''')))A'$$
$$-r\psi'A''(C'+2rA'')\Big\}+rC^3(-5C'\psi'+A'(\psi'+2r\psi'')+r(A''(3\psi'-2r\psi'')+r\psi'A'''))-r^4A'^2C'(A'+C')\psi'$$
$$+C^4(6\psi'+4r\psi'')\Big\}+AC\Big\{\psi'\Big(2A'^3(A'+2C')r^4+CA'^2(3A'+C'-2rA'')r^3+C^2A'(3A'+5C'+rA'')r^2$$
$$+C^3(A'+2C'-3rA'')r-C^4\Big)-2rC(C+rA')\left(C^2+r^2A'^2\right)\psi''\Big\}-C^2\left(C^2+r^2A'^2\right)\left(C^2+2rA'C+3r^2A'^2\right)\psi'\Big]. \tag{A1}$$

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[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. W. H. Freeman, San Francisco, 1973.
[2] P. T. Chrusciel, J. Lopes Costa, and M. Heusler, "Stationary Black Holes: Uniqueness and Beyond," *Living Rev. Rel.* **15** (2012) 7, arXiv:1205.6112 [gr-qc].
[3] C. A. R. Herdeiro and E. Radu, "Asymptotically flat black holes with scalar hair: a review," *Int. J. Mod. Phys. D* **24** (2015) no. 09, 1542014, arXiv:1504.08209 [gr-qc].
[4] W. Israel, "Event horizons in static electrovac space-times," *Commun. Math. Phys.* **8** (1968) 245–260.
[5] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom," *Phys. Rev. Lett.* **26** (1971) 331–333.
[6] R. Ruffini and J. A. Wheeler, "Introducing the black hole," *Phys. Today* **24** (1971) no. 1, 30.
[7] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 189–196, arXiv:physics/9905030.
[8] W. Israel, "Event horizons in static vacuum space-times," *Phys. Rev.* **164** (1967) 1776–1779.
[9] J. Chase, "Event horizons in static scalar-vacuum space-times," *Communications in Mathematical Physics* **19** (1970) no. 4, 276–288.
[10] J. D. Bekenstein, "Transcendence of the law of baryon-number conservation in black hole physics," *Phys. Rev. Lett.* **28** (1972) 452–455.
[11] M. Heusler and N. Straumann, "Scaling arguments for the existence of static, spherically symmetric solutions of self-gravitating systems," *Classical and Quantum Gravity* **9** (1992) no. 9, 2177.
[12] M. Heusler, "No hair theorems and black holes with hair," *Helv. Phys. Acta* **69** (1996) no. 4, 501–528, arXiv:gr-qc/9610019.
[13] M. Heusler, "A No hair theorem for selfgravitating nonlinear sigma models," *J. Math. Phys.* **33** (1992) 3497–3502.
[14] J. D. Bekenstein, "Novel ''no-scalar-hair'' theorem for black holes," *Phys. Rev. D* **51** (1995) no. 12, R6608.
[15] D. Sudarsky, "A Simple proof of a no hair theorem in Einstein Higgs theory,," *Class. Quant. Grav.* **12** (1995) 579–584.
[16] I. Pena and D. Sudarsky, "Do collapsed boson stars result in new types of black holes?," *Class. Quant. Grav.* **14** (1997) 3131–3134.
[17] C. Herdeiro and E. Radu, "Construction and physical properties of Kerr black holes with scalar hair," *Class. Quant. Grav.* **32** (2015) no. 14, 144001, arXiv:1501.04319 [gr-qc].
[18] C. A. R. Herdeiro and E. Radu, "Kerr black holes with scalar hair," *Phys. Rev. Lett.* **112** (2014) 221101, arXiv:1403.2757 [gr-qc].
[19] J. D. Bekenstein, "Black Holes with Scalar Charge," *Annals Phys.* **91** (1975) 75–82.
[20] A. A. H. Graham and R. Jha, "Stationary Black Holes with Time-Dependent Scalar Fields," *Phys. Rev. D* **90** (2014) no. 4, 041501, arXiv:1407.6573 [gr-qc].
[21] A. Saa, "New no scalar hair theorem for black holes," *J. Math. Phys.* **37** (1996) 2346–2351, arXiv:gr-qc/9601021.
[22] A. Saa, "Searching for nonminimally coupled scalar hairs," *Phys. Rev. D* **53** (1996) 7377–7380, arXiv:gr-qc/9602061.
[23] T. P. Sotiriou and V. Faraoni, "Black holes in scalar-tensor gravity," *Phys. Rev. Lett.* **108** (2012) 081103, arXiv:1109.6324 [gr-qc].
[24] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, "Modified Gravity and Cosmology," *Phys. Rept.* **513** (2012) 1–189, arXiv:1106.2476 [astro-ph.CO].
[25] S. Capozziello and M. De Laurentis, "Extended Theories of Gravity," *Phys. Rept.* **509** (2011) 167–321, arXiv:1108.6266 [gr-qc].
[26] Y. Brihaye and L. Ducobu, "Hairy black holes, boson stars and non-minimal coupling to curvature invariants," *Phys. Lett. B* **795** (2019) 135–143, arXiv:1812.07438 [gr-qc].
[27] **CANTATA** Collaboration, E. N. Saridakis *et al.*, "Modified Gravity and Cosmology: An Update by the CANTATA Network," arXiv:2105.12582 [gr-qc].
[28] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, "Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution," *Phys. Rept.* **692** (2017) 1–104, arXiv:1705.11098 [gr-qc].
[29] A. Addazi *et al.*, "Quantum gravity phenomenology at the dawn of the multi-messenger era – A review," arXiv:2111.05659 [hep-ph].
[30] T. P. Sotiriou and S.-Y. Zhou, "Black hole hair in generalized scalar-tensor gravity," *Phys. Rev. Lett.* **112** (2014) 251102, arXiv:1312.3622 [gr-qc].

[31] T. P. Sotiriou and S.-Y. Zhou, "Black hole hair in generalized scalar-tensor gravity: An explicit example," *Phys. Rev. D* **90** (2014) 124063, arXiv:1408.1698 [gr-qc].

[32] Y. Brihaye and L. Ducobu, "Black holes with scalar hair in Einstein–Gauss–Bonnet gravity," *Int. J. Mod. Phys. D* **25** (2016) no. 07, 1650084, arXiv:1511.06897 [gr-qc].

[33] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, "Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling," *Phys. Rev. Lett.* **120** (2018) no. 13, 131104, arXiv:1711.02080 [gr-qc].

[34] A. Brandelet, Y. Brihaye, T. Delsate, and L. Ducobu, "Nutty black holes in galileon scalar-tensor gravity," *Int. J. Mod. Phys. A* **33** (2018) no. 32, 1850189, arXiv:1705.06145 [gr-qc].

[35] D. D. Doneva and S. S. Yazadjiev, "New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Scalar-Tensor Theories," *Phys. Rev. Lett.* **120** (2018) no. 13, 131103, arXiv:1711.01187 [gr-qc].

[36] G. Antoniou, A. Bakopoulos, and P. Kanti, "Evasion of No-Hair Theorems and Novel Black-Hole Solutions in Gauss-Bonnet Theories," *Phys. Rev. Lett.* **120** (2018) no. 13, 131102, arXiv:1711.03390 [hep-th].

[37] E. Babichev and C. Charmousis, "Dressing a black hole with a time-dependent Galileon," *JHEP* **08** (2014) 106, arXiv:1312.3204 [gr-qc].

[38] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction*. Springer, 2013.

[39] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. L. Said, J. Mifsud, and E. Di Valentino, "Teleparallel Gravity: From Theory to Cosmology," arXiv:2106.13793 [gr-qc].

[40] M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer, and A. A. Coley, "Teleparallel theories of gravity: illuminating a fully invariant approach," *Class. Quant. Grav.* **36** (2019) no. 18, 183001, arXiv:1810.12932 [gr-qc].

[41] C.-Q. Geng, C.-C. Lee, E. N. Saridakis, and Y.-P. Wu, ""Teleparallel" dark energy," *Phys. Lett. B* **704** (2011) 384–387, arXiv:1109.1092 [hep-th].

[42] S. Bahamonde and M. Wright, "Teleparallel quintessence with a nonminimal coupling to a boundary term," *Phys. Rev. D* **92** (2015) no. 8, 084034, arXiv:1508.06580 [gr-qc]. [Erratum: Phys.Rev.D 93, 109901 (2016)].

[43] M. Hohmann, "Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants," *Phys. Rev. D* **98** (2018) no. 6, 064004, arXiv:1801.06531 [gr-qc].

[44] M. Hohmann, "Scalar-torsion theories of gravity I: general formalism and conformal transformations," *Phys. Rev. D* **98** (2018) no. 6, 064002, arXiv:1801.06528 [gr-qc].

[45] M. Hohmann and C. Pfeifer, "Scalar-torsion theories of gravity II: $L(T, X, Y, \phi)$ theory," *Phys. Rev. D* **98** (2018) no. 6, 064003, arXiv:1801.06536 [gr-qc].

[46] G. Kofinas, E. Papantonopoulos, and E. N. Saridakis, "Self-Gravitating Spherically Symmetric Solutions in Scalar-Torsion Theories," *Phys. Rev. D* **91** (2015) no. 10, 104034, arXiv:1501.00365 [gr-qc].

[47] S. Bahamonde, U. Camci, S. Capozziello, and M. Jamil, "Scalar-Tensor Teleparallel Wormholes by Noether Symmetries," *Phys. Rev. D* **94** (2016) no. 8, 084042, arXiv:1608.03918 [gr-qc].

[48] K. Flathmann and M. Hohmann, "Post-Newtonian Limit of Generalized Scalar-Torsion Theories of Gravity," *Phys. Rev. D* **101** (2020) no. 2, 024005, arXiv:1910.01023 [gr-qc].

[49] M. Hohmann, L. Järv, M. Krššák, and C. Pfeifer, "Teleparallel theories of gravity as analogue of nonlinear electrodynamics," *Phys. Rev.* **D97** (2018) no. 10, 104042, arXiv:1711.09930 [gr-qc].

[50] M. Krššák and E. N. Saridakis, "The covariant formulation of f(T) gravity," *Class. Quant. Grav.* **33** (2016) no. 11, 115009, arXiv:1510.08432 [gr-qc].

[51] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7, 2019.

[52] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984.

[53] C. Pfeifer, "A quick guide to spacetime symmetry and symmetric solutions in teleparallel gravity," 1, 2022. arXiv:2201.04691 [gr-qc].

[54] S. Bahamonde, A. Golovnev, M.-J. Guzmán, J. L. Said, and C. Pfeifer, "Black Holes in $f(T, B)$ Gravity: Exact and Perturbed Solutions," arXiv:2110.04087 [gr-qc].

[55] M. Hohmann, L. Järv, M. Krššák, and C. Pfeifer, "Modified teleparallel theories of gravity in symmetric spacetimes," arXiv:1901.05472 [gr-qc].

[56] N. Tamanini and C. G. Boehmer, "Good and bad tetrads in f(T) gravity," *Phys. Rev. D* **86** (2012) 044009, arXiv:1204.4593 [gr-qc].

[57] D. Horvat, S. Ilijić, A. Kirin, and Z. Narančić, "Nonminimally coupled scalar field in teleparallel gravity: boson stars," *Class. Quant. Grav.* **32** (2015) no. 3, 035023, arXiv:1407.2067 [gr-qc].

[58] J. D. Bekenstein, "Exact solutions of Einstein conformal scalar equations," *Annals Phys.* **82** (1974) 535–547.

[59] N. Bocharova, K. Bronnikov, and V. Melnikov, "An exact solution of the system of einstein equations and mass-free scalar field, vestn," *Mosk. univ. Fiz. astron* **6** (1970) 706.

[60] K. A. Bronnikov and Y. N. Kireev, "Instability of Black Holes with Scalar Charge," *Phys. Lett. A* **67** (1978) 95–96.

[61] D. Sudarsky and T. Zannias, "Spherical black holes cannot support scalar hair," *Phys. Rev. D* **58** (1998) 087502, arXiv:gr-qc/9712083.

[62] S. Bahamonde, K. Flathmann, and C. Pfeifer, "Photon sphere and perihelion shift in weak $f(T)$ gravity," *Phys. Rev. D* **100** (2019) no. 8, 084064, arXiv:1907.10858 [gr-qc].

[63] S. Bahamonde, J. Levi Said, and M. Zubair, "Solar system tests in modified teleparallel gravity," *JCAP* **10** (2020) 024, arXiv:2006.06750 [gr-qc].

[64] C. Pfeifer and S. Schuster, "Static spherically symmetric black holes in weak f(T)-gravity," *Universe* **7** (2021) no. 5, 153, arXiv:2104.00116 [gr-qc].

[65] S. Bahamonde and C. Pfeifer, "General teleparallel modifications of Schwarzschild geometry," *Int. J. Geom. Meth. Mod.*

Phys. **18** (2021) no. supp01, 2140001, arXiv:2010.02161 [gr-qc].
[66] S. Bahamonde and C. G. Böhmer, "Modified teleparallel theories of gravity: Gauss–Bonnet and trace extensions," *Eur. Phys. J. C* **76** (2016) no. 10, 578, arXiv:1606.05557 [gr-qc].
[67] S. Bahamonde, M. Marciu, S. D. Odintsov, and P. Rudra, "String-inspired Teleparallel cosmology," *Nucl. Phys. B* **962** (2021) 115238, arXiv:2003.13434 [gr-qc].
[68] S. Bahamonde, U. Camci, and S. Capozziello, "Noether symmetries and boundary terms in extended Teleparallel gravity cosmology," *Class. Quant. Grav.* **36** (2019) no. 6, 065013, arXiv:1807.02891 [gr-qc].
[69] M. Hohmann and C. Pfeifer, "Teleparallel axions and cosmology," *Eur. Phys. J. C* **81** (2021) no. 4, 376, arXiv:2012.14423 [gr-qc].
[70] S. Bahamonde, J. Gigante Valcarcel, L. Järv, and C. Pfeifer, "Exploring Axial Symmetry in Modified Teleparallel Gravity," *Phys. Rev. D* **103** (2021) no. 4, 044058, arXiv:2012.09193 [gr-qc].
[71] L. Järv, M. Hohmann, M. Krššák, and C. Pfeifer, "Flat connection for rotating spacetimes in extended teleparallel gravity theories," *Universe* **5** (2019) 142, arXiv:1905.03305 [gr-qc].

Bibliography

[Aldrovandi and Pereira, 2014] Aldrovandi, R. and Pereira, J. (2014). *Teleparallel gravity. An introduction*.

[Ascher et al., 1979a] Ascher, U., Christiansen, J., and Russell, R. D. (1979a). A collocation solver for mixed order systems of boundary value problems. *Mathematics of Computation*, 33(146):659–679.

[Ascher et al., 1979b] Ascher, U., Christiansen, J., and Russell, R. D. (1979b). Colsys—a collocation code for boundary-value problems. In *Codes for Boundary-Value problems in ordinary differential equations*, pages 164–185. Springer.

[Babichev and Charmousis, 2014] Babichev, E. and Charmousis, C. (2014). Dressing a black hole with a time-dependent Galileon. *JHEP*, 08:106.

[Bahamonde et al., 2021] Bahamonde, S., Dialektopoulos, K. F., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., Hohmann, M., Said, J. L., Mifsud, J., and Di Valentino, E. (2021). Teleparallel Gravity: From Theory to Cosmology.

[Bahamonde et al., 2022] Bahamonde, S., Ducobu, L., and Pfeifer, C. (2022). Scalarized black holes in teleparallel gravity. *JCAP*, 04(04):018.

[Blagojevic and Hehl, 2012] Blagojevic, M. and Hehl, F. W. (2012). Gauge Theories of Gravitation.

[Brandelet et al., 2018] Brandelet, A., Brihaye, Y., Delsate, T., and Ducobu, L. (2018). Nutty black holes in galileon scalar-tensor gravity. *Int. J. Mod. Phys. A*, 33(32):1850189.

[Brihaye and Ducobu, 2016] Brihaye, Y. and Ducobu, L. (2016). Black holes with scalar hair in Einstein–Gauss–Bonnet gravity. *Int. J. Mod. Phys. D*, 25(07):1650084.

[Brihaye and Ducobu, 2018] Brihaye, Y. and Ducobu, L. (2018). Spinning-charged-hairy black holes in 5D Einstein gravity. *Phys. Rev. D*, 98(6):064034.

[Brihaye and Ducobu, 2019] Brihaye, Y. and Ducobu, L. (2019). Hairy black holes, boson stars and non-minimal coupling to curvature invariants. *Phys. Lett. B*, 795:135–143.

[Brihaye et al., 2020] Brihaye, Y., Ducobu, L., and Hartmann, B. (2020). Boson and neutron stars with increased density. *Phys. Lett. B*, 811:135906.

[Brihaye et al., 2014] Brihaye, Y., Herdeiro, C., and Radu, E. (2014). Myers–Perry black holes with scalar hair and a mass gap. *Phys. Lett. B*, 739:1–7.

[Brihaye et al., 2016] Brihaye, Y., Herdeiro, C., and Radu, E. (2016). Inside black holes with synchronized hair. *Phys. Lett. B*, 760:279–287.

[Carroll, 1997] Carroll, S. M. (1997). Lecture notes on general relativity.

[Cisterna et al., 2016] Cisterna, A., Delsate, T., Ducobu, L., and Rinaldi, M. (2016). Slowly rotating neutron stars in the nonminimal derivative coupling sector of Horndeski gravity. *Phys. Rev. D*, 93(8):084046.

[Cisterna et al., 2015] Cisterna, A., Delsate, T., and Rinaldi, M. (2015). Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling. *Phys. Rev. D*, 92(4):044050.

[de Boor, 2001] de Boor, C. (2001). *A Practical Guide to Splines*. Applied Mathematical Sciences. Springer New York.

[Deffayet et al., 2011] Deffayet, C., Gao, X., Steer, D. A., and Zahariade, G. (2011). From k-essence to generalised Galileons. *Phys. Rev. D*, 84:064039.

[Deffayet and Steer, 2013] Deffayet, C. and Steer, D. A. (2013). A formal introduction to Horndeski and Galileon theories and their generalizations. *Class. Quant. Grav.*, 30:214006.

[Deuflhard, 2011] Deuflhard, P. (2011). *Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms*. Springer Series in Computational Mathematics. Springer Berlin Heidelberg.

[Dvali et al., 2000] Dvali, G. R., Gabadadze, G., and Porrati, M. (2000). 4-D gravity on a brane in 5-D Minkowski space. *Phys. Lett. B*, 485:208–214.

[Herdeiro and Radu, 2014] Herdeiro, C. A. R. and Radu, E. (2014). Kerr black holes with scalar hair. *Phys. Rev. Lett.*, 112:221101.

[Herdeiro and Radu, 2015] Herdeiro, C. A. R. and Radu, E. (2015). Asymptotically flat black holes with scalar hair: a review. *Int. J. Mod. Phys. D*, 24(09):1542014.

[Hohmann, 2018a] Hohmann, M. (2018a). Scalar-torsion theories of gravity I: general formalism and conformal transformations. *Phys. Rev. D*, 98(6):064002.

[Hohmann, 2018b] Hohmann, M. (2018b). Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants. *Phys. Rev. D*, 98(6):064004.

[Hohmann and Pfeifer, 2018] Hohmann, M. and Pfeifer, C. (2018). Scalar-torsion theories of gravity II: $L(T, X, Y, \phi)$ theory. *Phys. Rev. D*, 98(6):064003.

[Horndeski, 1974] Horndeski, G. W. (1974). Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space. *International Journal of Theoretical Physics*, 10(6):363–384.

[Hui and Nicolis, 2013] Hui, L. and Nicolis, A. (2013). No-Hair Theorem for the Galileon. *Phys. Rev. Lett.*, 110:241104.

[Kanti et al., 1996] Kanti, P., Mavromatos, N. E., Rizos, J., Tamvakis, K., and Winstanley, E. (1996). Dilatonic black holes in higher curvature string gravity. *Phys. Rev. D*, 54:5049–5058.

[Kobayashi and Nomizu, 1996] Kobayashi, S. and Nomizu, K. (1996). *Foundations of Differential Geometry, Volume 1 & 2*. A Wiley Publication in Applied Statistics. Wiley.

[Kobayashi, 2019] Kobayashi, T. (2019). Horndeski theory and beyond: a review. *Rept. Prog. Phys.*, 82(8):086901.

[Kobayashi et al., 2011] Kobayashi, T., Yamaguchi, M., and Yokoyama, J. (2011). Generalized G-inflation: Inflation with the most general second-order field equations. *Prog. Theor. Phys.*, 126:511–529.

[Langlois, 2019] Langlois, D. (2019). Dark energy and modified gravity in degenerate higher-order scalar–tensor (DHOST) theories: A review. *Int. J. Mod. Phys. D*, 28(05):1942006.

[Misner et al., 1973] Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation / Charles W. Misner, Kip S. Thorne, John Archibald Wheeler*. W. H. Freeman San Francisco.

[Nicolis and Rattazzi, 2004] Nicolis, A. and Rattazzi, R. (2004). Classical and quantum consistency of the DGP model. *JHEP*, 06:059.

[Nicolis et al., 2009] Nicolis, A., Rattazzi, R., and Trincherini, E. (2009). The Galileon as a local modification of gravity. *Phys. Rev. D*, 79:064036.

[Ostrogradsky, 1850] Ostrogradsky, M. (1850). Mémoires sur les équations différentielles, relatives au problème des isopérimètres. *Mem. Acad. St. Petersbourg*, 6(4):385–517.

[Papageorgiou et al., 2022] Papageorgiou, A., Park, C., and Park, M. (2022). Rectifying No-Hair Theorems in Gauss-Bonnet theory.

[Penrose, 2005] Penrose, R. (2005). *The Road to Reality: A Complete Guide to the Laws of the Universe*. Vintage.

[Semay and Silvestre-Brac, 2016] Semay, C. and Silvestre-Brac, B. (2016). *Relativité restreinte - 3e éd.* Sciences Sup. Dunod [written in French].

[Silva et al., 2018] Silva, H. O., Sakstein, J., Gualtieri, L., Sotiriou, T. P., and Berti, E. (2018). Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling. *Phys. Rev. Lett.*, 120(13):131104.

[Sotiriou and Zhou, 2014a] Sotiriou, T. P. and Zhou, S.-Y. (2014a). Black hole hair in generalized scalar-tensor gravity. *Phys. Rev. Lett.*, 112:251102.

[Sotiriou and Zhou, 2014b] Sotiriou, T. P. and Zhou, S.-Y. (2014b). Black hole hair in generalized scalar-tensor gravity: An explicit example. *Phys. Rev. D*, 90:124063.

[Spindel, 2001] Spindel, P. (2001). *Mécanique: Mécanique newtonienne*, volume 1 of *Mécanique*. Archives Contemporaines Editions [written in French].

[Touboul et al., 2017] Touboul, P. et al. (2017). MICROSCOPE Mission: First Results of a Space Test of the Equivalence Principle. *Phys. Rev. Lett.*, 119(23):231101.

[Townsend, 1997] Townsend, P. K. (1997). Black holes: Lecture notes.

[Vainshtein, 1972] Vainshtein, A. (1972). To the problem of nonvanishing gravitation mass. *Physics Letters B*, 39(3):393–394.

[Vandeev and Semenova, 2022] Vandeev, V. P. and Semenova, A. N. (2022). Geodesic deviation on symmetry axis in Taub-NUT metric.

[Visser, 2007] Visser, M. (2007). The Kerr spacetime: A Brief introduction. In *Kerr Fest: Black Holes in Astrophysics, General Relativity and Quantum Gravity*.

[Wald, 1984] Wald, R. (1984). *General Relativity*. University of Chicago Press.