UMONS - Université de Mons
Faculté des Sciences

Differential algebra, ordered fields and model theory

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Une dissertation présentée en vue de l'obtention
du grade académique de docteur en sciences.

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Septembre 2015

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Note de l'auteur

Ce travail est le résultat d'un parcours de plusieurs années de travail, de doutes, d'émotions, d'aventures et même parfois de mésaventures qui ont heureusement toujours bien fini.

Depuis le début, Françoise Point a eu un rôle déterminant. Elle m'a accompagné et conseillé dans mon parcours et mes activités de mathématicien: lecture d'articles, rédaction de résultats, participation à des conférences. C'est avec beaucoup de patience qu'elle a lu et relu de nombreuses fois certaines preuves, parfois fausses ou mal écrites. Sa contribution dans l'élaboration de ce travail est inestimable.

Christian Michaux, on pourrait croire qu'il a tout fait pour retarder mon travail en me demandant de donner des cours, rédiger des questions d'examens, corriger des copies,... mais travailler avec lui a sans aucun doute été à la fois enrichissant et amusant. J'ai pris un certain plaisir dans les tâches d'enseignement qui, par ailleurs, ont été l'occasion de transmettre aux étudiants ce que j'avais appris dans mon parcours.

Mes voyages scientifiques ont été l'occasion de nombreuses rencontres avec des mathématiciens de tous poils, de tous âge, de tous tempéraments. Je me réjouis des liens d'amitié que j'ai pu créer avec certains d'entre-eux. Parmi ces voyages, ressortent plusieurs petits séjours en Angleterre, destination qui se distingue des autres par un charme assez unique, ainsi que des séjours de plus longue durée à Berkeley, Konstanz et Lyon. J'en garde des souvenirs impérissables.

Marcus Tressl et Piotr Kowalski m'ont fait le plaisir de faire partie du jury de cette thèse et de se déplacer jusque Mons spécialement pour participer aux défenses. Les remarques formulées par les différents membres du jury m'ont permis d'améliorer certains passages de la thèse. Je leur en suis très reconnaissant.

Il serait long et vain de tenter d'énumérer les nombreuses personnes qui ont joué un rôle dans la réalisation de cette thèse ou de raconter tous les rebondissements de cette histoire. Je préfère donc en rester là et passer au vif du sujet: les corps différentiels et la théorie des modèles.

Preface

Differential algebra, the branch of mathematics concerning differential rings, fields and algebras provides a suitable background for an algebraic study of differential equations and differential varieties. The subject's history has started with the works of E. Picard and E. Vessiot on extending Galois theory to the context of differential linear equations (published in several articles during the last decades of the 19th century).

Later, the study of differential ideals and a basis theorem (an analogue of Hilbert's basis theorem on ordinary polynomial rings) by J. F. Ritt [51] and H. W. Raudenbush [49] in the 1930s was an important progress on the subject, especially on the understanding of differentially algebraic sets. The first book dedicated to differential algebraic equations [51] was published by J. F. Ritt in 1932. During the next decades, E. Kolchin made extensive publications. His works provide quite a general presentation of Galois theory of differential fields through the theory of strongly normal extensions (see for instance his monograph [30]).

Differentially closed fields of characteristic 0 were studied by A. Robinson in [53] where an axiomatisation and some basic model theoretic properties are given. In [4], L. Blum established a more concise axiomatisation. Those works marked the very beginning of the interplay between model theory and differential algebra. The theory of differentially closed fields is denoted DCF_0. The interest in DCF_0 was probably increased by the fact that it is an ω-stable theory where interesting phenomena occur, like the non-minimality of the differential closure [54, 31] and the property that Lascar and Morley ranks differ [23]. Elimination of imaginaries in DCF_0 was proved by B. Poizat in [47] on the purpose of investigating differential Galois theory with the tools of model theory. By the use of algebraic methods, E. Kolchin gave another definition of differentially closed fields, called constrainedly closed differential fields in his paper [31].

Topological differential fields were considered in several contexts leading to different kinds of interactions between the derivation and the topology.

In 1980, M. Rosenlicht introduced the notion of differential valuation [55]. A differential valuation naturally occurs in Hardy fields, defined in [6] (see also [56]) as fields of germs of real-valued functions on neighbourhoods of $+\infty$ in $\mathbb{R}$ that are closed under the usual differentiation (and endowed with the usual addition and multiplication of germs). Very recently, M. Aschenbrenner, L. van den Dries and J. van der Hoeven [3] explored the model theory of Hardy fields (generalised as H-fields in [2]) and the model theory of the ordered differential field T of transseries. They made significant progress on a conjecture stating that the theory of T is model complete and is the model companion of the theory of H-fields with small derivations.

About twenty years after Rosenlicht's first works on differential valuations and Hardy fields, T. Scanlon defined and studied valued D-fields [58] from the view point of model theory. Unlike Hardy fields whose valuation is trivial on the subfield of constants, valued D-fields have the property that any element of the value group is the valuation of some constant element.

In this thesis, we investigate ordered differential fields. In particular, real closed differential fields will have a central role. So our approach and our intuitions mainly come from real closed fields theory and from differential algebra, in particular differential Galois theory.

A specific class of ordered differential fields called closed ordered differential fields will be ubiquitous. It was introduced and studied, in the late 1970s, by M. Singer [65] who presented it via a first-order axiomatisation and showed quantifier elimination. We refer to the theory of closed ordered differential fields by CODF. If K is a model of CODF (which in particular is a real closed field) then the algebraic closure $K(i)$ of K is a model of DCF_0. That property of models of CODF will play a role in some proofs on differential Galois theory of formally real fields where we will consider extensions of differential fields inside a saturated model of CODF. Note that it shows that models of CODF are examples of differential fields whose differential closure is minimal. In contrast with Hardy fields, the derivation of a model of CODF interacts with the order topology in a way which is similar to Scanlon's valued D-fields since the subfield of constants is dense.

The work of M. Singer on CODF aroused the attention of some model theorists. T. Brihaye, C. Michaux and C. Rivière exploited Singer's axiomatisation in order to adapt to CODF some properties of real closed fields [42, 7, 52]. In particular, they showed that CODF does not have the independence property [42]. We will use similar methods in several proofs, in order to transfer or adapt some properties of real closed fields to models of CODF.

A generalisation of CODF, called uniform companion of large differential fields, was presented by M. Tressl [71]. It generalises CODF in two ways, firstly it is suitable for fields with several commuting derivations, secondly it does not only give a description of closed ordered differential fields but also of differentially closed p-adic fields and differentially closed pseudo-finite fields. A paper of N. Guzy and F. Point [19] on topological differential fields provides a generalisation of CODF where the case of several derivations is not considered but the fields may be endowed with several topologies (unlike in M. Tressl's paper where the topology has to be definable in the ring language). The recent work of N. Solanki [67] is a generalisation of [71]. It is possible that this generalises [19] as well, then providing a global approach which unifies [71] and [19], but this has not been established yet.

This thesis is organised in four chapters. The first one is an introduction to the objects and the background of model theory, differential algebra and real closed fields. In the following chapters, which may be read independently from one another, we present algebraic and model theoretic results on ordered differential fields.

The second chapter deals with differential ideals of the ring of differential polynomials with coefficients in a model of CODF and their relationship with differentially algebraic sets. We establish a nullstellensatz which describes that relationship. We then investigate the question of describing differential polynomials that are positive on a given "differentially semialgebraic" set (i.e. a set of solutions of a system of differential algebraic inequalities). Note that the problem of characterising positive polynomials (here there is no derivation) was already the object of Hilbert's 17th problem. That problem is the following. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$. Suppose that f takes only non-negative values over the reals, can f be represented as a sum of squares of rational functions?* It was solved by E. Artin who gave it an affirmative answer in 1927. There were afterwards other works on improving Artin's theorem and notably a result by G. Stengle which he named positivstellensatz by analogy to the well-known term nullstellensatz refering to characterisations of polynomials vanishing on some given set (instead of polynomials which are only assumed to be positive or non-negative).

Chapter 3 goes through differential Galois theory of formally real fields. We consider strongly normal extensions L/K (in the sense of Kolchin's definition [30]) assuming that the field of constants of K is real closed and L is formally real. We denote by $\mathrm{gal}(L/K)$ the differential Galois group of L/K. We prove that $\mathrm{gal}(L/K)$ is a definable group in the field of constants C_K of K. Since C_K is real closed, it is equivalent to say that the differential Galois group is semialgebraic. In the special cases of Picard-Vessiot extensions and Weierstrass extensions, we have a more explicit description of the differential Galois groups which are linear groups in the first case and elliptic curves in the second case. For an enough saturated model $\mathscr{U}$ of CODF containing L, we consider $\mathrm{Gal}(L/K) := \mathrm{gal}(\langle L, C_{\mathscr{U}}\rangle/\langle K, C_{\mathscr{U}}\rangle)$ called the full differential Galois group of L/K. Assuming that L/K is a regular strongly normal extension, we show that $E \mapsto \mathrm{Gal}(L/E)$ is an injective map from the set of intermediate extensions E of L/K such that L/E is regular, into the set of definable subgroups of $\mathrm{Gal}(L/K)$. We exhibit a regular strongly normal extension L/K where that map is not surjective. Throughout this chapter, model theoretic methods are used substantially. For instance, the proof of the definability of the differential Galois group is based on the fact that L is generated by a tuple of elements whose type in DCF_0 is isolated and on elimination of imaginaries in real closed fields. A recent result of elimination of imaginaries in CODF by F. Point [46] also plays a role.

Finally, chapter 4 is an investigation of model theoretic questions on the theory CODF. We first study definable types in the Stone space of CODF and in analogy with the case of real closed fields, we get a characterisation of definable types via the notion of cut. The statement of our characterisation is the following. Assume that A is a real closed differential subfield of a model of CODF, the type of $\bar{u}$ over A is definable if and only if A is Dedekind complete in the real closure of $A\langle\bar{u}\rangle$. A consequence of that result is the density of definable types in the Stone space of CODF. We consider the dp-rank of types and show that CODF is not a strongly dependent theory (i.e. types may have infinite dp-rank). Bounds on the VC-density in some NIP theories are studied by M. Aschenbrenner et al. in [1]. We consider the question of finding bounds for the VC-density in CODF and its possible relation to bounds on the dp-rank.

Chapter 1

Model theory, differential fields and ordered fields

In this chapter, we give the background on differential fields, ordered fields and their model theory. The results presented here are well known, so they are generally stated without proof. Nonetheless, when no proof is available in the litterature, we write the proof.

Notations

For a set A when we write $\bar{a} \in A$, the overline means that $\bar{a}$ is a tuple of elements of A of finite length (i.e. $\bar{a}$ belongs to the cartesian power A^n of A, for some $n \in \mathbb{N} \setminus \{0\}$). We denote variables of formulas and polynomials by capital lettres X,Y, ... and tuples of finitely many variables by $\bar{X}$, $\bar{Y}$, ...

1.1 Model Theory

We recall here some basic notions of model theory involved later. For a more detailed text on model theory where proofs are included we refer the reader to [57], [20] or [70].

We assume that the reader is familiar with the following notions: $\mathcal{L}$-structure (see [70], Definition 1.1.2), (atomic) $\mathcal{L}$-formula (see [70], Definition 1.2.8), definable set (see [70], page 9), $\mathcal{L}$-theory (see [70], Definition 1.3.1), complete $\mathcal{L}$-theory (see [70], Definition 1.3.5) and $\mathcal{L}$-embedding (see [70], Definition 1.1.3). By a quantifier-free $\mathcal{L}$-formula, we mean an $\mathcal{L}$-formula where no quantifier occurs and $\mathcal{L}$-sentence means $\mathcal{L}$-formula without free variable.

Let M and N be $\mathcal{L}$-structures, we use the following notations. We write $M \subseteq N$ to mean that M is an $\mathcal{L}$-substructure of N (see [70], Definition 1.1.5). For an $\mathcal{L}$-formula $\theta(X_1, \ldots, X_n)$ and $m_1, \ldots, m_n \in M$, we write $M \models \theta(m_1, \ldots, m_n)$ to mean that M satisfies $\theta(m_1, \ldots, m_n)$ (see [70], Definition 1.2.9). For an $\mathcal{L}$-theory T and an $\mathcal{L}$-sentence ϕ, $T \vdash \phi$ means that T proves ϕ (see [70], Definition 1.3.3). For an $\mathcal{L}$-formula $\phi(X, Y_1, \ldots, Y_n)$ and $m_1, \ldots, m_n \in M$, $\phi(M, \bar{m}) := \{x \in M : M \models \phi(x, m_1, \ldots, m_n)\}$.

Remark 1.1.1. Note that a map $\sigma : M \to N$ is an $\mathcal{L}$-embedding iff for any atomic $\mathcal{L}$-formula $\phi(X_1, \ldots, X_n)$ and every $\bar{m} := (m_1, \ldots, m_n) \in M$, $M \models \phi(m_1, \ldots, m_n)$ iff $N \models \phi(\sigma(m_1), \ldots, \sigma(m_n))$.

We will on many occasions use the compactness theorem (see [70], Theorem 2.2.1).

Whenever we will consider $\mathcal{L}$-definable sets, we will mention whether we allow some parameters in the definition.

Definition 1.1.2. Let M and N be $\mathcal{L}$-structures and $\sigma : M \to N$. We say that σ is an elementary embedding iff for every $\mathcal{L}$-formula $\phi(X_1, \ldots, X_n)$, and every $\bar{m} := (m_1, \ldots, m_n) \in M$,

$$M \models \phi(m_1, \ldots, m_n) \text{ iff } N \models \phi(\sigma(m_1), \ldots, \sigma(m_n)).$$

Definition 1.1.3. Let N be an $\mathcal{L}$-structure and M be an $\mathcal{L}$-substructure of N, M is an elementary substructure of N iff the identity map $\sigma : M \to N : m \mapsto m$ is an elementary embedding.

Let T be an $\mathcal{L}$-theory.

Definition 1.1.4. T is said to be model complete if for any models M, N of T such that M is an $\mathcal{L}$-substructure of N, for any $\mathcal{L}$-formula $\phi(\bar{X})$ and any $\bar{m} \in M$, $M \models \phi(\bar{m})$ iff $N \models \phi(\bar{m})$.

In other words, T is model complete iff for any models M, N of T such that M is an $\mathcal{L}$-substructure of N, it holds that M is an elementary substructure of N.

Definition 1.1.5. T is said to eliminate quantifiers if for any $\mathcal{L}$-formula $\phi(\bar{X})$ there is a quantifier-free $\mathcal{L}$-formula $\theta(\bar{X})$ such that

$$T \vdash \forall \bar{X}(\phi(\bar{X}) \leftrightarrow \theta(\bar{X})).$$

If T eliminates quantifiers then T is model complete.

Definition 1.1.6. Let T_1, T_2 be $\mathcal{L}$-theories, T_2 is a model completion of T_1 iff

- Any model of T_2 is a model of T_1.
- Any model of T_1 is a substructure of some model of T_2.
- For any model A of T_1, for any models B, C of T_2 such that A is an $\mathcal{L}$-substructure of B and C, for any $\mathcal{L}$-formula $\phi(\bar{X})$ and any $\bar{a} \in A$, $B \models \phi(\bar{a})$ iff $C \models \phi(\bar{a})$.

If T has a model completion, then it is unique. The model completion of any theory is model complete.

The model theoretic notions of algebraic and definable elements are defined as follows. Let B be an $\mathcal{L}$-structure and A be a subset of B. We say that $x \in B$ is algebraic over A iff there exists an $\mathcal{L}$-formula ϕ and a tuple $\bar{a}$ in A such that x belongs to $\phi(B, \bar{a})$ and $\phi(B, \bar{a})$ is finite. If there is an $\mathcal{L}$-formula ϕ such that $\{x\} = \phi(B, \bar{a})$ then x is said to be definable over A. The set of algebraic elements over A in B is denoted $\mathrm{acl}^B(A)$ and called the algebraic closure of A in B. Similarly the set of definable elements over A in B is denoted $\mathrm{dcl}^B(A)$ and called the definable closure of A in B. When there are several languages on B, we have to specify which language the closures are relative to. Note that if B is an algebraically closed field (the language is $\{+, \cdot, -, ^{-1}, 0, 1\}$), then $\mathrm{dcl}^B(A)$ is the subfield of B generated by A and $\mathrm{acl}^B(A)$ is the classical algebraic closure of $\mathrm{dcl}^B(A)$.

Types

Let M be an $\mathcal{L}$-structure and A be a set of elements of M. We denote $\mathcal{L}(A) := \mathcal{L} \cup \{c_a : a \in A\}$, an extended language where c_a is a new constant symbol for each element $a \in A$. For ease of notation, we will use a instead of the constant symbol c_a. If $\bar{m}$ is a tuple of elements of M, we denote by $\mathrm{tp}^M_{\mathcal{L}}(\bar{m}/A)$, the type of $\bar{m}$ in M over A in the language $\mathcal{L}$, that is the set of $\mathcal{L}(A)$-formulas satisfied by $\bar{m}$ in M.

Note that if N is an $\mathcal{L}$-elementary extension of M, then for any $\bar{m} \in M$, $\mathrm{tp}^M_{\mathcal{L}}(\bar{m}/A) = \mathrm{tp}^N_{\mathcal{L}}(\bar{m}/A)$.

When the context is clear enough, we omit the subscript $\mathcal{L}$ or the exponent M in the notation $\mathrm{tp}^M_{\mathcal{L}}(\bar{m}/A)$.

Definition 1.1.7. If M, N are $\mathcal{L}$-structures, then a partial elementary map is a map $\sigma : A \to N$ where A is a subset of M and such that for any $a_1, \ldots, a_n \in A$ it holds that $\mathrm{tp}^M(a_1, \ldots, a_n) = \mathrm{tp}^N(\sigma(a_1), \ldots, \sigma(a_n))$.

Note that an elementary embedding is an elementary map.

Let $n \in \mathbb{N} \setminus \{0\}$. We denote $S_n^M(A) := \{\mathrm{tp}_{\mathcal{L}}^N(\bar{u}/A) : N$ is a model of the $\mathcal{L}(A)$-theory of M, $\bar{u} \in N$ and the length of $\bar{u}$ is $n\}$. The elements of $S_n^M(A)$ are called n-types. We endow $S_n^M(A)$ with the topology whose basic open sets are $[\phi] := \{p \in S_n^M(A) : \phi \in p\}$ where ϕ is an $\mathcal{L}(A)$-formula with n free variables. That topology is called the Stone topology. Isolated points in that topological space will have a central role in Chapter 3. One now give an explicit (non-topological) description of isolated n-types: let N be a model of the $\mathcal{L}(A)$-theory of M and $\bar{u} \in N$, $\mathrm{tp}_{\mathcal{L}}^N(\bar{u}/A)$ is isolated in $S_n^M(A)$ iff there is an $\mathcal{L}(A)$-formula ϕ such that for all $\bar{v} \in N$,

$$N \models \phi(\bar{v}) \text{ iff } \mathrm{tp}_{\mathcal{L}}^N(\bar{u}/A) = \mathrm{tp}_{\mathcal{L}}^N(\bar{v}/A).$$

If T is complete, we define S_n^T by letting $S_n^T := S_n^M(\varnothing)$ for some (any) model M of T. We call S_n^T the (nth) Stone space of T.

Let κ be a cardinal.

Definition 1.1.8. We say that an $\mathcal{L}$-structure M is κ-saturated iff for all $n \in \mathbb{N} \setminus \{0\}$, all $A \subseteq M$ such that $|A| < \kappa$ and all type $p \in S_n^M(A)$, there is a realisation of p in M.

We say that M is saturated iff M is $|M|$-saturated.

Then when M is κ-saturated and $|A| < \kappa$, $S_n^M(A) = \{\mathrm{tp}_{\mathcal{L}}^M(\bar{u}/A) : \bar{u} \in M$ and the length of $\bar{u}$ is $n\}$. We will need two properties of saturated structures: homogeneity and universality.

Definition 1.1.9. An $\mathcal{L}$-structure M is κ-universal iff for any model N of the $\mathcal{L}$-theory of M such that $|N| < \kappa$, there is an elementary embedding $\sigma : N \to M$.

Proposition 1.1.10. *If M is κ-saturated then M is κ-universal.*

Definition 1.1.11. An $\mathcal{L}$-structure M is κ-homogeneous iff for any subset A of M of cardinality $|A| < \kappa$ and $m \in M$, any partial elementary map $\sigma : A \to M$ is the restriction of a partial elementary map $\tau : A \cup \{m\} \to M$.

Proposition 1.1.12. *If M is κ-saturated then M is κ-homogeneous.*

Corollary 1.1.13. *Let M be saturated. For any $\bar{m}, \bar{n} \in M$, if $\mathrm{tp}^M(\bar{m}) = \mathrm{tp}^M(\bar{n})$ then there is an automorphism σ of M such that $\sigma(\bar{m}) = \bar{n}$.*

Suppose that T is a complete theory in a countable language then T has κ-saturated models for all cardinal κ (see [36], Theorem 4.3.12). Moreover, if κ is uncountable and strongly inaccessible then there is a saturated model of T of cardinality κ (see [36], Corollary 4.3.14).

Remark 1.1.14. We will make use of the fact that when M and N are substructures of saturated model $\mathscr{U}$ of a theory eliminating quantifiers such that both M and N are of cardinality $< |\mathscr{U}|$ then every $\mathcal{L}$-embedding $\sigma : M \to N$ is a partial $\mathcal{L}$-elementary map of $\mathscr{U}$ (Remark 1.1.1 combined with quantifier elimination) which extends to an $\mathcal{L}$-automorphism of $\mathscr{U}$ (Corollary 1.1.13).

We will give later a model theoretic description of real closures and differential closures using the notion of prime model:

Definition 1.1.15. A prime model of T is a model M of T such that for any model N of T, there is an elementary embedding $\sigma : M \to N$. If A is a subset of M then M is said to be prime over A iff every partial elementary map $A \to N$ extends to an elementary embedding $M \to N$.

Note that when A is a subset of M then M is prime over A iff M is a prime model of its $\mathcal{L}(A)$-theory.

Definition 1.1.16. A model N of T is atomic over A for some $A \subseteq N$ iff for every tuple $\bar{a}$ of elements of N, $\mathrm{tp}^N(\bar{a}/A)$ is isolated in $S_n^N(A)$.

Lemma 1.1.17. *If N is a prime model of T then N is atomic over $\varnothing$.*

We refer the reader to [70], sections 4.5 and 5.3 for more informations and details on prime and atomic models.

Elimination of imaginaries

Some theories have the convenient behavior that for any definable equivalence relation one may choose in a definable way a single element in any equivalence class (called imaginary element by S. Shelah). This leads to the notions of elimination of imaginaries. It will be an important notion in chapter 3.

Definition 1.1.18. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be languages and M be an $\mathcal{L}_2$-structure. An $\mathcal{L}_1$-structure N is definable in M iff

- the set N is an $\mathcal{L}_2$-definable subset of M^k (for some $k \in \mathbb{N} \setminus \{0\}$):
- for any n-ary relation symbol r in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m}_1, \ldots, \bar{m}_n \in M$, $M \models \phi(\bar{m}_1, \ldots, \bar{m}_n)$ iff
$$\bar{m}_1, \ldots, \bar{m}_n \in N \text{ and } N \models r(\bar{m}_1, \ldots, \bar{m}_n);$$
- for any n-ary function symbol f in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1} \in M$, $M \models \phi(\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1})$ iff
$$\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1} \in N \text{ and } N \models f(\bar{m}_1, \ldots, \bar{m}_n) = \bar{m}_{n+1};$$

- For any constant symbol c in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m} \in M$, $M \models \phi(\bar{m})$ iff

$$\bar{m} \in N \text{ and } N \models \bar{m} = c.$$

For instance a definable group in M is a definable subset of a cartesian power M^k of M with a group operation whose graph is a definable subset of M^{3k}.

Definition 1.1.19. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be languages and M be an $\mathcal{L}_2$-structure. An $\mathcal{L}_1$-structure N is interpretable in M iff

- the set N is the quotient of an $\mathcal{L}_2$-definable subset D of M^k by an equivalence relation R which is $\mathcal{L}_2$-definable in M;

- for any n-ary relation symbol r in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m}_1, \ldots, \bar{m}_n \in M$, $M \models \phi(\bar{m}_1, \ldots, \bar{m}_n)$ iff

$$\bar{m}_1, \ldots, \bar{m}_n \in D \text{ and } N \models r([\bar{m}_1], \ldots, [\bar{m}_n])$$

  where $[\bar{m}_1], \ldots, [\bar{m}_n]$ are the R-equivalence classes of $\bar{m}_1, \ldots, \bar{m}_n$;

- for any n-ary function symbol r in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1} \in M$, $M \models \phi(\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1})$ iff

$$\bar{m}_1, \ldots, \bar{m}_n, \bar{m}_{n+1} \in D \text{ and } N \models f([\bar{m}_1], \ldots, [\bar{m}_n]) = [\bar{m}_{n+1}];$$

- For any constant symbol c in $\mathcal{L}_1$, there is an $\mathcal{L}_2$-formula ϕ such that for any $\bar{m} \in M$, $M \models \phi(\bar{m})$ iff

$$\bar{m} \in D \text{ and } N \models [\bar{m}] = c.$$

For example if G is a definable group in a structure M and H is a definable normal subgroup of G, then G/H is an interpretable group in M.

Definition 1.1.20. The theory T eliminates the imaginaries iff for any model M of T, any $\bar{a} \in M$ and any $\mathcal{L}$-formula $\phi(\bar{X}, \bar{Y})$ there is a formula $\psi(\bar{X}, \bar{Y})$ such that there is in M a unique $\bar{b}$ such that $M \models \forall \bar{X} (\phi(\bar{X}, \bar{a}) \leftrightarrow \psi(\bar{X}, \bar{b}))$.

We say that T has uniform elimination of imaginaries if ψ does not depend on $\bar{a}$ and M.

Below we write $\exists^{=1}$ as an abbreviation of *there exists a unique.*

Theorem 1.1.21. *The theory T has uniform elimination of imaginaries iff for every $\mathcal{L}$-formula $\theta(\bar{X}, \bar{Y})$ where $\bar{X}$ and $\bar{Y}$ have the same length, there is an $\mathcal{L}$-formula $\phi(\bar{X}, \bar{Y})$ such that if θ defines an equivalence relation on any model of T then $T \vdash \forall \bar{Y}\, \exists^{=1} \bar{Z}\, \forall \bar{X}(\theta(\bar{X}, \bar{Y}) \leftrightarrow \phi(\bar{X}, \bar{Z}))$.*

In simple words, T has uniform elimination of imaginaries iff for every formula θ, there is a formula ϕ such that if θ defines an equivalence relation in any model of T then for any equivalence class C (in any model of T), there is a unique element $\bar{z} \in C$ such that $\bar{x} \in C$ iff $\phi(\bar{x}, \bar{z})$ holds.

We will make use of the fact that if the theory of M eliminates the imaginaries and N is interpretable in M then N is definable in M.

Theorem 1.1.22 (Existence of a canonical parameter)**.** *Suppose that T has elimination of imaginaries. Let M be a saturated model of T and W be a definable subset of M^n, there is $\bar{d} \in \mathrm{dcl}^M(W)$ such that the following equivalence holds for any $\mathcal{L}$-automorphism σ of M:*

$$\sigma(\bar{d}) = \bar{d} \text{ iff } \sigma(W) = W.$$

The tuple $\bar{d}$ is called a canonical parameter of W.

Definition 1.1.23. We say that T has definable Skolem functions iff for every $\mathcal{L}$-formula $\phi(\bar{X}, Y)$ there is a formula $\psi(\bar{X}, Y)$ such that

$$T \vdash \forall \bar{X}(\exists Y \phi(\bar{X}, Y) \to (\exists^{=1} Y \psi(\bar{X}, Y) \wedge \forall Y(\psi(\bar{X}, Y) \to \phi(\bar{X}, Y)))).$$

Intuitively, T has definable Skolem functions if for every formula $\phi(\bar{X}, Y)$ there is a definable function f (given by $y = f(\bar{x})$ iff $\psi(\bar{x}, y)$ in the formal definition above) such that for any $\bar{x}$, $\phi(\bar{x}, f(\bar{x}))$ holds (whenever there is y such that $\phi(\bar{x}, y)$ holds).

The theory CODF, which will play the central role in this thesis, has elimination of imaginaries but does not have definable Skolem functions. Note that there are examples of theories having definable Skolem functions but that do not eliminate imaginaries. For instance, let p be a prime number and pCF be the theory of p-adically closed fields, i.e., the theory of the field $\mathbb{Q}_p$ in the language of fields with a binary relation $|$ defined by $x|y$ iff $v_p(x) \leq v_p(y)$. By Theorem 3.2 of [14], pCF has definable Skolem functions. Moreover, pCF does not eliminates the imaginaries. A description of an expanded language on $\mathbb{Q}_p$ where elimination of imaginaries holds is given in [21].

1.2 Differential fields

All fields and rings will be commutative and of characteristic 0.

1.2.1 Differential rings and fields

A differential ring is a ring R endowed with a map $R \to R : x \mapsto x'$, called a derivation and such that for all $x, y \in R$,

- $(x+y)' = x' + y'$;
- $(xy)' = x'y + xy'$.

We take the following notation: $x^{(0)} := x$ and for any $j \in \mathbb{N}$, $x^{(j+1)} := (x^{(j)})'$. Moreover, if $\bar{a} := (a_1, \dots, a_n)$ is a tuple of elements of R then we write $\bar{a}'$ for $(a'_1, \dots, a'_n)$ and $\bar{a}^{(j)}$ for $(a_1^{(j)}, \dots, a_n^{(j)})$.

We use the language $\mathcal{L}_{df} := \{+, \cdot, -, ^{-1}, 0, 1, '\}$ to axiomatise and study differential fields[1]. Since a differential field is in particular a field, we will also use the language of fields $\mathcal{L}_{\text{fields}} := \{+, \cdot, -, ^{-1}, 0, 1\}$. We denote by DF_0 the first-order $\mathcal{L}_{df}$-theory whose models are differential fields (of characteristic 0).

Let F be a differential ring, we let $C_F := \{x \in F : x' = 0\}$, i.e., C_F is the kernel of the derivation.

Theorem 1.2.1 ([37], Lemma 2.1)**.** *Let F be a differential field, then C_F is a differential subfield of F which is relatively algebraically closed in F.*

Then for a differential field F, C_F is called the field of constants of F.

In the special case of one differential variable X, a differential polynomial is a polynomial $p(X, X^{(1)}, \dots, X^{(n)})$ in the ring $R[X, X^{(1)}, X^{(2)}, \dots]$ which is endowed with a derivation in the natural way. We now give a formal definition of the differential ring of differential polynomials.

Definition 1.2.2. Let R be a differential ring and I be a set (which will be finite in most cases), the ring $R[X_i^{(j)} : i \in I, j \in \mathbb{N}]$ is endowed with a derivation by letting $(X_i^{(j)})' := X_i^{(j+1)}$. This differential ring is denoted $R\{X_i : i \in I\}$ and is called the ring of differential polynomials with variables $X_i : i \in I$.

Suppose R is integral and let F be the fraction field of R. The fraction field of $R\{X_i : i \in I\}$ is denoted $F\langle X_i : i \in I\rangle$.

For a subset A of a differential field F, we denote $\langle A\rangle$ the differential subfield of F generated by A.

Definition 1.2.3. Let F be a differential field and $f \in F\{X_1, \dots, X_k\}$. We say that the order of f is the smallest $n \in \mathbb{N}$ such that $f \in F[\bar{X}, \dots, \bar{X}^{(n)}]$.

[1] Our notations do not make any distinction between a symbol of some language and its interpretation in a structure. The symbols $+, \cdot$ and $-$ are binary function symbols, whereas $^{-1}$ and $'$ are unary function symbols. The symbols 0 and 1 are constant symbols.

We denote $\mathrm{ord}(f)$ the order of f.

The fact that a differential polynomial may be seen as an ordinary polynomial will be used on many occasions. So we introduce the following notation. Let $\bar{X} := (X_1, \ldots, X_k)$. For a differential polynomial $f \in F\{\bar{X}\}$, f^* will be the unique element of $F[X_1, \ldots, X_{k\cdot(\mathrm{ord}(f)+1)}]$ such that

$$f(\bar{X}) = f^*(\bar{X}, \ldots, \bar{X}^{(\mathrm{ord}(f))}).$$

Definition 1.2.4. Let $f \in F\{X\}$ and $n := \mathrm{ord}(f)$, the separant of f is

$$s_f := \frac{\partial f}{\partial X^{(n)}}.$$

The separant will be used in the description of prime differential ideals provided by Lemma 1.2.7 as well as in the axiomatisation of CODF in section 1.4.

Definition 1.2.5. Let R be a differential ring and I be an ideal of R. Then I is called differential iff $I' \subseteq I$.

Lemma 1.2.6. *When I is a differential ideal of the ring differential ring R, we may define a derivation on the quotient ring R/I by letting $(a+I)' := a' + I$.*

For a polynomial $f \in F\{X\}$, we denote $[f]$ the differential ideal of $F\{X\}$ generated by f. Moreover, $I(f)$ denotes

$$\{g \in F\{X\} : s_f^k g \in [f] \text{ for some } k \in \mathbb{N}\}.$$

Lemma 1.2.7 ([37], Lemma 1.8)**.** *If I is a non-zero prime differential ideal of $F\{X\}$. Then there is $f \in F\{X\}$ such that $I = I(f)$ and $s_f \notin I$.*

We call the polynomial f from Lemma 1.2.7 a minimal differential polynomial of I. Note that f is an irreducible element of $F\{X\}$ such that for all $g \in I \setminus \{0\}$, either $\mathrm{ord}(g) > \mathrm{ord}(f)$ or

$$\mathrm{ord}(g) = \mathrm{ord}(f) \text{ and } \deg(g) \geq \deg(f)$$

where the degree is considered with regard to $X^{(\mathrm{ord}(f))}$.

On many occasions, particularly in Chapter 3, we will be interested in transcendency. Let $K \subseteq L$ be differential fields, we say that $r \in L$ is differentially algebraic over K iff there is a differential polynomial $p \in K\{X\}$, such that $p(r) = 0$. Otherwise we say that r is differentially transcendental over K. Note that if r is differentially transcendental over K then the transcendence degree of $K\langle r\rangle$ over K is infinite, since for all n, no non-zero ordinary polynomial $p(X_0, \ldots, X_n) \in K[X_0, \ldots, X_n]$ vanishes at $(r, r', \ldots, r^{(n)})$. The converse is actually true, by virtue of the following lemma.

Lemma 1.2.8 ([37], Lemma 1.9). *Let K be a differential field, $f \in K\{X\}$, $n := \mathrm{ord}(f)$ and r be an element of some differential field extension of K. For any $l \in \mathbb{N}$, there is $p_l(\bar{X}) \in K(X_0, X_1, \ldots, X_n)$ such that if $f(r) = 0$ then $r^{(l)} = p_l(r, r', \ldots, r^{(n)})$.*

We give now a few lemma explaining how the derivation of a given differential field (or ring) may extend to a field (or ring) extension.

Lemma 1.2.9 ([35], Page 2). *Let R be a differential integral domain then the derivation of R extends in a unique way to the fraction field of R.*

Lemma 1.2.10 (A special case of [24], Chapter IV, Theorem 14). *Let K be a differential field, $L := K(t)$ be a transcendental extension of K and $a \in L$. There is a unique derivation $'$ on L extending the derivation of K and such that $t' = a$.*

Lemma 1.2.11 ([35], Example 1.14). *Let K be a differential field and L be an algebraic field extension of K. Then there is a unique derivation on L such that K is a differential subfield of L.*

Radical Differential ideals

The material of this subsection can be found in [26] and [37].

Definition 1.2.12. Let R be a ring and I an ideal. The radical of I is $\sqrt{I} := \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$.

Definition 1.2.13. An ideal I in R is said to be radical iff $\sqrt{I} = I$.

Lemma 1.2.14. *The radical of an ideal is a radical ideal.*

Unfortunately, the radical of a differential ideal is in general not a differential ideal. But one can show that in a ring containing $\mathbb{Q}$, the radical of a differential ideal is actually differential. More precisely:

Definition 1.2.15. A Ritt algebra is a differential ring having $\mathbb{Q}$ among its subrings.

For example, any differential field of characteristic zero is a Ritt algebra.

Lemma 1.2.16 ([26], Lemma 1.8). *Let R be a Ritt algebra and I a differential ideal of R, $\sqrt{I}$ is a differential ideal.*

Theorem 1.2.17 (Ritt-Raudenbush). *Let R be a Ritt algebra whose radical differential ideals are finitely generated. Then, every radical differential ideal in $R\{X_1, \ldots, X_n\}$ is finitely generated (as a radical differential ideal).*

Proof. See [37], Theorem 1.16. □

Theorem 1.2.18 (Ritt-Raudenbush)**.** *Let R be a Ritt algebra whose radical differential ideals are finitely generated. For any radical differential ideal I in $R\{X_1, \ldots, X_n\}$, there is a natural number m and prime differential ideals $P_i \subseteq R\{X_1, \ldots, X_n\}$ such that $I = \bigcap_{i=1}^m P_i$.*

Proof. See [37], Theorem 1.18. □

Note that the hypotheses of these two theorems are fulfilled when R is a differential field.

Remark 1.2.19. In the theorem above, one may assume that the P_i's are distinct and minimal (for the property of being prime and extending I). With this assumption, the P_i's are unique.

1.2.2 Differentially closed fields

Differentially closed fields were introduced by Robinson [53]. Their $\mathcal{L}_{df}$-theory was axiomatised later by Blum [4]. The scheme of axioms is the following:

- Axioms for DF_0;
- Axioms for algebraically closed fields;
- For any differential polynomials f, g, if $\mathrm{ord}(f) > \mathrm{ord}(g)$, then the system $f(X) = 0 \wedge g(X) \neq 0$ has a solution.

The theory of differentially closed fields of characteristic 0 is denoted DCF_0.

Theorem 1.2.20 ([57], Theorem 40.2)**.** *DCF_0 is the model completion of DF_0.*

Theorem 1.2.21 ([57], Corollary 40.3)**.** *DCF_0 is complete and eliminates quantifiers.*

Theorem-Definition 1.2.22 ([57], Theorem 41.3 and Theorem 41.4)**.** For any differential field K, there is a differential extension L of K which is prime over K and is a model of DCF_0. We call L a differential closure of K.

L. Blum showed that DCF_0 is ω-stable. Therefore Shelah's result that in an ω-stable theory, prime models are unique up to isomorphism implies that all differential closures of a given differential field K are K-isomorphic (see [61]).

However the differential closure is in general non minimal. This fact was apparently proved independently by Rosenlicht (see [54]) and Kolchin (see [31], section 8).

In the theory DCF_0, the algebraic and definable closures have a simple description (see Lemma 5.1 in [37]). Let L be a differentially closed field and $A \subseteq L$, $\mathrm{dcl}^L(A)$ is the differential subfield of L generated by A and $\mathrm{acl}^L(A)$ is the algebraic closure (as a field) of $\mathrm{dcl}^L(A)$.

On isolated types and constrained elements in differentially closed fields

In [31], E. Kolchin does not use our terminology (isolated type, differential closure) which comes from logicians, but he gives equivalent definitions of the same notions and uses another vocabulary (constrained element, constrained closure). We explain below the relation between his definitions and ours. It does not play a determinant role in the sequel but we believe it may help the reader who wishes to consult Kolchin's works, for instance [30] and [31].

Let K be a differential field. For any $\bar{z}$ in some differential field extension of K, $\mathcal{I}_K(\bar{z})$ denotes $\{f \in K\{X_1, \ldots, X_n\} : f(\bar{z}) = 0\}$. It is easy to see that $\mathcal{I}_K(\bar{z})$ is a prime differential ideal of $K\{X_1, \ldots, X_n\}$.

Let $\bar{x}, \bar{y} \in L$ where L is some differentially closed extension of K. One says that $\bar{x}$ is a differential specialisation of $\bar{y}$ over K iff $\mathcal{I}_K(\bar{y}) \subseteq \mathcal{I}_K(\bar{x})$. Moreover $\bar{x}$ is a generic differential specialisation of $\bar{y}$ over K iff $\mathcal{I}_K(\bar{y}) = \mathcal{I}_K(\bar{x})$.

One says that $\bar{x}$ is constrained over K iff there exists $g \in K\{X\}$ such that $g(\bar{x}) \neq 0$ and for all non-generic differential specialisation $\bar{y}$ of $\bar{x}$ over K, it holds that $g(\bar{y}) = 0$.

We relate this definition to the notion of isolated type in differentially closed fields. Note that since DCF_0 eliminates quantifiers, $\mathrm{tp}^L(\bar{x}/K)$ is determined by the set of atomic $\mathcal{L}_{df}(K)$-formulas satisfied by $\bar{x}$. Therefore, $\mathrm{tp}^L(\bar{x}/K) = \mathrm{tp}^L(\bar{y}/K)$ iff $\mathcal{I}_K(\bar{x}) = \mathcal{I}_K(\bar{y})$ iff $\bar{x}$ is a generic differential specialisation of $\bar{y}$ over K.

Lemma 1.2.23. *Let $\bar{x} \in L$, $\mathrm{tp}^L(\bar{x}/K)$ is isolated iff $\bar{x}$ is constrained over K.*

Proof. [$\Rightarrow$] Since DCF_0 eliminates quantifiers, $\mathrm{tp}^L(\bar{x}/K)$ is isolated by a formula ψ which we suppose to be quantifier-free. So without loss of generality, $\psi(\bar{X}) := \bigwedge_{i=1}^m f_i(\bar{X}) = 0 \wedge \bigwedge_{i=1}^n g_i(\bar{X}) \neq 0$. Take $g = \prod_{i=1}^n g_i$. Clearly $g(\bar{x}) \neq 0$.

Let $\bar{y} \in L$ be a non-generic differential specialisation of $\bar{x}$. Since $\mathcal{I}_K(\bar{x}) \subseteq \mathcal{I}_K(\bar{y})$, then $L \models \bigwedge_{i=1}^m f_i(\bar{y}) = 0$. Moreover, because $\mathcal{I}_K(\bar{x}) \neq \mathcal{I}_K(\bar{y})$, then $\mathrm{tp}^L(\bar{x}/K) \neq \mathrm{tp}^L(\bar{y}/K)$. So $L \not\models \psi(\bar{y})$, since ψ isolates $\mathrm{tp}^L(\bar{x}/K)$. Therefore $g(\bar{y}) = 0$.

[$\Leftarrow$] Let g be the differential polynomial given by the fact that $\bar{x}$ is constrained over K (see definition above). Take $\psi(\bar{X}) := g(\bar{X}) \neq 0 \wedge \bigwedge_{i=1}^m f_i(\bar{X}) = 0$ where $\mathcal{I}_K(\bar{x})$ is generated by $f_1, \ldots, f_m$ as a radical differential ideal (see Theorem

1.2.17). Then for all $\bar{y} \in L$, $L \models \bigwedge_{i=1}^{m} f_i(\bar{y}) = 0$ iff $\bar{y}$ is a differential specialisation of $\bar{x}$. So if $L \models \psi(\bar{y})$ then $\bar{y}$ is a generic differential specialisation of $\bar{x}$, equivalently, $\mathrm{tp}^L(\bar{x}/K) = \mathrm{tp}^L(\bar{y}/K)$. In other words, $\mathrm{tp}^L(\bar{x}/K)$ is isolated by ψ.
□

One says that a differential field extension M of K is constrained if and only if each tuple of elements of M is constrained over K. A differential field M is called constrainedly closed iff it has no proper constrained extension. Finally a constrainedly closed extension of K which can be embedded in every constrainedly closed extension of K will be called a constrained closure of K.

Let $\bar{x} \in L$. The correspondence between Kolchin's terminology of and ours is summarised as follows (see [31])

- $\bar{x}$ is constrained over K means $\mathrm{tp}^L(\bar{x}/K)$ is isolated;
- constrained extension of K means atomic extension of K;
- constrainedly closed means differentially closed (i.e. model of DCF_0);
- constrained closure means differential closure.

Now, we explain the relationship between (differential) transcendency and isolated types.

Lemma 1.2.24. *Let E be a field, F be an algebraically closed extension of E and $z \in F$. Then $\mathrm{tp}^F_{\mathcal{L}_{\text{fields}}}(z/E)$ is isolated iff z is algebraic over E.*

Proof. It follows from the correspondence between the elements of $S_1^F(E)$ and prime ideals of $E[X]$, which is a consequence of quantifier elimination in algebraically closed fields. That correspondence is given in Proposition 4.1.16 of [36].
□

Proposition 1.2.25. *Let $K \subseteq L$ be differential fields, L be differentially closed and $z \in L$.*

1. *If z is algebraic over K then $\mathrm{tp}^L(z/K)$ is isolated.*
2. *If $\mathrm{tp}^L(z/K)$ is isolated then z is differentially algebraic over K.*

Proof.

1. Since z is algebraic over K, by Lemma 1.2.11, $\mathrm{tp}^L_{\mathcal{L}_{\text{fields}}}(z/K) \vdash \mathrm{tp}^L_{\mathcal{L}_{df}}(z/K)$. Then the statement follows from the fact that L is algebraically closed and Lemma 1.2.24.
2. See Lemma 2.12 of [37].

□

Corollary 1.2.26. *Let $K \subseteq L$ be differential fields, L be differentially closed and $z \in L$. If z is transcendental over K and $z' = 0$ then $\mathrm{tp}^L(z/K)$ is not isolated.*

Proof. Suppose to the contrary that $\mathrm{tp}^L(z/K)$ is isolated by a formula $\psi(X)$, assumed to be equivalent to $\bigwedge_i f_i(X) = 0 \wedge \bigwedge_i g_i(X) \neq 0$. Let $\psi^*(X_0, X_1, \ldots, X_n)$ be $\bigwedge_i f_i^*(X_0, X_1, \ldots, X_n) = 0 \wedge \bigwedge_i g_i^*(X_0, X_1, \ldots, X_n) \neq 0$. Then the $\mathcal{L}_{\text{fields}}$-formula $\psi^*(X_0, X_1, \ldots, X_n) \wedge \bigwedge_{i=1}^n (X_i = 0)$ isolates $\mathrm{tp}^L_{\mathcal{L}_{\text{fields}}}(z, z', \ldots, z^{(n)}/K)$. So $\psi^*(X, 0, \ldots, 0)$ isolates $\mathrm{tp}^L_{\mathcal{L}_{\text{fields}}}(z/K)$. A contradiction with Lemma 1.2.24. □

1.3 Ordered fields

We consider the languages $\mathcal{L}_{or} := \{+, \cdot, -, 0, 1, <\}$ and

$$\mathcal{L}_{of} := \{+, \cdot, -, ^{-1}, 0, 1, <\}$$

called respectively the language of ordered rings and the language of ordered fields[2].

Definition 1.3.1. An ordered ring is an $\mathcal{L}_{or}$-structure $(R, +, \cdot, -, 0, 1, <)$ such that

- $(R, +, \cdot, -, 0, 1)$ is a commutative ring,
- $<$ is a total order on R,
- for any $x, y, z \in R$, $x < y$ iff $x + z < y + z$,
- for any $x, y, z \in R$, ($x < y$ and $0 < z$) implies $x \cdot z < y \cdot z$.

If moreover $(R, +, \cdot, -, 0, 1)$ is a field, we say that $(R, +, \cdot, -, 0, 1, <)$ is an ordered field. Then one endows R with an $\mathcal{L}_{of}$-structure in the natural way.

Example 1.3.2. The field $\mathbb{Q}$ of rational numbers and the field $\mathbb{R}$ of real numbers, endowed with the usual order, are examples of ordered fields.

Remark 1.3.3. Any ordered ring has characteristic 0 (it is a special case of the fact that any ordered group is torsion free). Since this thesis is dedicated to ordered fields, we made the choice to assume from the begining that all rings and fields are of characteristic 0.

[2]The symbol $<$ is a binary relation symbol.

Definition 1.3.4. A field $(F, +, \cdot, -, ^{-1}, 0, 1)$ is formally real iff there exists an order $<$ on F such that $(F, +, \cdot, -, ^{-1}, 0, 1, <)$ is an ordered field.

Lemma 1.3.5 (see [48], Theorem 1.8)**.** *A field F is formally real iff -1 is not a sum of squares of F.*

Note that it is possible to endow some formally real fields with several orders. But we won't play that game here.

Definition 1.3.6. A formally real field $(F, +, \cdot, -, ^{-1}, 0, 1)$ is real closed iff it has no proper formally real algebraic extension.

Theorem 1.3.7 (see [48], Theorem 3.3)**.** *A field F is real closed iff the algebraic closure of F is $F(i)$ where i is a square root of -1 and $F(i) \neq F$.*

Theorem 1.3.8 (see [48], Theorem 3.3)**.** *Let $(F, +, \cdot, -, ^{-1}, 0, 1, <)$ be an ordered field. Then F is real closed iff any positive element of F is a square of an element of F and any polynomial of odd degree with coefficients from F has a root in F.*

This theorem provides a first order $\mathcal{L}_{of}$-axiomatisation of the class of real closed fields. We denote RCF the first order $\mathcal{L}_{of}$-theory of real closed fields.

Theorem 1.3.9 (see [20], Theorem 8.4.4)**.** *RCF has quantifier elimination in $\mathcal{L}_{of}$.*

It follows from the quantifier elimination that RCF is a complete theory.

Note that in real closed fields the relation $<$ is definable in the language $\mathcal{L}_{\text{fields}}$. More precisely,

$$\text{RCF} \vdash \forall X \forall Y (X < Y) \leftrightarrow ((\exists Z \ (Y = X + Z^2)) \wedge \neg(X = Y)).$$

Then there may not be two distincts orders on a real closed field. However, RCF fails to have quantifier elimination in $\mathcal{L}_{\text{fields}}$.

Quantifier elimination in RCF provides a simple description of definable sets. In a real closed field L, for a subfield K of L, definable sets with parameters in K are boolean combinations of sets of the form

$$\{\bar{x} \in L : p(\bar{x}) \geq 0\}$$

where $p \in K[\bar{X}]$. Definable sets in a real closed field are called semialgebraic sets by the algebraists.

Remark 1.3.10. In chapter 3, we will meet semialgebraic groups. Semialgebraic groups are well studied.

If G is semialgebraic in $\mathbb{R}$, by [43], Remark 2.6, G is isomorphic to a real Lie group.

Moreover, any semialgebraic group is isomorphic to a Nash group. In [22] (see Proposition 3.1), they show that for any affine Nash group over any real closed field F there is a Nash homomorphism with finite kernel from G into some algebraic group $H(F)$ defined over F.

Theorem 1.3.11 (see [20], Theorem 4.4.4)**.** *RCF has uniform elimination of imaginaries in* $\mathcal{L}_{\text{fields}}$.

Theorem 1.3.12 (see [20], Section 3.1, Example 3)**.** *The theory RCF of real closed fields in the language* $\mathcal{L}_{of}$ *has definable Skolem functions.*

Definition 1.3.13. Let $(F, +, \cdot, -, ^{-1}, 0, 1)$ be a formally real field, a real closure of F is a maximal formally real algebraic extension of F.

Unlike the algebraic closure and the differential closure, two real closures of F do not need to be F-isomorphic as fields. However, it is known that if we fix an order of F then a real closure of F whose order coincides with the order of F is prime over F (in the language $\mathcal{L}_{of}$). All such real closures of F are $\mathcal{L}_{of}(F)$-isomorphic. There is a precise description of the real closure of F inside a fixed real closed extension:

Theorem 1.3.14. *Let* $(L, +, \cdot, -, ^{-1}, 0, 1)$ *be a real closed field and* F *be any subfield of* L*. Then the relative algebraic closure of* F *inside* L *is real closed and moreover is the unique real closure of* F *inside* L*.*

Proof. This follows from Theorem 1.3.8 and from the definition of real closure. □

Under the hypotheses of Theorem 1.3.14 and when L is fixed, we denote F^{rc} the real closure of F inside L.

Let B be an $\mathcal{L}$-structure such that there is a total order on B which is $\mathcal{L}$-definable. For any subset A of B, $\text{acl}^B(A) = \text{dcl}^B(A)$. Hence in any real closed field, it holds that algebraic and definable closures coincide.

Therefore we get the following description of the closures. Suppose F is real closed and A is a subfield of F. Then $\text{dcl}^F(A) = \text{acl}^F(A)$ is the relative algebraic closure (as a field) of A in F, i.e. the real closure of A inside F.

1.4 Ordered differential fields and CODF

1.4.1 Definitions and generalities

An ordered differential field is a differential field with a total order which makes it an ordered field. We do not suppose any particular relation between the order and the derivation.

Example 1.4.1. Let K be a formally real field and t be transcendental over K, then $K(t)$ is formally real. So one may endow $K(t)$ with an order in such a way that $K(t)$ is an ordered field. Starting with the trivial derivation on K (or with any other derivation on K, if there is one) and using Lemma 1.2.10, we get a derivation on $K(t)$. The field $K(t)$ has been endowed with a structure of ordered differential field.

Let $\mathcal{L}_{odf}$ be the language $\{+, \cdot, -, ^{-1}, 0, 1, <, '\}$, called the language of ordered differential fields. The first-order $\mathcal{L}_{odf}$-theory whose models are ordered differential fields is called ODF.

In [65], M. Singer defines an $\mathcal{L}_{odf}$-theory called CODF (acronym for closed ordered differential fields). An $\mathcal{L}_{odf}$-structure K is a model of CODF iff K satisfies the following first-order axioms:

1. Axioms for real closed fields
2. Axioms for the derivation
3. Let $f, g_1, \ldots, g_m \in K\{X\}$ be such that for all $i \in \{1, \ldots, m\}$, $n := \text{ord}(f) \geq \text{ord}(g_i)$. If there are $a_0, \ldots, a_n \in K$ such that
$$f^*(a_0, \ldots, a_n) = 0, s_f^*(a_0, \ldots, a_n) \neq 0$$
and $g_1^*(a_0, \ldots, a_n) > 0, \ldots, g_m^*(a_0, \ldots, a_n) > 0$, then there is $z \in K$ such that $f(z) = 0$ and $g_1(z) > 0, \ldots, g_m(z) > 0$.

M. Singer shows that CODF is the model completion of ODF. In particular, the following property is proved (see the proof of the theorem in [65], page 85).

Lemma 1.4.2. *Let K be a model of ODF. Let $f, g_1, \ldots, g_m \in K\{X\}$ be such that for all $i \in \{1, \ldots, m\}$, $n := \text{ord}(f) \geq \text{ord}(g_i)$ and there are $a_0, \ldots, a_n \in K$ such that $f^*(a_0, \ldots, a_n) = 0$, $s_f^*(a_0, \ldots, a_n) \neq 0$ and*
$$g_1^*(a_0, \ldots, a_n) > 0, \ldots, g_m^*(a_0, \ldots, a_n) > 0.$$
Let $t_0, \ldots, t_{n-1}$ be algebraically independent over K and infinitesimal with respect to K. Then we may endow any real closure $K(t_0, \ldots, t_{n-1})^{rc}$ of $K(t_0, \ldots, t_{n-1})$

with a derivation in such a way that there is $z \in K(t_0, \ldots, t_{n-1})^{rc}$ *such that* $f(z) = 0$ *and* $g_1(z) > 0, \ldots, g_m(z) > 0$.

By definition of the model completion of a theory, any ordered differential field has an extension which is a model of CODF. This fact ensures the consistency of CODF. However we do not know any natural example of model of CODF.

A theory is called universal iff any substructure of a model of T is a model of T. A theorem of Robinson (see [57], Theorem 13.2) states that the model completion of a universal theory eliminates quantifiers. Since ODF is a universal theory we have the following theorem:

Theorem 1.4.3. *The theory CODF eliminates quantifiers.*

A consequence of the elimination of quantifiers in CODF is the fact that CODF is complete.

Since any model K of CODF is a real closed field, $K(i)$ is an algebraically closed field. One can say even more on $K(i)$:

Theorem 1.4.4 (see [66])**.** *Let* K *be a model of CODF and* i *be a square root of -1,* $(K(i), +, \cdot, -, ^{-1}, 0, 1, ')$ *is a model of* DCF_0.

As far as we know, the following question is still open. Let K be an ordered differential field, can we find a model of CODF containing K and whose constant field is the real closure of C_K?

Moreover, we do not know any analogue for differential fields of Artin's result that any algebraically closed field L is of the shape $K(i)$ for some real closed subfield K of L. The following conjecture is a natural analogous statement:

Conjecture 1.4.5. *Let* L *be a model of* DCF_0*, there exists a model* K *of CODF such that* $K(i) = L$.

Now we present a negative result on CODF.

Theorem 1.4.6 ([70],Theorem 4.5.7)**.** *Let* T *be a countable complete theory with infinite models. Then* T *has a prime model iff for all* n*, the isolated* n*-types are dense in* S_n^T.

Remark 1.4.7. M. Singer proves in [65] that isolated 1-types are not dense in the Stone space S_1^{CODF}. Then the fact that CODF has no prime model results from Theorem 1.4.6.

Hence, there is no analogue of the differential closure (or of the real closure) in the theory CODF.

1.4.2 Definable Closure

We will use in an argument below the fact that when F is a model of CODF, C_F is dense in F (with respect to the topology induced by $<$). That fact is mentionned without proof in [7], it is proved (in a more general setting) in [19], Corollary 3.13. Indeed it follows easily from the axioms of CODF: to show that there is a constant element c between given elements $b_1 < b_2$ of a model of CODF, take $f := X'$, $g_1 := X - b_1$ and $g_2 := b_2 - X$ in the axiomatisation of CODF.

Let F be a model of CODF and A be a subset of F. As $<$ is $\mathcal{L}_{odf}$-definable (actually it is even $\mathcal{L}_{df}$-definable), then $\mathrm{acl}^F(A) = \mathrm{dcl}^F(A)$. Using the axiomatisation of CODF, we show below that $\mathrm{dcl}^F(A)$ is the relative algebraic closure (as a field) of $\langle A\rangle$ in F, i.e. the real closure of $\langle A\rangle$ in F.

Suppose to the contrary, that $t \in \mathrm{acl}^F(A) = \mathrm{dcl}^F(A)$ and t is transcendental over $\langle A\rangle$.

If t is differentially transcendental over $\langle A\rangle$: Since $t \in \mathrm{dcl}^F(A)$, t is the only realisation of an $\mathcal{L}_{odf}$-formula $\psi(X)$ with parameters in A. Because CODF eliminates quantifiers and t is differentially transcendental, $\psi(X)$ is equivalent to a formula of the shape $\bigwedge_{i=1}^{k} g_i(X) > 0$ where for any $i \in \{1,\dots,k\}$, $g_i(X) \in \langle A\rangle\{X\}$. It is obvious that for all $i \in \{1,\dots,k\}$, $g_i(t) > 0$ is equivalent to $g_i^*(t,t',\dots) > 0$ and for any constant $c \in C_F$, $g_i(t+c) > 0$ iff $g_i^*(t+c,t',\dots) > 0$. Since C_F is dense in F and the map $F \to F : x \mapsto g_i^*(x)$ is continuous, we get $g_i^*(t+c) > 0$ for a constant $c \in C_F \setminus \{0\}$, close enough to 0. So $F \models \psi(t) \wedge \psi(t+c)$. Since $c \neq 0$, it contradicts the fact that t is definable over $\langle A\rangle$.

Otherwise: Let $\mathcal{I}(t)$ be the set elements of $\langle A\rangle\{X\}$ vanishing at t, $\mathcal{I}(t) \neq \{0\}$ and $\mathcal{I}(t)$ is a prime differential ideal of $\langle A\rangle\{X\}$. Let f be a minimal polynomial of $\mathcal{I}(t)$ (it does exist by Lemma 1.2.7). It follows that s_f does not vanish at t. Since t is definable over A, t is the only realisation in F of a formula $\psi(X)$ which one may assume to be

$$f(X) = 0 \bigwedge_{i=1}^{k} g_i(X) > 0$$

for some $g_i(X) \in \langle A\rangle\{X\}$ since CODF eliminates quantifiers. Let $n := \mathrm{ord}(f)$, by Lemma 1.2.8, for any $l \in \mathbb{N}$, there is a rational fraction $p_l(\bar{X}) \in \langle A\rangle(X_0, X_1, \dots, X_n)$ such that whenever $f(r) = 0$, $r^{(l)} = p_l(r, r', \dots, r^{(n)})$. Hence one may suppose that for any $i \in \{1,\dots,k\}$, $\mathrm{ord}(g_i) \leq \mathrm{ord}(f)$.

Let $\phi(\bar{X})$ be the formula

$$(f^*(\bar{X}) = 0) \bigwedge_{i=1}^{k} (g_i^*(\bar{X}) > 0) \wedge (s_f^*(\bar{X}) \neq 0).$$

As $\phi(t, \ldots, t^{(n)})$ holds, $\bigwedge_{i=1}^{k}(g_i^*(\bar{X}) > 0) \wedge (s_f^*(\bar{X}) \neq 0)$ defines a non empty open subset of F^{n+1}, moreover there are infinitely many realisations of ϕ in F^{n+1}. (It is a consequence of implicit functions Theorem, see [5], Corollary 2.9.6.)

The axioms of CODF imply that there is $u \in F$ such that $\psi(u)$ holds and $u \neq t$.

1.4.3 Definable Skolem functions and imaginaries

We will make use of a result of L. van den Dries proved in [14]. We review it below.

Let $\mathcal{L}$ be a language containing at least one constant symbol and T be a first order $\mathcal{L}$-theory. Let B be an $\mathcal{L}$-structure and A be a subset of B.

We say that B is algebraic over A iff $\mathrm{acl}^B(A) = B$. Moreover, we say that B is rigid over A iff any A-automorphism of B is the identity.

We denote $T_\forall$ the universal theory of T, that is the set of $\mathcal{L}$-sentences satisfied by the class of $\mathcal{L}$-substructures of any model of T.

Theorem 1.4.8 ([14], Theorem 2.1). *If T eliminates quantifiers, then the following are equivalent:*

1. *T has definable Skolem functions;*
2. *Any model A of $T_\forall$ has an extension that models the theory T and that is algebraic and rigid over A.*

Lemma 1.4.9. *The theory CODF does not have definable Skolem functions.*

Proof. Let F be a real closed field that is not a model of CODF (for example any real closed field with the trivial derivation). Clearly, $F \models \mathrm{CODF}_\forall$ and F has no ordered algebraic proper extension. By Theorem 1.4.8, CODF does not have definable Skolem functions. □

Theorem 1.4.10 ([46], Theorem 0.3). *The theory CODF eliminates imaginaries.*

Chapter 2

Stellensätze

Introduction

The purpose of this chapter it to extend some results of real geometry and real algebra to the context of ordered differential fields and rings.

The first results are motivated by the works on a nullstellensatz for real closed fields by J.-L. Krivine [34], D. W. Dubois [16] and J.-J. Risler [50]. In section 2.1, we recall the definition of the real radical of an ideal I and use it to state Dubois-Krivine-Risler nullstellensatz.

In section 2.2, we prove that if I is a real differential ideal, I is an intersection of prime differential real ideals and we show that the real radical of a differential ideal I is the smallest real differential ideal containing I. Using the fact that CODF is model complete we prove a nullstellensatz for models of CODF (see Theorem 2.2.8).

In section 2.3, we study the behaviour of real differential ideals under restriction. Let $A \subseteq B$ be differential rings and I be a real differential ideal of B. We show that if $I \cap A$ is prime, then I is an intersection of prime differential real ideals and moreover the ideals involved in the intersection can be chosen to all have the same intersection with A.

In section 2.4, we make use of the fact that in a model of CODF, differential tuples are dense (see Corollary 2.4.2) in order to transfer Stengle's positivstellensatz for models of RCF to models of CODF (see Theorem 2.4.3). However, to use this result of density, we need to make a particular topological assumption in the statement of our positivstellensatz. By a different approach using the model completeness of CODF, we get an algebraic variation of that positivstellensatz where the topological assumption does not appear.

An improvement of Stengle's positivstellensatz for the field of real numbers was proved by Schmüdgen in [60]. That result had an important impact and is known as Schmüdgen's theorem. We show, in section 2.5, that $\mathbb{R}$ can be endowed with a derivation in such a way that $\mathbb{R} \models$ CODF and we prove a differential analogue of Schmüdgen's theorem using a transfer method, which is similar to the one we used for a differential version of Stengle's positivstellensatz.

In [69], G. Stengle investigates the real differential spectrum of a differential ring and shows that the real radical of a differential ideal is still differential. In that paper, there is an attempt to prove a nullstellensatz and a positivstellensatz for CODF, presented as a corollary of an abstract result for differential rings. This abstract theorem (see Theorem 4.2 of [69]) is unfortunately false: T. Grill gave a counterexample in his Ph.D. thesis (see [17], p.47).

Part of this chapter was published in Mathematical Logic Quarterly [8]. Some of the results were investigated during a stay at the university of Konstanz where I was invited by Salma Kuhlmann in 2013. I also had interesting discussions with Marcus Tressl who communicated me Grill's counterexample to Theorem 4.2 of [69].

Conventions and notations

Let F be a differential field. If L is a differential field extension of F, for any $A \subseteq F\{X_1, \ldots, X_n\}$, $\mathcal{V}_L(A) := \{\bar{a} \in L^n : \forall f \in A, f(\bar{a}) = 0\}$ and for any $B \subseteq L^n$, $\mathcal{I}_F(B) := \{f \in F\{X_1, \ldots, X_n\} : \forall \bar{a} \in B, f(\bar{a}) = 0\}$. Note that $\mathcal{I}_F(B)$ is a radical differential ideal of $F\{X_1, \ldots, X_n\}$. The notation $\mathcal{I}_F(\bar{z})$ for $\bar{z} \in L$, introduced previously, corresponds to the special case where B is a singleton. Recall that $\mathcal{I}_F(\bar{z})$ is a prime differential ideal of $F\{X_1, \ldots, X_n\}$. We sometimes omit the subscripts L and F and so we write $\mathcal{V}(A)$, $\mathcal{I}(B)$ and $\mathcal{I}(\bar{z})$.

If $A \subseteq F^{n\cdot(k+1)}$ where $n, k \in \mathbb{N}$ and $n \neq 0$, $\mathrm{diff}_n(A)$ will denote the set of differential tuples in A i.e.,

$$\mathrm{diff}_n(A) = \{(\bar{u}, \bar{u}', \ldots, \bar{u}^{(k)}) \in A : \bar{u} \in F^n\}.$$

Definition 2.0.1. Let R be a ring and a subset P of R. We say that P is a cone of R (or a preordering of R) iff for all $p_1, p_2 \in P$ and $a \in R$,

- $p_1 + p_2 \in P$,
- $p_1 \cdot p_2 \in P$,
- $a^2 \in P$.

Let S be a finite subset of $F\{\bar{X}\}$. We use the notation $W_S := \{\bar{x} \in F^n :$ for all $g \in S$, $g(\bar{x}) \geq 0\}$ and T_S denotes the cone of $F\{X_1, \ldots, X_n\}$ generated by S. Note that we will sometimes use this notation for an ordinary field F and for $S \subset F[\bar{X}]$, i.e., when the derivation of F is trivial. In that case, T_S will be seen as a cone of $F[\bar{X}]$.

We will denote $W_S^* := W_{S^*}$ where $S^* = \{g^* : g \in S\}$. Explicitely, $W_S^* := \{\bar{x} \in F^{n\cdot(d+1)} : \text{for all } g \in S, g^*(\bar{x}) \geq 0\}$ for $d = \max_{g \in S} \operatorname{ord}(g)$. So $W_S^* \subseteq F^{n\cdot(d+1)}$. Moreover, $T_S^* = T_{S^*}$, i.e., T_S^* is the cone (in the ring of ordinary polynomials) generated by $\{g^* : g \in S\}$.

We will make use of the model completeness of CODF, more explicitely the following fact:

Fact 2.0.2. *Let K be a model of CODF and L an ordered differential field extension of K (not assumed to be a model of CODF). For any $g_i \in K\{\bar{X}\}$ and $\Box_i \in \{=, \neq, >, \geq, <, \leq\}$, if $L \models \exists \bar{X} \bigwedge_{i=1}^k g_i(\bar{X}) \Box_i 0$ then $K \models \exists \bar{X} \bigwedge_{i=1}^k g_i(\bar{X}) \Box_i 0$.*

2.1 The nullstellensatz for ordered fields

Our nullstellensatz is inspired by a result on real closed fields proved independently by J.-L. Krivine in [34], D. W. Dubois in [16] and J.-J. Risler in [50] (proofs can also be found in [5] and [48]).

In this section, rings and fields are not necessarily differential. For convenience of the reader, before stating the main result, we define real ideals. We also state some well-known results. They can be found in [48] and [5].

Definition 2.1.1. Let R be a ring. An ideal I of R is said to be real if for any $u_1, \ldots, u_n \in R$ such that $\sum_{i=1}^n u_i^2 \in I$, it holds that $u_i \in I$ for all i.

Lemma 2.1.2 ([5], lemme 4.1.6)**.** *Let R be a ring and I be a prime ideal of R, I is real iff the fraction field of R/I is formally real (and so it is possible to define an order on it).*

Lemma 2.1.3 ([5], lemme 4.1.5)**.** *Let R be a ring, if I is a real ideal of R then I is radical.*

For any ring R, S_R denotes the set of sums of squares of R.

Definition 2.1.4. Let R be a ring and I an ideal of R, the real radical of I is

$$\mathcal{R}(I) := \{f \in R : f^{2m} + s \in I \text{ for some } m \in \mathbb{N} \text{ and } s \in S_R\}.$$

Lemma 2.1.5 ([5], Proposition 4.1.7)**.** *Suppose R is a ring and I is an ideal in R. The real radical of I is the smallest real ideal containing I.*

Dubois-Krivine-Risler Theorem can be formulated as follows:

Theorem 2.1.6. *Let F be a real closed field, I be an ideal of $F[X_1, \ldots, X_n]$ and $f \in F[X_1, \ldots, X_n]$. Suppose that any zero of I in F^n is also a zero of f. Then $f \in \mathcal{R}(I)$.*

2.2 The nullstellensatz for ordered differential fields

Let R be a differential ring, an ideal of R will be called real differential if it is real and differential.

Here, $(K, +, \cdot, -, ^{-1}, 0, 1, <, ')$ is an ordered differential field. We first need to refine Ritt-Raudenbush Theorem (see 1.2.18) in the particular case of real ideals.

Lemma 2.2.1. *If I is a real differential ideal in $K\{X_1, \ldots, X_n\}$, then I is the intersection of a finite number of prime real differential ideals.*

Proof. By 1.2.18, $I = \bigcap_{i=1}^m P_i$ where P_i are prime differential ideals and by 1.2.19, we suppose that P_i are minimal.

Suppose P_1 is not real i.e., there are $u_1, \ldots, u_k \in K\{X_1, \ldots, X_n\} \setminus P_1$ s.t. $u_1^2 + \cdots + u_k^2 \in P_1$. By the assumption of minimality, for all $i = 2, \ldots, m$ there are $v_i \in P_i \setminus P_1$.

Let $v := \prod_{i=2}^m v_i$. Clearly, $v \notin P_1$ and $v \in \bigcap_{i=2}^m P_i$.

Then, $(u_1 v)^2 + \cdots + (u_k v)^2 \in \bigcap_{i=1}^m P_i = I$. Since I is real, $(u_1 v) \in I \subseteq P_1$. Finally, as P_1 is prime, $u_1 \in P_1$ or $v \in P_1$, in both cases, that gives a contradiction. □

Definition 2.2.2. Let I be a prime differential ideal in $K\{X_1, \ldots, X_n\}$, $L \supseteq K$ be a differential extension of K and $\bar{z} \in L$. We will say that $\bar{z}$ is a generic zero of I iff $I = \mathcal{I}_K(\bar{z})$.

Lemma 2.2.3. *Suppose K is real closed. Let I be a prime differential ideal in $K\{X_1, \ldots, X_n\}$, I is real iff I has a generic zero in some ordered differential extension L of K.*

Proof. [⇒] If I is real, then $L := \mathrm{Frac}(K\{X_1, \ldots, X_n\}/I)$ is formally real and so the order of K can be extended to all L (actually any order of L is an extension of the order of K, since K is real closed).

Consider $\bar{z} := \bar{X} + I \in L$. Let $f \in K\{\bar{X}\}$, $f \in I$ iff $f(\bar{X}) + I = 0$ iff $f(\bar{X} + I) = 0$ iff $f(\bar{z}) = 0$.

[⇐] If I has a generic zero in an ordered differential extension L of K (say $I = \mathcal{I}(\bar{z})$, $\bar{z} \in L^n$) and $\sum_{i=1}^n f_i^2 \in I$ for some $f_1, \ldots, f_n \in K\{X_1, \ldots, X_n\}$. Then $\sum_{i=1}^n f_i^2(\bar{z}) = 0$. So for all i, $f_i(\bar{z}) = 0$, namely $f_i \in I$. □

Remark 2.2.4. One easily see that if $\bar{z}$ is a generic zero of I and $\bar{z}_1$ has the same type as $\bar{z}$ ($\text{tp}(\bar{z}_1/K) = \text{tp}(\bar{z}/K)$), then $\bar{z}_1$ is also a generic zero of I. So, in a $|K|^+$-saturated ordered differential extension of K, any prime real differential ideal of $K\{X_1, \ldots, X_n\}$ has a generic zero.

Notation. Let R be a differential ring and $a_1, \ldots, a_n \in R$, $[a_1, \ldots, a_n]$ will denote the differential ideal generated by $a_1, \ldots, a_n$ in R. Similarly, if $E \subseteq R$, $[E]$ will denote the differential ideal generated by E in R. Moreover, $\{a_1, \ldots, a_n\}$ (resp. $\{E\}$) will denote the radical differential ideal generated by $a_1, \ldots, a_n$ in R (resp. by E in R).

From now on, K is a closed ordered differential field.
Below we write $\mathcal{I}$ for $\mathcal{I}_K$ and $\mathcal{V}$ for $\mathcal{V}_K$.

Proposition 2.2.5. *Let I be a differential ideal of $K\{X_1, \ldots, X_n\}$, I is real iff $\mathcal{I}(\mathcal{V}(I)) = I$.*

Proof. [⇒] Clearly $\mathcal{I}(\mathcal{V}(I)) \supseteq I$. For the other inclusion, we first consider a particular case.

If I is prime: By Ritt-Raudenbush Theorem 1.2.17, there are $g_1, \ldots, g_k \in K\{\bar{X}\}$ such that $I = \{g_1, \ldots, g_k\}$. Moreover, by Lemma 2.2.3 and the fact that I is real, there exists $\bar{z}$ in some ordered differential extension L of K s.t. $I = \mathcal{I}(\bar{z})$.

Let $f \notin I$, $L \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) = 0 \wedge f(\bar{X}) \neq 0$ (because $\bar{z} \in L$).

So by Fact 2.0.2, $K \models \exists \bar{X} \bigwedge_{i=1}^{k} g_i(\bar{X}) = 0 \wedge f(\bar{X}) \neq 0$. Hence, $f \notin \mathcal{I}(\mathcal{V}(I))$.

We showed that $\mathcal{I}(\mathcal{V}(I)) \subseteq I$.

General case: As I is real, by Lemma 2.2.1, there are prime real differential ideals P_i such that $I = \bigcap_{i=1}^{m} P_i$. So $\mathcal{I}(\mathcal{V}(I)) = \mathcal{I}(\mathcal{V}(\bigcap_{i=1}^{m} P_i)) \subseteq \bigcap_{i=1}^{m} \mathcal{I}(\mathcal{V}(P_i)) = \bigcap_{i=1}^{m} P_i = I$.

[⇐] Suppose $\mathcal{I}(\mathcal{V}((I)) = I$. Let $f_1, \ldots, f_l \in K\{X_1, \ldots, X_n\}$ be such that $f_1^2 + \cdots + f_l^2 \in I$, then $f_1^2 + \cdots + f_l^2 \in \mathcal{I}(\mathcal{V}(I))$. Equivalently, $f_1^2 + \cdots + f_l^2$ vanishes on $\mathcal{V}(I)$. Therefore, $f_1, \ldots, f_l$ vanish on $\mathcal{V}(I)$ and so $f_1, \ldots, f_l$ belong to $\mathcal{I}(\mathcal{V}(I)) = I$. □

Let I be a differential ideal in $K\{X_1, \ldots, X_n\}$. We consider the real radical of I defined in section 2.1 and denoted by $\mathcal{R}(I)$.

Lemma 2.2.6 ([69], Proposition 3.3)**.** *Let $I \subseteq K\{X_1, \ldots, X_n\}$ be a differential ideal, then $\mathcal{R}(I)$ is a real differential ideal.*

The lemma above, showed by G. Stengle, implies the following corollary which is the last step before reaching the nullstellensatz.

Corollary 2.2.7. *Let $I \subseteq K\{X_1, \dots, X_n\}$ be a differential ideal. The real radical of I is the smallest (i.e., intersection of all) real differential ideal(s) of $K\{X_1, \dots, X_n\}$ containing I.*

Proof. Follows evidently from Lemma 2.1.5 and Lemma 2.2.6. □

Here, K is still a model of CODF.

Theorem 2.2.8 (Nullstellensatz)**.** *Let I be a differential ideal of the differential ring $K\{X_1, \dots, X_n\}$, then $\mathcal{I}(\mathcal{V}(I)) = \mathcal{R}(I)$.*

Proof. Firstly, as $\mathcal{R}(I)$ is real, by Proposition 2.2.5, $\mathcal{I}(\mathcal{V}(\mathcal{R}(I))) = \mathcal{R}(I)$.

Moreover, $I \subseteq \mathcal{R}(I)$, so $\mathcal{I}(\mathcal{V}(I)) \subseteq \mathcal{I}(\mathcal{V}(\mathcal{R}(I))) = \mathcal{R}(I)$.

Secondly, we have that $\mathcal{I}(\mathcal{V}(I))$ is real and contains I. So by Corollary 2.2.7, $\mathcal{R}(I) \subseteq \mathcal{I}(\mathcal{V}(I))$. □

2.3 Prime real differential ideals

The results in this section are adjustments to real differential ideals of some classical results of differential algebra (see for example [26], Chapter II).

For some technical reasons, we will need the following three lemmas:

Lemma 2.3.1 ([26], Lemma 1.6)**.** *Let S and T be any subsets of a differential ring. Then $\{S\} \cdot \{T\} \subseteq \{S \cdot T\}$.*

Lemma 2.3.2. *Let I_1, I_2 be differential ideals of $K\{X_1, \dots, X_n\}$,*

$$\mathcal{R}(I_1) \cdot \mathcal{R}(I_2) \subseteq \mathcal{R}([I_1 \cdot I_2]).$$

Proof. In this proof $S := S_{K\{X_1,\dots,X_n\}}$.

Let $h \in \mathcal{R}(I_1) \cdot \mathcal{R}(I_2)$, $h = f \cdot g$ where $f^{2k} + s_1 \in I_1$ and $g^{2l} + s_2 \in I_2$ with $k, l \in \mathbb{N}, l \leq k$ and $s_1, s_2 \in S$. Hence, $(f^{2k} + s_1)(g^{2l} + s_2) = f^{2k} g^{2l} + t \in I_1 \cdot I_2$ (for some $t \in S$). Therefore, $f^{2k} g^{2k} + t \cdot g^{2(k-l)} \in [I_1 \cdot I_2]$. Finally, we get $h^{2k} + u \in [I_1 \cdot I_2]$ with $u = t \cdot g^{2(k-l)} \in S$. □

Lemma 2.3.3. *Consider a multiplicatively closed set M in $K\{X_1, \dots, X_n\}$ and J a proper real differential ideal which is maximal among the proper real differential ideals I such that $I \cap M = \varnothing$. Then J is prime.*

Proof. Looking for a contradiction, suppose $b_1, b_2 \notin J$ and $b_1 b_2 \in J$. By maximality of J and by Lemma 2.2.6, there exists $a_1, a_2 \in M$, $a_1 \in \mathcal{R}(\{J, b_1\})$ and $a_2 \in \mathcal{R}(\{J, b_2\})$. By Lemma 2.3.2, $a_1 a_2 \in \mathcal{R}(\{\{J, b_1\} \cdot \{J, b_2\}\})$. Now we use Lemma 2.3.1 and get $a_1 a_2 \in \mathcal{R}(\{J\}) = \mathcal{R}(J) = J$. Therefore, as M is multiplicatively closed, $a_1 a_2$ belongs to both M and J. A contradiction. □

The proposition below easily follows from Theorem 2.2.1 and Lemma 2.2.7. However, we prove it as a consequence of Lemma 2.3.3.

Proposition 2.3.4. *Let I be a proper differential ideal in $K\{X_1, \dots, X_n\}$. Then the real radical of I is the intersection of all prime real differential ideals of $K\{X_1, \dots, X_n\}$ containing I (if I is not contained in any prime real differential ideal, then $\mathcal{R}(I) = K\{X_1, \dots, X_n\}$).*

Proof. Let $a \notin \mathcal{R}(I)$ and M the multiplicatively closed set $\{a^k : k \in \mathbb{N} \setminus \{0\}\}$. Let J be a proper real differential ideal which is maximal among the ones which contain I and are disjoint to M (by Zorn Lemma, such an ideal does exist). By Lemma 2.3.3, J is prime and the proof is complete. □

We work now in the following situation: let $B := K\{X_1, \dots, X_n\}$ and $A \subseteq B$ a differential subring, P a prime differential ideal in A and I a differential ideal in B.

Definition 2.3.5. We will say that I contracts to P iff $I \cap A = P$.

If $I \subseteq B$ is real and contracts to P in A, can we find a prime real differential ideal J of B extending I which contracts to P? The next proposition gives a positive answer under a suitable hypothesis on I.

Proposition 2.3.6. *Let I be a proper real differential ideal in B which contracts to P and such that for all $a \in A$ and $b \in B$, $a \cdot b \in I$ implies $a \in I$ or $b \in I$. Then I is an intersection of prime real differential ideals also contracting to P.*

Proof. Let $x \in B \setminus I$ and $M := \{a \cdot x^k : a \in A \setminus P, k \in \mathbb{N}\}$ the multiplicatively closed subset of B generated by $A \setminus P$ and x. By the hypothesis made on I, $I \cap M = \varnothing$. Consider a real differential ideal J which is maximal among the ones which also contract to P and are disjoint to M. By Lemma 2.3.3, J is prime. Moreover, J doesn't contain x. Finally, let us show that $J \cap A = P$: if $a \in J \cap A$, then $a \notin M$, so $a \notin A \setminus P$, therefore $a \in P$. □

2.4 The positivstellensatz

2.4.1 Topological version

The positivstellensatz we will prove for models of CODF is based on Lemma 2.4.1, a result of real geometry called Stengle's positivstellensatz (originally proved in [68] by G. Stengle). We take here the formulation of [5].

Lemma 2.4.1 ([5], Corollaire 4.4.3)**.** *Let F be a real closed field and S be a finite subset of $F[X_1, \ldots, X_n]$. Let $f \in F[X_1, \ldots, X_n]$, $W := W_S$ and $T := T_S$.*

$$\forall \bar{x} \in W, f(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g, h \in T : f \cdot g = f^{2m} + h.$$

We will use the following consequence of the axiomatisation of CODF, in order to transfer Lemma 2.4.1 in differential fields.

Corollary 2.4.2 ([7], Lemma 2.2)**.** *If K be a model of CODF, then the set $\mathrm{diff}_n(K^{n \cdot d})$ is dense in $K^{n \cdot d}$ (with respect to the topology induced by $<$ on K).*

Theorem 2.4.3 (Positivstellensatz)**.** *Let K be a closed ordered differential field. Let S be a finite subset of $K\{X_1, \ldots, X_n\}$ and $W_S := \{\bar{x} \in K^n$: for all $g \in S$, $g(\bar{x}) \geq 0\}$. Let $f \in K\{X_1, \ldots, X_n\}$ and T_S be the cone of $K\{X_1, \ldots, X_n\}$ generated by S.*

Suppose moreover that there exists an open set $O \subseteq K^{n \cdot (d+1)}$ such that $O \subseteq W_S^* \subseteq \mathrm{cl}(O)$, where $d := \max_{g \in S} \mathrm{ord}(g)$. Then

$$\forall \bar{x} \in W_S, f(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g, h \in T_S : f \cdot g = f^{2m} + h.$$

Proof. In this proof, we will need to consider $f^*(\bar{x})$ for $\bar{x} \in W_S^*$. If $\mathrm{ord}(f) > d$, this is of course meaningless since $W_S^* \subseteq K^{n \cdot (d+1)}$, so we will see W_S^* as a subset of $K^{n \cdot (e+1)}$ where $e := \max\{d, \mathrm{ord}(f)\}$. One may say in other words that we consider the polynomials of S^* as polynomials with $n \cdot (e+1)$ variables via the inclusion embedding $K[X_1, \ldots, X_{n \cdot (d+1)}] \subseteq K[X_1, \ldots, X_{n \cdot (e+1)}]$.

Moreover, we replace O by $U := O \times K^{n(e-d)}$ which is a dense open subset of W_S^* in $K^{n \cdot (e+1)}$. We are now ready to make the argument.

By Lemma 2.4.1, one knows that

$$\forall \bar{x} \in W_S^*, f^*(\bar{x}) \geq 0 \Leftrightarrow \exists m \in \mathbb{N}, g^*, h^* \in T_S^* : f^* \cdot g^* = (f^*)^{2m} + h^*.$$

If $\forall \bar{x} \in W_S, f(\bar{x}) \geq 0$ then $\forall \bar{x} \in W_S^*, f^*(\bar{x}) \geq 0$ because Lemma 2.4.2 implies that $\mathrm{diff}_n(U)$ is dense in W_S^*. Moreover, the converse is obviously true.

So, $\forall \bar{x} \in W_S, f(\bar{x}) \geq 0$ iff $\forall \bar{x} \in W_S^*, f^*(\bar{x}) \geq 0$.

Finally, using continuity of the polynomials and density of $\text{diff}_n(K^{n\cdot(e+1)})$ in $K^{n\cdot(e+1)}$, one obtains the following equivalence

$$\exists m \in \mathbb{N}, g^*, h^* \in T_S^* : f^* \cdot g^* = (f^*)^{2m} + h^*$$
$$\Leftrightarrow \exists m \in \mathbb{N}, g, h \in T : f \cdot g = f^{2m} + h.$$

□

As mentioned in the introduction, T. Grill has found a counterexample to an abstract and more general version of the theorem above. This explains the fact that we had to put some conditions on W_S^*. Moreover, note that in the proof, we only use the fact that $\text{diff}_n(W_S^*)$ is dense in W_S^* and so this weaker condition would be sufficient to get the result.

2.4.2 Algebraic version

We will prove a slightly different positivstellensatz for models of CODF.

The proof of our positivstellensatz (Theorem 2.4.3) only applies to differential polynomials that are non-negative on a set of the shape W_S, where S is any finite set of differential polynomials satisfying a particular topological condition.

Here, we take a different approach that leads us to a result under purely algebraic hypotheses. We will make use of some results from [69] and [17]. Moreover, unlike in the proof of Theorem 2.4.3, we will not use Corollary 2.4.2 but we will need the model completeness of CODF.

Definition 2.4.4. Let A be a ring and T be a cone of A, $T \cap -T$ is called the support of T.

Below we always assume that $\frac{1}{2} \in A$. That assumption is used in the proof of the following lemma.

Lemma 2.4.5 (See [39], Proposition 2.1.2)**.** *Let T be any cone of a ring A then the support of T is an ideal of A. Moreover, $-1 \in T$ iff $T = A$.*

Note that if A is a field, T is a cone of A and $T \neq A$, then the support of T is $\{0\}$.

Definition 2.4.6. Let T be a cone of a ring A. We say that T is proper iff $T \neq A$ (equivalently $-1 \notin T$).

Definition 2.4.7. Let A be a ring and T be a cone of A, we will say that T is a prime cone (or an ordering) of A iff the support of T is a prime ideal of A and $T \cup -T = A$.

If moreover A is a differential ring, we will say that T is differential if its support is a differential ideal.

Definition 2.4.8. Let A be a ring, I be an ideal of A and T be a cone of A, I is T-convex iff for any $a_1, a_2 \in T$,

$$a_1 + a_2 \in I \text{ implies } a_1, a_2 \in I.$$

Lemma 2.4.9 (See [39], Proposition 2.1.7)**.** *Let T be any cone of a ring A, the support of T is T-convex.*

Definition 2.4.10. Let A be a differential ring, T be a cone of A and I be an ideal of A. The differential T-radical of I is the intersection of all T-convex radical differential ideals of A containing I. It is denoted $\sqrt[d,T]{I}$.

Note that if T is the set of sums of squares of A then a real ideal of A is a T-convex ideal of A. Moreover, a radical T-convex ideal of A is a real ideal. Then for any differential ideal I of A, $\sqrt[d,T]{I}$ is the real radical of I (see Corollary 2.2.7).

Proposition 2.4.11 ([17], Proposition 9)**.** *Let A be a differential ring and T be a proper cone of A. The following are equivalent*

1. *The differential T-radical of the zero ideal is proper.*
2. *T is contained in a proper differential cone.*

Theorem 2.4.12. *Any proper differential cone is contained in a proper prime differential cone.*

Proof. Follows from Zorn Lemma and Theorem 3.7 from [69]. □

Theorem 2.4.13. *Let K be a model of CODF and S be a finite subset of $K\{\bar{X}\}$. We denote $T := T_S$ and $W := W_S$. Suppose that there is a T-convex proper differential ideal in $K\{\bar{X}\}$. Then*

$$W = \varnothing \textit{ iff } -1 \in T.$$

Proof. It is obvious that $-1 \in T$ implies that W is empty.

Assume now that $-1 \notin T$ and let us show that W is nonempty. By the hypothesis on T the differential T-radical of the zero ideal is proper, so by Proposition 2.4.11, T is contained in a proper differential cone. Moreover, by Theorem 2.4.12, T is also contained in a proper prime differential cone P.

By Lemma 2.4.9, the support I of P is a proper prime differential P-convex ideal.

Let $L := \mathrm{Frac}(K\{\bar{X}\}/I)$. Obviously, as I is proper K can be embedded in L and as I is differential one can endow L with a derivation making it a differential field extension of K. Furthermore, the extension of P to L is a proper cone of L (see page 21 of [39] for the definition of the extension and by Proposition 2.1.6 of [39], the extension is proper). By [39], Lemma 1.4.4. (or [39], 2.5.1), P is contained in an ordering P_L of L.

Let $\geq$ be the order relation defined on L by P_L.

Taking $\bar{z} := \bar{X} + I$; one checks that for all $g \in S, g(\bar{z}) \geq 0$.

So $L \models \exists \bar{Y} \bigwedge_{g \in S} g(\bar{Y}) \geq 0$.

By model completeness of CODF (Fact 2.0.2), $K \models \exists \bar{Y} \bigwedge_{g \in S} g(\bar{Y}) \geq 0$ and so $W \neq \varnothing$. □

Remark 2.4.14. The hypotheses of Theorem 2.4.3 and Theorem 2.4.13 are of different nature. We do not know whether one of them is stronger than the other one.

Remark 2.4.15. In Theorem 2.4.13, the hypothesis that there is a T-convex proper differential ideal in $K\{\bar{X}\}$ is equivalent to say that there is a differential cone $\tilde{T} \supseteq T$ such that $\tilde{T} \cap -\tilde{T}$ is a proper (differential) ideal of $K\{\bar{X}\}$ (use Proposition 2.4.11).

Remark 2.4.16. The hypothesis that a T-convex proper differential ideal exists (in Theorem 2.4.13) is satisfied for the following examples.

Take $S := \{X\}$, since the sums of squares of K are exactly the non-negative elements of K (which is a semiring and denoted $K^{\geq 0}$), $T = K^{\geq 0}[X]$. The support of T is the zero ideal. It is a T-convex differential ideal.

If we take $S := \{X, X'\}$ then $T = K^{\geq 0}[X, X']$. The support of T is again the zero ideal.

By the usual tricks (see [39], Section 2.3), we get

Theorem 2.4.17 (Positivstellensatz)**.** *Let K be a model of CODF and S be a finite subset of $K\{\bar{X}\}$. We let T and W as in Theorem 2.4.13. Suppose that there is a T-convex proper differential ideal in $K\{\bar{X}\}$.*

For any $f \in K\{\bar{X}\}$, $f \geq 0$ on K iff there are $p, q \in T$ and $m \in \mathbb{N}$ such that $f^{2m} + p = qf$.

2.5 Schmüdgen's theorem

The notion of archimedean ordered field occurs in this section. An archimedean ordered field is an ordered field where there is no infinitesimal (nonzero) element. More precisely, an ordered field F is archimedean iff for any $u \in F$, if $0 < u$ then there is a rational number q such that $0 < q < u$. For instance, the field $\mathbb{R}$ of real numbers is archimedean.

2.5.1 Schmüdgen's theorem for the real field

Let S be a finite set of polynomials in $\mathbb{R}[\bar{X}]$.

Theorem 2.5.1 (Schmüdgen)**.** *If W_S is compact, then for any $f \in \mathbb{R}[\bar{X}]$,*

$$(\forall \bar{x} \in W_S, f(\bar{x}) > 0) \Rightarrow f \in T_S.$$

The proof of Schmüdgen's theorem uses the fact that $\mathbb{R}$ is a locally compact, real closed field. Moreover, any locally compact, formally real field is isomorphic to the field of real numbers (see the classification of locally compact fields in [72], Chapter 1, Theorem 5). Therefore the proof only works for that field.

Schmüdgen's theorem is in general false for arbitrary real closed fields. A counter-example is provided in [39], Example 6.3.3 for non-archimedean fields. Moreover, the compactness assumption on W_S as well as the assumption that $f > 0$ and not only $f \geq 0$ may not be removed since examples of non-negative polynomials which are not sums of squares are known (see for instance [59] for an explicit example and note that the existence of such a polynomial was already proved by D. Hilbert).

2.5.2 Endowing $\mathbb{R}$ with a structure of CODF

Since Schmüdgen's theorem is a result on the field $\mathbb{R}$ of real numbers, in order to get a similar result on a model of CODF, we are interested in endowing $\mathbb{R}$ with a derivation in such a way that $\mathbb{R}$ is a model of CODF.

C. Michaux showed in his thesis ([41]) that the theory CODF has countable archimedean models. We will use a similar method to prove the existence of the derivation on $\mathbb{R}$ that we are seeking.

We will need the following lemma which is a variation of Lemma 1.4.2, used by M. Singer in [65] to show the existence of a model of CODF containing a given model of ODF.

Lemma 2.5.2 ([41], Chapter 2, Lemma 2.3.4)**.** *Let K be a countable archimedean model of ODF. Let $f, g_1, \ldots, g_m \in K\{Y\}$ be such that for all $i \in \{1, \ldots, m\}$, $n := \operatorname{ord}(f) \geq \operatorname{ord}(g_i)$.*
If $K \models \exists A_0, \ldots, A_n(f^*(\bar{A}) = 0 \wedge s_f^*(\bar{A}) \neq 0 \wedge \bigwedge_{i=1}^m g_i^*(\bar{A}) > 0)$. Then there exists a countable archimedean model of ODF such that

$$L \models \exists Y(f(Y) = 0 \wedge s_f(Y) \neq 0 \wedge \bigwedge_{i=1}^m g_i(Y) > 0).$$

The proof of this lemma can be effortlessly generalised to any (countable or uncountable) archimedean model K of ODF such that the transcendence degree of $\mathbb{R}$ over K is greater than $|K|$.

In fact in his proof C. Michaux shows that for any $f, g_1, \ldots, g_m \in K\{Y\}$ and $\bar{c} := (c_0, \ldots, c_n) \in K$ such that $K \models (f^*(\bar{c}) = 0 \wedge s_f^*(\bar{c}) \neq 0 \wedge \bigwedge_{i=1}^m g_i^*(\bar{c}) > 0)$, there is a neighborhood V of 0 in $\mathbb{R}$ such that for any $t_0, \ldots, t_{n-1} \in V$, there is $t_n \in K(t_0, \ldots, t_{n-1})^{rc}$ such that

$$K(t_0, \ldots, t_n) \models (f^*(\bar{v}) = 0 \wedge s_f^*(\bar{v}) \neq 0 \wedge \bigwedge_{i=1}^m g_i^*(\bar{v}) > 0)$$

where $\bar{v} = (c_0 + t_0, \ldots, c_n + t_n)$. The t_i's ($i < n$) can be chosen algebraically independent over K. Consequently, t_n is transcendental over K and one can define $v_0' = v_1, \ldots, v_{n-1}' = v_n$ and one finally compute $v_n', v_n^{(2)}, \ldots$ from the identity $f^*(\bar{v}) = 0$ (using Lemma 1.2.8).

Theorem 2.5.3. *Let $\mathbb{R}$ be the field of real numbers, there exists a derivation $'$ on $\mathbb{R}$ such that $(\mathbb{R}, +, \cdot, -, ^{-1}, 0, 1, <, ')$ is a model of CODF.*

Proof. We will start with the field $\mathbb{Q}$ of rational numbers. We consider for all $k \in \mathbb{N}$ the set $T_k := \{r \in \mathbb{R} : \frac{1}{k+2} < |r| < \frac{1}{k+1}\}$. Let $B := \{b_\lambda : \lambda \in 2^{\aleph_0}\}$ be a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ such that for all $k \in \mathbb{N}$, $|B \cap T_k| = 2^{\aleph_0}$.

Remark. $2^{\aleph_0}$ has no bounded subset of cardinality $2^{\aleph_0}$.

One will now built by induction on $\alpha \in 2^{\aleph_0}$ a chain of subfields F_α of $\mathbb{R}$. In the end, we will get $\bigcup_{\alpha \in 2^{\aleph_0}} F_\alpha = \mathbb{R}$. We will put a derivation on the F_α's in such a way that $\bigcup_{\alpha \in 2^{\aleph_0}} F_\alpha$ is a model of CODF. Moreover, for any $\alpha \in 2^{\aleph_0}$, $F_{\alpha+1}$ will be generated as a field extension of F_α by a set of cardinality at most $|F_\alpha|$, hence we will have for any α, $|F_\alpha| < 2^{\aleph_0}$ and so the transcendence degree of $\mathbb{R}$ over F_α (i.e. the cardinality of $B \setminus F_\alpha$) is $2^{\aleph_0}$. Furthermore, for any $k \in \mathbb{N}$, $|(B \cap T_k) \setminus F_\alpha| = 2^{\aleph_0}$.

Let $F_0 := \mathbb{Q}$ endowed with the trivial derivation.

For any $\alpha \in 2^{\aleph_0}$. If $b_\alpha \in F_\alpha$, then we let $F_{\alpha+1} := F_\alpha$.

Otherwise we let $F_{\alpha,0} := F_\alpha(b_\alpha)$ and $b_\alpha' = 0$.

Then we enumerate all systems of differential (in)equalities $S_\lambda, \lambda \in |F_\alpha|$ of the shape

$$f(Y) = 0 \wedge s_f(Y) \neq 0 \wedge \bigwedge_{i=1}^m g_i(Y) > 0$$

where $f, g_1, \ldots, g_m \in F_\alpha\{Y\}$, $n = \mathrm{ord}(f) \geq \max\{\mathrm{ord}(g_i) : i \in \{1, \ldots, m\}\}$ and there are $c_0, \ldots, c_n \in K$ such that

$$K \models (f^*(\bar{c}) = 0 \wedge s_f^*(\bar{c}) \wedge \bigwedge_{i=1}^{m} g_i^*(\bar{c}) > 0).$$

For any $\lambda \in |F_\alpha|$**.** We take $t_0, \ldots, t_{n-1}$ in $|(B \cap T_k) \setminus F_\alpha|$ for a big enough k in order to get, using Lemma 2.5.2, a transcendental element t_n over F_α such that the system S_λ has a solution in $F_{\alpha,\lambda}(t_0, \ldots, t_n)$, putting a derivation on $F_{\alpha,\lambda}(t_0, \ldots, t_n)$ in the same way as in Lemma 2.5.2 (i.e, $t_0' = t_1, \ldots, t_{n-1} = t_n$ and $t_n^{(l)}$ following from the fact that t_0 is a solution of S_λ).

We let $F_{\alpha,\lambda+1} := F_{\alpha,\lambda}(t_0, \ldots, t_n)$.

If $\lambda \in |F_\alpha|$ **is limit.** We let $F_{\alpha,\lambda} := \bigcup_{\kappa \in \lambda} F_{\alpha,\kappa}$.

Then we let $F_{\alpha+1}$ to be the real closure of $\bigcup_{\lambda \in |F_\alpha|} F_{\alpha,\lambda}$.

For any limit ordinal $\alpha \in 2^{\aleph_0}$**.** We let $F_\alpha := \bigcup_{\beta \in \alpha} F_\beta$.

Note that by construction, for any $0 < \alpha < 2^{\aleph_0}$, the field F_α is real closed.
Finally, we consider $F := \bigcup_{\alpha \in 2^{\aleph_0}} F_\alpha$.
As $F \subseteq \mathbb{R}$ is real closed and $B \subset F$, $F = \mathbb{R}$.
By construction, it is clear that F is a model of CODF. □

2.5.3 Schmüdgen's theorem for differential fields

We endow the field of real numbers with the derivation provided by Theorem 2.5.3. Then $\mathbb{R}$ is a model of CODF.

The idea of our proof of an analog of Schmüdgen's theorem for that model of CODF is similar to the one of the proof of Theorem 2.4.3.

Let S be a finite subset of $\mathbb{R}\{\bar{X}\}$ where $\bar{X} := (X_1, \ldots, X_n)$.

Theorem 2.5.4. *If W_S^* is compact and there is an open set $O \subseteq \mathbb{R}^{n\cdot(d+1)}$ such that $O \subseteq W_S^* \subseteq \mathrm{cl}(O)$, where $d := \max_{g \in S} \mathrm{ord}(g)$. Then for any $f \in \mathbb{R}\{\bar{X}\}$ of order e and any rational number q*

$$(\forall \bar{x} \in W_S, f(\bar{x}) > q) \Rightarrow f \in T_E,$$

where $E := S \cup \{\pm X_i^{(j)} + r : i \in \{1, \ldots, n\}, j \in \{d+1, \ldots, e+1\}\}$ for a real number $r > 0$ (that may be chosen arbitrarily).

Proof. Let $e := \max\{d, \mathrm{ord}(f)\}$. As in the proof of Theorem 2.4.3, we need to embed W_S^* in $\mathbb{R}^{n\cdot(e+1)}$ but this time in such a way that the set we obtain is compact. So for an element $r \in \mathbb{R}$ such that $r > 0$, we let $W := W_S \times [-r, r]^{e-d}$ i.e. $W = W_E^*$ and $U := O\times]-r, r[^{e-d}$.

If $\forall \bar{x} \in W_S, f(\bar{x}) > q$ then $\forall \bar{x} \in U, f^*(\bar{x}) > q$. By the density assumption and the continuity of f^*, for all $\bar{x} \in W$, $f^*(\bar{x}) \geq q$ and so is positive on W and by Theorem 2.5.1, $f^* \in T_E^*$. Hence, $f \in T_E$. □

Remark 2.5.5. Since we are working with the usual topology of the reals (which is a topology making the polynomials continuous), W_S^* is always closed and so the compactness reduces to the boundedness. If W_S^* is bounded then W_S is "d-differentially" bounded, i.e. there is $r \in \mathbb{R}$ such that for any $\bar{x} := (x_1, \ldots, x_n) \in W_S$ for any $1 \leq i \leq n$ and $0 \leq j \leq d$, $|x_i^{(j)}| < r$. However, we have to be careful that the converse may fail to occur. For instance, take $S := \{X, -X, X'+1\}$, then $d = 1$, $W_S^* = \{0\} \times [-1, +\infty[$ even though $W_S := \{0\}$ is obviously 1-differentially bounded.

Chapter 3

Differential Galois Theory

Introduction

Strongly normal extensions have been introduced by E. Kolchin, generalising both Picard-Vessiot extensions and Weierstrass extensions [29]. In this chapter, we consider strongly normal extensions in the class of formally real differential fields. We use model theoretic methods like B. Poizat [47] and A. Pillay [45] in order to investigate differential Galois theory of formally real strongly normal extensions. More specifically the theories CODF and DCF_0 will play a central role, keeping in mind M. Singer's result that when M is a model of CODF and $i^2 = -1$, then $M(i)$ is a model of DCF_0 (see Theorem 1.4.4).

We start by a section on the special cases of Picard-Vessiot extensions and their differential Galois groups. Picard-Vessiot extensions will provide a wealth of examples and counter-examples. We also state the recent result on the existence of formally real Picard-Vessiot extensions (see Theorem 3.1.20).

Let K be a formally real differential field. Under the hypothesis that the field of constants C_K of K is real closed, we show that the differential Galois group $\mathrm{gal}(L/K)$ of a formally real, strongly normal extension L of K is isomorphic to a definable group G in C_K (in other words, it is isomorphic to a semialgebraic group). Moreover, if we denote $\mathscr{U}$ a sufficiently saturated model of CODF containing L and $\mathrm{Gal}(L/K)$ the group $\mathrm{gal}(\langle L, C_{\mathscr{U}}\rangle/\langle K, C_{\mathscr{U}}\rangle)$ then we get a group isomorphism $\eta : \mathrm{Gal}(L/K) \to G$ where G is definable in $C_{\mathscr{U}}$.

We give then examples of extensions of formally real differential fields which are strongly normal, namely Picard-Vessiot extensions and Weierstrass extensions of formally real differential fields. We describe explicitely their differential Galois groups.

Let E be an intermediate extension ($K \subseteq E \subseteq L$), we denote G_E the image by

η of the subgroup of the elements of $\mathrm{Gal}(L/K)$ fixing E. Under the assumption that L is a regular extension of K, we get that $E \mapsto G_E$ is an injective map from the set of intermediate extensions E of L/K such that L/E is regular into the set of definable subgroups of G.

As the elements of $\mathrm{Gal}(E/K)$ are not supposed to respect the order induced on E by the one of $\mathscr{U}$, they do not need to have an extension in $\mathrm{Gal}(L/K)$. Therefore, we do not get a 1-1 Galois correspondence like in the well-studied case of an algebraically closed field of constants (see for instance [45]). However, one may consider the subgroup $\mathrm{Aut}(L/K)$ of the elements of $\mathrm{Gal}(L/K)$ respecting the order of $\mathscr{U}$ (the increasing automorphisms). We show that if G_0 is a definable subgroup of G, then we can find a tuple $\bar{d}$ in the real closure of L in $\mathscr{U}$, such that $\mathrm{Aut}(L(\bar{d})/K(\bar{d}))$ is isomorphic to $G_0 \cap \eta(\mathrm{Aut}(L/K))$.

We end the chapter considering non-finitely generated extensions of formally real differential fields. The notion of strongly normal extension has been extended to non-finitely generated extensions by J.J. Kovacic in [32]. We show in that context that the differential Galois group is isomorphic to a subgroup of a projective limit of definable groups, in analogy to the non-formally real case considered in [32].

Part of this chapter is a joint paper with F. Point [9].

Conventions and notations

Let F be a differential field. Recall that basic formulas are atomic formulas and their negations. The set of all basic $\mathcal{L}_{df}(F)$-sentences true in F is the diagram of F and will be denoted by $\mathrm{Diag}(F)$. Note that a model of $\mathrm{Diag}(F)$ is $\mathcal{L}_{df}$-isomorphic to an $\mathcal{L}_{df}$-extension of F (it is an $\mathcal{L}_{df}$-structure which is not necessarily a differential field).

For any differential field M and any differential subfield E of F, we denote the set of $\mathcal{L}_{df}$-embeddings from F into M fixing E pointwise by $\mathrm{Hom}_E(F, M)$. By $\mathrm{Isom}_E(F, M)$, we denote the set of $\mathcal{L}_{df}$-isomorphisms from F onto M and fixing E pointwise.

Definition 3.0.1. Let F/E be a differential field extension. The differential Galois group of F/E is the group $\mathrm{Isom}_E(F, F)$ and is denoted $\mathrm{gal}(F/E)$.

Throughout this chapter, unless otherwise specified, by embedding (resp. isomorphism) we mean differential field embedding (resp. differential field isomorphism), i.e. $\mathcal{L}_{df}$-embedding (resp. $\mathcal{L}_{df}$-isomorphism). By definable, we mean $\mathcal{L}_{df}$-definable possibly over some parameters. Types are also considered in the language $\mathcal{L}_{df}$.

3.1 Picard-Vessiot Extensions

We define Picard-Vessiot extensions and consider their differential Galois groups. We give in section 3.1.4 an example of Picard-Vessiot extensions L/K and K/R such that L is formally real and L is not a Picard-Vessiot extension of R. Finally, in section 3.1.5, we briefly review recent works on the question of the existence of Picard-Vessiot extensions of formally real fields.

From section 3.1.1 to section 3.1.4, the results are mostly taken from chapter 2 and 3 of [35], possibly with a few minor variations related to the fact that in our context we do not assume that the fields of constants are algebraically closed.

3.1.1 Generalities on linear differential equations and Picard-Vessiot extensions

Let K be a differential field. Let us consider a linear differential homogeneous equation $\mathscr{L}(Y) = 0$ defined on K, i.e. $\mathscr{L}(Y) := a_nY^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1Y^{(1)} + a_0Y$ with $a_n \in K^\times$ and $a_{n-1}, \ldots, a_0 \in K$. We will say that n is the order of $\mathscr{L}$.

Let L be an extension of K. Since the derivation is C_L-linear, the set of all solutions of $\mathscr{L}(Y) = 0$ in L is a C_L-vector space. In order to study the dimension of the vector space of solutions, we need to introduce the Wronskian of $u_1, \ldots, u_n$, denoted $wr(u_1, \ldots u_n)$, which is by definition

$$wr(u_1, \ldots, u_n) := \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

Lemma 3.1.1 ([35], Proposition 2.8)**.** *Let $u_1, \ldots, u_n \in L$, then $u_1, \ldots, u_n$ are linearly independent over C_L iff $wr(u_1, \ldots, u_n) \neq 0$.*

Moreover, as $wr(u_1, \ldots, u_n)$ only depends on $u_1, \ldots, u_n$ and not on the extension L of K where the u_i's lie, we have the following lemma:

Lemma 3.1.2. *If L and F are extensions of K such that $u_1, \ldots, u_n \in K$, then $u_1, \ldots, u_n$ are linearly independent over C_L iff they are linearly independent over C_F.*

Lemma 3.1.3 ([35], Theorem 2.9)**.** *If elements $u_1, \ldots, u_n, u_{n+1}$ of a differential field extension of K are solutions of $\mathscr{L}(Y) = 0$, then they are linearly dependent* *over the field of constants. In other words, the dimension over the constants of the vector space of solutions of $\mathscr{L}(Y) = 0$ in any extension of K is at most n.*

We will say that $u_1, \ldots, u_n$ is a fundamental system of solutions of the equation $\mathscr{L}(Y) = 0$ if $u_1, \ldots, u_n$ are linearly independent over C_L and for all $i \in \{1, \ldots, n\}$, $\mathscr{L}(u_i) = 0$.

Therefore whenever K has a fundamental system of solutions of $\mathscr{L}(Y) = 0$ and L is an extension of K, the vector space of solutions of $\mathscr{L}(Y) = 0$ in L only depends on C_L and not on L itself.

We now state the definition of Picard-Vessiot extensions. We still assume $\mathscr{L}(Y) = 0$ is a linear differential homogeneous equation with coefficients in K.

Definition 3.1.4. An extension L of K is said to be a Picard-Vessiot extension (PV extension for short) for the equation $\mathscr{L}(Y) = 0$ of order n iff

1. $C_K = C_L$;
2. $L = K\langle u_1, \ldots, u_n\rangle$ where $u_1, \ldots, u_n$ is a fundamental system of solutions of the equation $\mathscr{L}(Y) = 0$.

Lemma 3.1.5 ([35], Chapter 3)**.** *Let L be an extension of K and F be an extension of L such that L contains a fundamental system of solutions of $\mathscr{L}(Y) = 0$, if F contains a solution of $\mathscr{L}(Y) = 0$ which does not belong to L, then F contains a constant which does not belong to L.*

Proof. Let us denote $\bar{u} := (u_1, \ldots, u_n)$, a fundamental system of solutions of $\mathscr{L}(Y) = 0$ in L. Suppose that $C_F = C_L$ then as $\bar{u}$ are linearly independent over C_L, they are still linearly independent over C_F. So if $x \in F$ and $\mathscr{L}(x) = 0$, then x depends linearly on $\bar{u}$ over $C_F = C_L$, which implies that $x \in L$. It contradicts the hypothesis that F contains a solution of $\mathscr{L}(Y) = 0$ which is not in L. □

Therefore the condition $C_K = C_L$ required in the definition of PV extension implies the minimality of PV extensions:

Corollary 3.1.6 ([35], Chapter 3)**.** *PV extensions of K for $\mathscr{L}(Y) = 0$ are minimal among the extensions containing a fundamental system of solutions of $\mathscr{L}(Y) = 0$.*

Under some hypotheses on the field of constants, one shows the existence of PV extensions:

Theorem 3.1.7 ([35], Theorem 3.4)**.** *Suppose C_K is algebraically closed. For any linear homogeneous equation $\mathscr{L}(Y) = 0$ with coefficients in K, a PV extension of K does exist.*

We will see in section 3.1.5 that the existence of PV extensions when K is formally real and C_K is real closed is a recent result.

3.1.2 On the differential Galois group of a Picard-Vessiot extension

Theorem 3.1.8 ([26], Theorem 5.5.). *Let L be a Picard-Vessiot extension of K then* $\mathrm{gal}(L/K)$ *is isomorphic to a linear group over C_K.*

In the literature, number of proofs of this theorem use the fact that C_K is algebraically closed. The proof of [26] has the advantage to apply to any Picard-Vessiot extension (without particular assumption on the field of constants). Note that by a linear group over C_K, we mean the points in C_K of a linear algebraic group over C_K.

Lemma 3.1.9 (See [35], Lemma 3.19). *Let L/K be a differential field extension and let $x_1, \ldots, x_n \in C_L$ be algebraically dependent over K. Then $x_1, \ldots, x_n$ are algebraically dependent over C_K.*

Corollary 3.1.10. *Let L/K be a differential field extension. Whenever a constant $x \in L$ is algebraic over K, x is algebraic over C_K.*

Definition 3.1.11. Let L/K be a differential field extension and $\Sigma \subseteq \mathrm{gal}(L/K)$. Then L/K is Σ-normal iff for all $u \in L \backslash K$, there exists $\sigma \in \Sigma$ such that $\sigma(u) \neq u$.

A characterisation of PV extensions with algebraically closed fields of constants is provided by Proposition 3.9 of [35]. Using the same arguments without assumption on the fields of constants, we obtain a sufficient condition for an extension to be Picard-Vessiot.

Proposition 3.1.12. *Let L/K be a differential field extension. Suppose that the following properties hold:*

1. *$C_K = C_L$,*
2. *There is a C_K-vector space V of finite dimension such that $L = K\langle V \rangle$,*
3. *There is a subgroup Σ of* $\mathrm{gal}(L/K)$*, leaving V invariant and such that L/K is Σ-normal.*

Then L/K is a Picard-Vessiot extension.

Proof. Suppose L/K satisfies the above conditions. Let $y_1, \ldots, y_n$ be a basis of the C_K-vector space V. By Lemma 3.1.1, $wr(\bar{y}) \neq 0$. Hence one may consider the linear differential operator $\mathscr{L}$ defined by

$$\mathscr{L}(Y) := \frac{wr(Y, \bar{y})}{wr(\bar{y})} \in L\{Y\}.$$

Let $\mathscr{L}(Y) := Y^{(n)} + b_{n-1}Y^{(n-1)} + \cdots + b_0 Y^{(0)}$. Before showing that L is a PV extension of K for $\mathscr{L}(Y) = 0$, we need to show that $\mathscr{L} \in K\{Y\}$, equivalently, for any $i \in \{0, \ldots, n-1\}$, any $\sigma \in \Sigma$, $\sigma(b_i) = b_i$.

By Example 2.5 from [35], for all i, $b_i := a_i/a_n$, where $a_i = (-1)^i. \det M_i$ and

$$M_i := \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(i-1)} & y_2^{(i-1)} & \cdots & y_n^{(i-1)} \\ y_1^{(i+1)} & y_2^{(i+1)} & \cdots & y_n^{(i+1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

Let $\sigma \in \Sigma$, since V is σ invariant, there are $c_{kj} \in C_K$ such that $\sigma(y_k) = \sum_{j=1}^n c_{jk} y_j$. Using Proposition 2.6 from [35] with $z_k = \sigma(y_k)$, we get

$$\det(\sigma M_i) = \det(c_{jk}). \det(M_i).$$

Therefore

$$\sigma(b_i) = \frac{\sigma(a_i)}{\sigma(a_n)} = (-1)^{i-n} \frac{\sigma(\det(M_i))}{\sigma(\det(M_n))}$$
$$= (-1)^{i-n}. \frac{\det(\sigma(M_i))}{\det(\sigma(M_n))} = (-1)^{i-n}. \frac{\det(M_i)}{\det(M_n)} = \frac{a_i}{a_n} = b_i.$$

Clearly, for any $i \in \{1, \ldots, n\}$, $wr(y_i, \bar{y}) = 0$. So $\mathscr{L}(y_i) = 0$ and as the order of $\mathscr{L}(Y)$ is n then $y_1, \ldots, y_n$ is a fundamental system of solutions. Thus $L = K\langle y_1, \ldots, y_n \rangle$ is a Picard-Vessiot extension of K for the linear equation $\mathscr{L}(Y) = 0$. □

Assuming that C_K is algebraically closed, the converse of Proposition 3.1.12 holds. However as shown in the following example, it is in general false even under the assumption that C_K is real closed and L is formally real.

Example 3.1.13. Let $K := \mathbb{R}(t)$ where t is transcendental over $\mathbb{R}$, $t' = 1$ and for all $r \in \mathbb{R}$, $r' = 0$. Let $L := \mathbb{R}(u)$ where $u^3 = t$. Then L is a Picard Vessiot extension of K for the equation $Y' - \frac{1}{3t}Y = 0$ and L/K is not gal(L/K)-normal.

3.1.3 Picard-Vessiot extensions with relatively algebraically closed fields of constants

Corollary 3.1.14. *Let K be a differential field and L be a finite Galois extension of K such that C_K is relatively algebraically closed in L. Then L is a Picard-Vessiot extension of K.*

Proof. By Lemma 1.2.11, the derivation of K extends in a unique way to a derivation of L. Any field automorphism of L fixing K pointwise is an element of $\mathrm{gal}(L/K)$.

We make use of Proposition 3.1.12 to verify that L/K is Picard-Vessiot. Since L/K is a Galois extension, L is a splitting field of a polynomial $p(X) \in K[X]$. Let R be the set of roots of $p(X)$ in L, then R is left invariant by $\mathrm{gal}(L/K)$ and $L = K\langle R\rangle$. Let V be the C_K-vector space generated by the roots of $p(X)$, then $L = K\langle V\rangle$ and V is left invariant by $\mathrm{gal}(L/K)$. Moreover, as L/K is a Galois extension, the subfield of L fixed by $\mathrm{gal}(L/K)$ is K.

It remains to show that $C_K = C_L$. Let x be a constant of L, then x is algebraic over K. So x is algebraic over C_K. Hence $x \in C_K$ because C_K is relatively algebraically closed in L. □

3.1.4 Examples of Picard-Vessiot extensions with real closed fields of constants

If K_1 is a differential field, K_2 is a Picard-Vessiot extension of K_1 and K_3 is Picard-Vessiot extension of K_2, then K_3 does not need to be a Picard-Vessiot extension of K_1. An example of this phenomenon is given in [35], Example 3.33. In that example, the differential fields are not formally real, since the fields of constants are assumed to be algebraically closed. The purpose of this section is to provide an example showing that inside the class of formally real fields, Picard-Vessiot extensions are not preserved by extension. More precisely, we will show that there exists formally real differential fields $R \subset K \subset L$ such that K is Picard-Vessiot extension of R, L is a Picard-Vessiot extension of K and L is not contained in any Picard-Vessiot extension of K.

We will consider differential field extensions of formally real fields whose constant fields are real closed. Therefore, Corollary 3.1.14 may apply.

In the sequel, it may be useful to the reader to keep in mind the obvious fact that if L is a PV extension of K (where L is formally real or not) and $i^2 = -1$ then $L(i)$ is a PV extension of $K(i)$.

From now to the end of this section, we fix the following. Let R be a real closed field, $t_1, \dots, t_n$ be algebraically independent over R and $\alpha_1, \dots, \alpha_n \in R$ be linearly independent over the rationals. We endow R with the trivial derivation and we let $t_j' := \alpha_j t_j$. As the t_j's are algebraically independent, it induces a (unique) derivation on $K := R(t_1, \dots, t_n)$ (see Lemma 1.2.10).

We will follow the strategy of the proof given in [35] where the following Lemma is shown for an algebraically closed field which is taken here to be $R(i)$ where i is a square root of -1.

Lemma 3.1.15 ([35], Example 3.33)**.** *Suppose $n \geq 3$. Let M be the splitting field of $X^n + t_1X^{n-1} + \cdots + t_n \in K(i)[X]$. Then $M/K(i)$ and $K(i)/R(i)$ are Picard-Vessiot extensions and M is not contained in any Picard-Vessiot extension of $R(i)$.*

Lemma 3.1.16. *$C_K = C_R$.*

Proof. As it is noted in [35], page 11, it would be enough to assume that C_R is relatively algebraically closed in K, which is the case here because $t_1, \ldots, t_n$ are algebraically independent over R. The proof is exactly similar to Proposition 1.20 from [35]. □

Lemma 3.1.17. *Let L be the splitting field for the polynomial $X^n + t_1X^{n-1} + \cdots + t_n \in K[X]$. Then L is formally real and the extension L/K is a Picard-Vessiot extension.*

Proof. Let $L := K(r_1, \ldots, r_n)$ where $X^n + t_1X^{n-1} + \cdots + t_n = \prod_{j=1}^{n}(X - r_j)$. Therefore for all $j \in \{1, \ldots, n\}$, $t_j \in \mathbb{Q}(r_1, \ldots, r_n)$. So $L = R(r_1, \ldots, r_n)$. Since $t_1, \ldots, t_n \in L$ are algebraically independent over R, $r_1, \ldots, r_n$ are algebraically independent over R as well. Hence, -1 is not a sum of squares of L. By Lemma 1.3.5, L is formally real.

Since $C_K = C_R$, then C_K is real closed and so C_K is relatively algebraically closed in L. By Corollary 3.1.14, L/K is a Picard-Vessiot extension. □

Lemma 3.1.18. *The extension K/R is a Picard-Vessiot extension.*

Proof. Let $a_1, \ldots, a_n$ be such that $\prod_{j=1}^{n}(X - \alpha_j) = X^n + a_1X^{n-1} + \cdots + a_n$. Let $\mathscr{L}(Y) := Y^{(n)} + a_1Y^{(n-1)} + \cdots + a_nY$. Then letting D_j be the linear operator given by $D_j(Y) := Y' - \alpha_jY$, we get $\mathscr{L} = D_1 \circ D_2 \circ \cdots \circ D_n = D_{\sigma(1)} \circ D_{\sigma(2)} \circ \cdots \circ D_{\sigma(n)}$ for any $\sigma \in S_n$.

For any $j \in \{1, \ldots, n\}$, $D_j(t_j) = 0$, so t_j is a solution of $\mathscr{L}(Y) = 0$. The t_j's are R-linearly independent and $\mathscr{L}$ has order n, so $t_1, \ldots, t_n$ is a fundamental system of solutions for $\mathscr{L}$. By Lemma 3.1.16, $C_R = C_K$. Hence K/R is a Picard-Vessiot extension. □

Lemma 3.1.19. *Suppose $n \geq 3$. Let $L := R(y_1, \ldots, y_n)$ be the splitting field of $X^n + t_1X^{n-1} + \cdots + t_n$. Then L is formally real, L/K and K/R are Picard-Vessiot and L is not contained in any Picard-Vessiot extension of R. Moreover if i is a square root of -1 then $L(i)$ is not contained in any Picard-Vessiot extension of $R(i)$.*

Proof. By Lemma 3.1.14 and Lemma 3.1.18, it is the fact that L is formally real, L/K and K/R are Picard-Vessiot.

Suppose L is contained in a Picard-Vessiot extension M of R. Then $M(i)$ is a Picard-Vessiot extension of $R(i)$ which is algebraically closed and $M(i)$ contains $L(i)$. A contradiction with Lemma 3.1.15 whenever $n \geq 3$. □

3.1.5 On the existence of formally real Picard-Vessiot extensions

Theorem 3.1.20 is a recent result on the existence of a formally real PV extensions. Two proofs are known. The proof given in [11] is quite complex and uses Tannakian categories. The proof of [25] uses model theory. We state the theorem in the case of formally real fields even though, in the two references, it was established and formulated for a larger class of fields.

Suppose K is formally real and C_K is real closed. Let $\mathscr{L}(Y) = 0$ be a linear differential homogeneous equation where $\mathscr{L}(Y) \in K\{Y\}$.

Theorem 3.1.20 (See Theorem 1.2 of [11] and Theorem 1.6 of [25])**.** *There is a PV extension L of K for $\mathscr{L}(Y) = 0$ such that L is formally real.*

Before the proofs of [11] and [25], there were several attempts to prove Theorem 3.1.20. As far as we know, two uncorrect or uncomplete proofs were published. The first proof is in C. Michaux's paper [40]. More recently, T. Crespo, Z. Hajto and E. Sowa showed the result as a corollary of Theorem 3.2 of [10]. A gap in the proof of that theorem was pointed out by M. Aschenbrenner. Since there is no mention of this gap in the literature (as far as we know), we outline below the proof of Theorem 3.2 proposed in [10] and we notice the gap. We then write down Aschenbrenner's argument.

Note that in [10], they use Kolchin's vocabulary (constrained element and constrained closure), whose relation with the notions of isolated type and differential closure is explained in section 1.2.2, page 12. However the notion of constrained extension defined in [10] is in general not equivalent to Kolchin's original one but assuming that $C_K \neq K$, the equivalence holds. We rephrase everything with our terminology.

For an ordered differential field M, we denote $\sigma : M(i) \to M(i)$ the conjugation, i.e, the differential field automorphism such that $\sigma\restriction_M$ is the identity and $\sigma(i) = -i$.

Later, we will make use of the following proposition:

Proposition 3.1.21 (Corollary 2 of section 3 of [31])**.** *Let $E \subseteq F$ be differential fields. Suppose F is a model of DCF_0. If for any element u of F such that $\mathrm{tp}^F(u/E)$ is isolated, it holds that u belongs to E, then E is a model of DCF_0.*

Statement of Theorem 3.2 of [10]. *Let K be a formally real differential field. There exists a formally real differential extension L of K such that $L(i)$ is differentially closed and for any element u of $L(i)$, it holds that* $\mathrm{tp}^{L(i)}(u/K(i))$ *is isolated.*

We review now the proof of Theorem 3.2 of [10].

Take a model M of CODF containing K. Then by Theorem 1.4.4, $M(i)$ is a model of DCF_0. By Zorn Lemma there is a maximal differential field L such that $K \subseteq L \subseteq M$ and for all $u \in E := L(i)$, $\mathrm{tp}^{M(i)}(u/K(i))$ is isolated.

It remains to show that E is a model of DCF_0.

By Proposition 3.1.21, it is sufficient to show that any element $v \in M(i)$ having an isolated type over E belongs to E.

It is the fact that $\sigma(E) = E$. So $\mathrm{tp}^{M(i)}(\sigma(v)/E)$ is isolated (if $\mathrm{tp}(v/E)$ is isolated by ψ, then $\mathrm{tp}(\sigma(v)/\sigma(E))$ is isolated by the formula $\sigma(\psi)$).

It is claimed that any element of $E\langle v, \sigma(v)\rangle$ has an isolated type over $K(i)$. Since $K \subseteq L \subseteq E\langle v, \sigma(v)\rangle \cap M \subseteq M$, the maximality of L implies that $v \in E$.

Aschenbrenner's example shows the fact that any element of $E\langle v, \sigma(v)\rangle$ has an isolated type over $K(i)$ does not hold in general.

Below, all types are considered in a model of DCF_0.

Example 3.1.22 (Aschenbrenner)**.** There exist a formally real differential field K and an element v of an extension of K such that

- v has an isolated type over $K(i)$
- $K(i)\langle v, \sigma(v)\rangle$ has an element whose type is not isolated over $K(i)$.

Proof. We take $K := \mathbb{R}(t)^{rc}$, a real closure of $\mathbb{R}(t)$ endowed with the trivial derivation on $\mathbb{R}$ and such that $t' = 1$. Since K is formally real, K may be endowed with an order.

Let M be a $|K|^+$-saturated model of CODF such that $K \subset M$. We will construct the required element $v \in M(i)$.

There is $c \in M$ such that $c' = 0$, $c \notin K$ and $0 < c < 1$ (it follows from the axioms of CODF and $|K|^+$-saturation of M). Let $d \in M$ be such that $c^2 + d^2 = 1$. As d is algebraic over C_M then $d \in C_M$ (see Theorem 1.2.1). Let $z := c + id$. It is clear that $z' = 0$ and $z \notin K(i)$.

There is (by the axioms of CODF again) an element $a \in M$ such that $a' = a$ and $a \neq 0$. Moreover, there is a unique $b \in M(i)$ such that $a + bi = z(a - bi)$. As $|z| = 1$, then $b \in M$ and $b' = b$.

Let $v := a + bi$. Note that $v \neq 0$ and $v' = v$.

- The type of v over $K(i)$ is isolated by the formula

$$\psi(X) := (X' = X) \wedge (X \neq 0).$$

To see this, let u be a realisation of $\psi(X)$ (in any extension of $K(i)$), u does not belong to $K(i)$. Firstly, since $K(i)$ is algebraically closed, u is transcendental over $K(i)$. Secondly, it follows from quantifier elimination in algebraically closed fields that all transcendental elements have the same $\mathcal{L}_{\text{fields}}$-type. Therefore $\psi(X) \vdash \text{tp}_{\mathcal{L}_{\text{fields}}}(u/K(i))$. Moreover,

$$\big(\{X = X'\} \cup \text{tp}_{\mathcal{L}_{\text{fields}}}(u/K(i))\big) \vdash \text{tp}(u/K(i)).$$

So $\psi(X) \vdash \text{tp}(u/K(i))$.

- $z \in K(i)\langle v, \sigma(v)\rangle$ and z is transcendental over the algebraically closed field $K(i)$. So by Corollary 1.2.26, the type of z is not isolated over $K(i)$.

□

By Proposition 1 of section 2 of [31], if F is a differential field and $\text{tp}(\bar{a}/F)$ is isolated then any tuple of elements of $F\langle\bar{a}\rangle$ has an isolated type over F. So in the example above, the fact that $\text{tp}(z/K(i))$ is not isolated implies that $\text{tp}(v, \sigma(v)/K(i))$ is not isolated, even though $\text{tp}(v/K(i))$ and $\text{tp}(\sigma(v)/K(i))$ are both isolated.

3.2 Strongly normal extensions

3.2.1 Definitions and framework

The notion of strongly normal extension has been defined and studied by E. Kolchin in [29] and then in a more general setting in [30]. He works inside a differential field extension $\mathcal{M}$ of K which is universal over K, in the sense that every finitely generated differential field extension of K may be embedded in $\mathcal{M}$. Here, $\mathcal{M}$ will be chosen to be a saturated model of DCF_0 of cardinality $\kappa > |K|$. Let $L \subseteq \mathcal{M}$ be a differential field extension of K and $\tau \in \text{Hom}_K(L, \mathcal{M})$, we say that τ is strong if τ is the identity on C_L and if $\langle L, C_{\mathcal{M}}\rangle = \langle \tau(L), C_{\mathcal{M}}\rangle$. An extension L of K is called strongly normal if L is a finitely generated extension of K and any element τ of $\text{Hom}_K(L, \mathcal{M})$ is strong (see [30], page 393).

Note that since $\mathcal{M}$ eliminates quantifiers, the elements of $\text{Hom}_K(L, \mathcal{M})$ are partial elementary maps in $\mathcal{M}$ and since $\mathcal{M}$ is saturated and $|\mathcal{M}| > |L|$, by Corollary 1.1.13, any element of $\text{Hom}_K(L, \mathcal{M})$ extends to an element of $\text{gal}(\mathcal{M}/K)$. Moreover, any element of $\text{gal}(\mathcal{M}/K)$ restricts to an element of $\text{Hom}_K(L, \mathcal{M})$. So the definition of strongly normal extension may be formulated with $\tau \in \text{gal}(\mathcal{M}/K)$ instead of $\tau \in \text{Hom}_K(L, \mathcal{M})$.

By Proposition 9 from Chapter 6, section 3 of [30], if L is a strongly normal extension of K, then $C_K = C_L$.

Now we define a notion of normality which is relative to any subset of $\mathrm{Hom}_K(L, \mathcal{M})$ and extending Definition 3.1.11:

Definition 3.2.1. Let L/K be a differential field extension and Σ be any subset of $\mathrm{Hom}_K(L, \mathcal{M})$. We say that L/K is Σ-normal iff for any $x \in L \setminus K$ there is $\sigma \in \Sigma$ such that $\sigma(x) \neq x$.

The relationship between strongly normal and normal extensions is as follows. E. Kolchin shows that if L is a strongly normal extension of K, then L/K is $\mathrm{Hom}_K(L, \mathcal{M})$-normal (see [30], Theorem 3, section 4, Chapter 6). In [29], under the assumption that C_K is algebraically closed, he shows that L/K is $\mathrm{gal}(L/K)$-normal (see [29], Proposition 2, Chapter III). Hence in some subsequent works by A. Pillay and J.J. Kovacic, one considers strongly normal extensions of K when C_K is algebraically closed. In our context of formally real differential fields, the corresponding natural hypothesis is that C_K is real closed.

Before giving the framework for our study of strongly normal extensions, we state the following result of J.J. Kovacic which will be useful in section 3.5.

Proposition 3.2.2 ([33], Proposition 11.4)**.** *Let E be a differential field. We assume that C_E is algebraically closed. Suppose that F_1 is a Picard-Vessiot extension of E, and F_2 is a Picard-Vessiot of F_1. If F_2 is contained in a strongly normal extension of E, then it is contained in a Picard-Vessiot extension of E.*

We fix now our framework and then state the definition of strongly normal extension we will adopt.

We fix once for all in this chapter a formally real differential field K. For any order of K, one view K as an $\mathcal{L}_{odf}$-structure which is a model of ODF. So there exists a model M of CODF containing K as a (ordered) differential subfield. We may and will view M as an $\mathcal{L}_{df}$-structure (keeping in mind that the order of M is $\mathcal{L}_{df}$-definable). We fix an $\mathcal{L}_{df}$-saturated model $\mathscr{U}$ of CODF which has cardinality $\kappa > |K|$ and which contains K as a differential subfield. We fix a square root of -1 in $\mathscr{U}$ and denote it i. We will use the fact that $\mathscr{U}(i)$ (which is a model of DCF_0 by Theorem 1.4.4) is also saturated. The role of $\mathcal{M}$ in the definition of strongly normal extension will be played by $\mathscr{U}(i)$. So $C_{\mathscr{U}}(i) = C_{\mathscr{U}(i)} = C_{\mathcal{M}}$.

Definition 3.2.3. A differential field extension L/K is strongly normal iff

(1) the fields of constants C_K and C_L are equal and real closed;

(2) L/K is finitely generated;

(3) for any $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$, $\langle L, C_{\mathscr{U}}(i)\rangle = \langle \sigma(L), C_{\mathscr{U}}(i)\rangle$.

We will get that if L/K is strongly normal and K is relatively algebraically closed in L, then L is $\mathrm{Hom}_K(L, \mathscr{U})$-normal (see section 3.3.1).

Remark 3.2.4. Property (3) of definition 3.2.3 does not depend on the choice of $\mathscr{U}$. Actually, letting $L = K\langle \bar{a}\rangle$, this is equivalent to say that the following partial type $p(\bar{X})$, with $\bar{X}$ of the same length as $\bar{a}$, is not realised in $\mathscr{U}(i)$:

$$p(\bar{X}) := \mathrm{Diag}(L) \cup \mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) \cup s(\bar{X}, \bar{a})$$

where

$$s(\bar{X}, \bar{a}) := \left\{ \forall \bar{C}_1, \bar{C}_2 \left((\bar{C}_1' = 0 \wedge \bar{C}_2' = 0) \rightarrow \bar{a} \neq \frac{p_1(\bar{X}, \bar{C}_1)}{p_2(\bar{X}, \bar{C}_2)} \right) \right.$$
$$\left. \text{where } p_1, p_2 \in K\{\bar{X}, Y_1, Y_2, \dots\} \right\}.$$

As mentioned above, whenever M is a model of DCF_0 containing a copy of L, $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^M(\bar{a}/K)$. Moreover, $p(\bar{X})$ is realised in $\mathscr{U}(i)$ iff it is realised in any $|K|^+$-saturated model of DCF_0.

Remark 3.2.5. Let $\hat{F}$ be a differential closure of F. We will make use of the fact that $\hat{F}$ (which is a prime model of DCF_0 over F) is atomic over F (see Lemma 1.1.17), i.e. the $\mathcal{L}_{df}$-type over F of any tuple of elements of $\hat{F}$ is isolated by a (quantifier-free) $\mathcal{L}_{df}$-formula with parameters in F (see Definition 1.1.16). This implies in particular that $C_{\hat{F}}$ is the algebraic closure of C_F (see [37], Lemma 2.11). Conversely E. Kolchin showed that for a model M of DCF_0 containing F and $\bar{a} \in M$, if $\mathrm{tp}^M_{\mathcal{L}_{df}}(\bar{a}/F)$ is isolated then $\bar{a}$ belongs to some differential closure of F (see Proposition 1 in Section 2 of [31] and Corollary 2 in Section 7 of [31]).

Fact 3.2.6 ([31], Theorem 3, Section 9)**.** *Let L/K be a strongly normal extension, L is contained in a differential closure of K.*

Corollary 3.2.7. *If $L := K\langle \bar{a}\rangle$ is strongly normal, then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ is isolated.*

Proof. Follows from Fact 3.2.6 and Remark 3.2.5. □

Remark 3.2.8. Let $\bar{a}, \bar{b}$ be finite tuples of elements of $\mathscr{U}(i)$. Since DCF_0 admits quantifier elimination in $\mathcal{L}_{df}$, if there is an element of $\mathrm{Isom}_K(K\langle \bar{a}\rangle, K\langle \bar{b}\rangle)$ sending $\bar{a}$ to $\bar{b}$, we have that $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$ (see the notion of partial elementary map: Definition 1.1.7).

Moreover, since $\mathscr{U}(i)$ is saturated then it has the following homogeneity property. If $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$, then there is an $\mathcal{L}_{df}$-automorphism of $\mathscr{U}(i)$ fixing K and sending $\bar{a}$ to $\bar{b}$ (see Corollary 1.1.13).

Then it follows that when L is finitely generated over K, any element of $\mathrm{Hom}_K(L, \mathscr{U}(i))$ extends to an automorphism of $\mathscr{U}(i)$ fixing K. If L is a strongly normal extension of K, then any such automorphism restricts to an automorphism of $\langle L, C_{\mathscr{U}(i)}\rangle$.

3.2.2 The definability of the differential Galois group

From now on, we always suppose that $L \subset \mathscr{U}$, so L is formally real as well.

Lemma 3.2.9. *Let L be a strongly normal extension of K. Let σ be an embedding from L into $\mathscr{U}$ fixing K, then $\langle \sigma(L), C_{\mathscr{U}}\rangle = \langle L, C_{\mathscr{U}}\rangle$.*

Proof. By Remark 3.2.8, we can extend σ to an automorphism of $\mathscr{U}(i)$. As L/K is strongly normal, $\langle \sigma(L), C_{\mathscr{U}}(i)\rangle = \langle L, C_{\mathscr{U}}(i)\rangle$.

Moreover, as $L \subset \mathscr{U}$ and $\sigma(L) \subset \mathscr{U}$, $\langle L, C_{\mathscr{U}}(i)\rangle$ is an extension of degree 2 of $\langle L, C_{\mathscr{U}}\rangle$ and $\langle \sigma(L), C_{\mathscr{U}}(i)\rangle$ is an extension of degree 2 of $\langle \sigma(L), C_{\mathscr{U}}\rangle$. Hence $\langle L, C_{\mathscr{U}}(i)\rangle \cap \mathscr{U} = \langle L, C_{\mathscr{U}}\rangle$ and $\langle \sigma(L), C_{\mathscr{U}}(i)\rangle \cap \mathscr{U} = \langle \sigma(L), C_{\mathscr{U}}\rangle$.

So $\langle L, C_{\mathscr{U}}\rangle = \langle \sigma(L), C_{\mathscr{U}}\rangle$. □

Lemma 3.2.10. *Let $L = K\langle \bar{a}\rangle$ be a strongly normal extension of K and $\hat{L}$ be a differential closure of L in $\mathscr{U}(i)$. Let $\bar{b} \in \hat{L} \cap \mathscr{U}$ such that $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$, then $L = K\langle \bar{b}\rangle$.*

Proof. By Remark 3.2.8, there is $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$ such that $\sigma(\bar{a}) = \bar{b}$. By definition 3.2.3, $\bar{a} \in \langle K, \bar{b}, C_{\mathscr{U}}(i)\rangle$. So there are $p_1,\ p_2 \in K\{\bar{X}\}$ and $\bar{c} \in C_{\mathscr{U}}(i)$ such that $\bar{a} = \frac{p_1(\bar{b},\bar{c})}{p_2(\bar{b},\bar{c})}$. Recall that $C_{\mathscr{U}}(i) = C_{\mathscr{U}(i)}$.

By model completeness of DCF_0, one can find $\bar{d} \in C_{\hat{L}}$ such that $\bar{a} = \frac{p_1(\bar{b},\bar{d})}{p_2(\bar{b},\bar{d})}$. Finally, as $C_{\hat{L}} = C_L(i) = C_K(i)$, we get $\bar{a} \in \langle K, \bar{b}, C_K(i)\rangle \cap \mathscr{U} = K\langle \bar{b}\rangle$ (the last set equality comes from the fact that $\langle K, \bar{b}, C_K(i)\rangle$ is an algebraic extension of degree 2 of $K\langle \bar{b}\rangle$).

By interchanging $\bar{b}$ and $\bar{a}$, we get $\bar{b} \in \langle K, \bar{a}, C_{\mathscr{U}}(i)\rangle$ and then the same argument leads to $\bar{b} \in K\langle \bar{a}\rangle$. □

By Corollary 3.2.7, if $L = K\langle \bar{a}\rangle$ is a strongly normal extension of K, then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ is isolated by a formula ψ. As DCF_0 has quantifier elimination, we may choose ψ to be quantifier-free.

If ψ is a quantifier-free formula and $\bar{b}$ is any tuple of $\mathscr{U}(i)$, then $\mathscr{U}(i) \models \psi(\bar{b})$ iff $\psi(\bar{b})$ is true in any differential subfield of $\mathscr{U}(i)$ containing $\bar{b}$. So we will write $\psi(\bar{b})$ instead of $\mathscr{U}(i) \models \psi(\bar{b})$.

Lemma 3.2.11. *Let $L = K\langle \bar{a} \rangle$ be a strongly normal extension of K, ψ be a quantifier-free formula that isolates $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ and $\hat{L}$ be a differential closure of L. Let $\bar{b} \in \mathscr{U}(i)$.*

1. *The following equivalences hold*

$$\begin{aligned} \psi(\bar{b}) &\text{ iff there is } \sigma \in \mathrm{gal}(\mathscr{U}(i)/K) \text{ such that } \sigma(\bar{a}) = \bar{b} \\ &\text{ iff there is } \sigma \in \mathrm{Isom}_K(L, K\langle \bar{b} \rangle) \text{ such that } \sigma(\bar{a}) = \bar{b}. \end{aligned}$$

2. *Suppose $\bar{b} \in \hat{L} \cap \mathscr{U}$. If $\psi(\bar{b})$ then $L = K\langle \bar{b} \rangle$. Moreover,*

$$\begin{aligned} \psi(\bar{b}) &\text{ iff there is } \sigma \in \mathrm{gal}(\mathscr{U}(i)/K) \text{ such that } \sigma(\bar{a}) = \bar{b} \\ &\text{ iff there is } \sigma \in \mathrm{gal}(L/K) \text{ such that } \sigma(\bar{a}) = \bar{b}. \end{aligned}$$

Proof.

1. It is clear that $\psi(\bar{b})$ iff $\mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ iff there is $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$ such that $\sigma(\bar{a}) = \bar{b}$. This obviously implies that there is $\sigma \in \mathrm{Isom}_K(L, K\langle \bar{b} \rangle)$ such that $\sigma(\bar{a}) = \bar{b}$.

   Let $\sigma \in \mathrm{Isom}_K(L, K\langle \bar{b} \rangle)$ such that $\sigma(\bar{a}) = \bar{b}$, by Remark 3.2.8, σ extends to an automorphism of $\mathscr{U}(i)$.

2. Suppose $\bar{b} \in \hat{L} \cap \mathscr{U}$.

   If $\psi(\bar{b})$ then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$. So by Lemma 3.2.10, $L = K\langle \bar{b} \rangle$.

   If $\psi(\bar{b})$ then $\mathrm{Isom}_K(L, K\langle \bar{b} \rangle) = \mathrm{gal}(L/K)$. So by the first part of the lemma there is $\sigma \in \mathrm{gal}(L/K)$ such that $\sigma(\bar{a}) = \bar{b}$.

   Suppose $\sigma \in \mathrm{gal}(L/K)$ and $\sigma(\bar{a}) = \bar{b}$, by Remark 3.2.8, σ lifts to an element of $\mathrm{gal}(\mathscr{U}(i)/K)$.

□

In the proof of Theorem 3.2.13, we will need the fact that some particular externally definable subsets of C_K are definable in C_K. This property is well studied for instance in the general context of NIP theories but we state the result in the more particular form used later.

Lemma 3.2.12. *Let $\bar{l} \in \mathscr{U}(i)$ and $S := \{\bar{c} \in C_K : \mathscr{U}(i) \models \xi(\bar{l}, \bar{c})\}$ where ξ is a quantifier-free $\mathcal{L}_{df}$-formula without parameters. The set S is definable in C_K (with some parameters in C_K).*

Proof. W.l.o.g., $\xi(\bar{X}, \bar{Y})$ is an $\mathcal{L}_{df}$-formula of the following form

$$\bigvee_{\alpha} \bigwedge_{\beta} (h_{\alpha\beta}(\bar{X}, \bar{Y}) = 0 \wedge p_{\alpha\beta}(\bar{X}, \bar{Y}) \neq 0)$$

where for all α and β, $h_{\alpha\beta} \in \mathbb{Z}\{\bar{X}\}[\bar{Y}]$ and $p_{\alpha\beta} \in \mathbb{Z}\{\bar{X}\}[\bar{Y}]$.

Let $\bar{W}$ be a tuple of variable symbols of length $(n+1).s$ where s is the length of $\bar{X}$ and n is greater than the order of any $h_{\alpha\beta}$ and $p_{\alpha\beta}$. Let $\phi(\bar{W}, \bar{Y})$ be the $\mathcal{L}_{\text{fields}}$-formula

$$\bigvee_{\alpha} \bigwedge_{\beta} (h^*_{\alpha\beta}(\bar{W}, \bar{Y}) = 0 \wedge p^*_{\alpha\beta}(\bar{W}, \bar{Y}) \neq 0).$$

Clearly, $S := \{\bar{c} \in C_K : \mathscr{U}(i) \models \phi(\bar{\ell}, \bar{c})\}$ where $\bar{\ell} = (\bar{l}, \bar{l}', \ldots, \bar{l}^{(n)})$.

Let $I := \{p(\bar{W}) \in C_K[\bar{W}] : \ p(\bar{\ell}) = 0\}$. Note that I is a prime ideal of $C_K[\bar{W}]$. By Hilbert's Basis Theorem, there is $e \in \mathbb{N}$ and $f_1, \ldots, f_e \in C_K[\bar{W}]$ such that $I = (f_1, \ldots, f_e)$.

There is $\bar{d} \in C_K$ such that we may rewrite, for any $j \in \{1, \ldots, e\}$, $f_j(\bar{W})$ as $g_j(\bar{W}, \bar{d})$ with $g_j(\bar{W}, \bar{Z}) \in \mathbb{Z}[\bar{W}, \bar{Z}]$.

For any $q \in \mathbb{Z}[\bar{W}, \bar{Y}]$, we denote $\theta_q(\bar{Z}, \bar{Y})$ a quantifier-free $\mathcal{L}_{\text{fields}}$-formula which is equivalent in ACF_0 (and so in $\mathscr{U}(i)$) to the formula

$$\forall \bar{W}((\bigwedge_{j=1}^{e} g_j(\bar{W}, \bar{Z}) = 0) \rightarrow q(\bar{W}, \bar{Y}) = 0).$$

Claim 1. Let $q(\bar{W}, \bar{Y}) \in \mathbb{Z}[\bar{W}, \bar{Y}]$, $\bar{c} \in C_K$, then

$$\mathscr{U}(i) \models \theta_q(\bar{d}, \bar{c}) \text{ iff } q(\bar{W}, \bar{c}) \in I.$$

Let $\theta(\bar{Z}, \bar{Y})$ be the quantifier-free $\mathcal{L}_{\text{fields}}$-formula

$$\bigvee_{\alpha} \bigwedge_{\beta} (\theta_{h^\star_{\alpha\beta}}(\bar{Z}, \bar{Y}) \wedge \neg\theta_{p^\star_{\alpha\beta}}(\bar{Z}, \bar{Y})).$$

Let $D := \{\bar{c} \in C_K : C_K \models \theta(\bar{d}, \bar{c})\}$.

Claim 2. $S = D$.

This ends the proof of the Lemma. □

Proof of Claim 1. Let $q(\bar{W}, \bar{Y}) \in \mathbb{Z}[\bar{X}, \bar{Y}]$, $\bar{c} \in C_K$. Denote $p(\bar{W}) := q(\bar{W}, \bar{c})$.

Suppose $p(\bar{W}) \in I$. Since $f_1, \ldots, f_e$ generate this ideal, there are $p_1, \ldots, p_e \in C_K[\bar{W}]$ such that $p(\bar{W}) = \sum_{i=1}^{e} f_i(\bar{W}).p_i(\bar{W})$. Let $\bar{a} \in \mathscr{U}(i)$ be such that $\mathscr{U}(i) \models (\bigwedge_{k=1}^{e} f_k(\bar{a}) = 0)$, then $p(\bar{a}) = \sum_{i=1}^{e} f_i(\bar{a}).p_i(\bar{a}) = 0$. So,

$$\mathscr{U}(i) \models \forall \bar{W}((\bigwedge_{j=1}^{e} f_j(\bar{W}) = 0) \rightarrow q(\bar{W}, \bar{c}) = 0).$$

Conversely assume that $\mathscr{U}(i) \models \forall \bar{W}((\bigwedge_{j=1}^{e} f_j(\bar{W}) = 0) \rightarrow p(\bar{W}) = 0)$, then since $\bar{\ell} \in \mathscr{U}(i)$ and $f_1, \ldots, f_e \in I$, we get $p(\bar{\ell}) = 0$ and so $p(\bar{W}) \in I$. □

Proof of Claim 2. Let $\bar{c} \in C_K$. Then

$$\begin{aligned}
\bar{c} \in D &\text{ iff } C_K \models \theta(\bar{d}, \bar{c}) \\
&\text{ iff } C_K \models \bigvee_{\alpha} \bigwedge_{\beta} (\theta_{h^{\star}_{\alpha\beta}}(\bar{d}, \bar{c}) \wedge \neg\theta_{p^{\star}_{\alpha\beta}}(\bar{d}, \bar{c})) \\
&\text{ iff } \mathscr{U}(i) \models \bigvee_{\alpha} \bigwedge_{\beta} (\theta_{h^{\star}_{\alpha\beta}}(\bar{d}, \bar{c}) \wedge \neg\theta_{p^{\star}_{\alpha\beta}}(\bar{d}, \bar{c})) \\
(\text{by Claim 1}) &\text{ iff } \bigvee_{\alpha} \bigwedge_{\beta} \big(h^{\star}_{\alpha\beta}(\bar{W}, \bar{c}) \in I \wedge p^{\star}_{\alpha\beta}(\bar{W}, \bar{c}) \notin I\big) \\
&\text{ iff } \mathscr{U}(i) \models \phi(\bar{\ell}, \bar{c}) \\
&\text{ iff } \bar{c} \in S.
\end{aligned}$$

□

Theorem 3.2.13. *If L/K is strongly normal then the group* $\mathrm{gal}(L/K)$ *is isomorphic to a group G which is definable in C_K.*

Note that as the derivation is trivial on C_K, a definable group in C_K is simply a semialgebraic group in C_K (see Remark 1.3.10 and the paragraph above).

Proof. Let $L = K\langle\bar{a}\rangle$ and ψ be a quantifier-free formula isolating $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$. Since ψ is quantifier-free, $\psi(L) = \psi(\mathscr{U}(i)) \cap L^n$ and $\psi(\mathscr{U}) = \psi(\mathscr{U}(i)) \cap \mathscr{U}^n$.

Claim 3. There exists a quantifier-free definable function g with parameters in K such that for any $\bar{b} \in \psi(L)$, there exists $\bar{c} \in C_K$ such that $\bar{b} = g(\bar{a}, \bar{c})$.

An automorphism $\sigma \in \mathrm{gal}(L/K)$ is determined by $\sigma(\bar{a})$. So by Lemma 3.2.11, for any $\bar{b} \in \psi(L)$, there is a unique $\sigma \in \mathrm{gal}(L/K)$ s.t. $\sigma(\bar{a}) = \bar{b}$. Conversely (by Lemma 3.2.11 again) for any $\sigma \in \mathrm{gal}(L/K)$, $\bar{b} := \sigma(\bar{a}) \in \psi(L)$. So there is a bijection between $\mathrm{gal}(L/K)$ and $\psi(L)$. We let

$$A := \psi(L).$$

Let σ be an automorphism of L (fixing K) and $\bar{b} := \sigma(\bar{a})$. By Claim 3, there exists $\bar{c} \in C_K$ such that $\bar{b} = g(\bar{a}, \bar{c})$.

We will define a ternary relation R on A which will correspond to composition of the automorphisms in the sense that $R(\bar{b}, \bar{d}, \bar{e})$ iff $\sigma \circ \tau = \mu$ whenever $\sigma : \bar{a} \mapsto \bar{b}, \tau : \bar{a} \mapsto \bar{d}, \mu : \bar{a} \mapsto \bar{e}$. In other words, $\sigma(\bar{d}) = \bar{e}$ whenever $\sigma : \bar{a} \mapsto \bar{b}$. So we have $\bar{c} \in C_K$ such that $\bar{d} = g(\bar{a}, \bar{c})$. Thus $\sigma(\bar{d}) = \sigma(g(\bar{a}, \bar{c}))$. As g is a definable function, $\bar{e} = \sigma(g(\bar{a}, \bar{c})) = g(\sigma(\bar{a}), \sigma(\bar{c})) = g(\bar{b}, \bar{c})$. So we define R by letting

$$R(\bar{b}, \bar{d}, \bar{e}) \Leftrightarrow \psi(\bar{b}) \wedge \psi(\bar{d}) \wedge \psi(\bar{e}) \wedge \exists \bar{C}\big(\bar{C}' = 0 \wedge g(\bar{a}, \bar{C}) = \bar{d} \wedge g(\bar{b}, \bar{C}) = \bar{e}\big).$$

Let $\bar{b}, \bar{d} \in A$, we define $\bar{b} \cdot \bar{d} = \bar{e}$ iff $R(\bar{b}, \bar{d}, \bar{e})$. By construction of R, it is clear that the structure $(A, \cdot)$ is isomorphic to the group $\mathrm{gal}(L/K)$. Moreover, $(A, \cdot)$ is a definable group in L.

Now we will show that $(A, \cdot)$ is isomorphic as a group to an interpretable group in C_K.

Let $B := \{\bar{c} \in C_K : L \models \psi(g(\bar{a}, \bar{c}))\}$. We define an equivalence relation $\sim$ on B by letting $\bar{c}_1 \sim \bar{c}_2$ iff $g(\bar{a}, \bar{c}_1) = g(\bar{a}, \bar{c}_2)$. Moreover, we define a ternary relation $R^{\sim}$ on B such that $R^{\sim}(\bar{c}_1, \bar{c}_2, \bar{c}_3)$ iff

$$R(g(\bar{a}, \bar{c}_1), g(\bar{a}, \bar{c}_2), g(\bar{a}, \bar{c}_3)).$$

As the relation $R^{\sim}$ is $\sim$-invariant, we may endow the quotient $B/\sim$ with the group law induced by $R^{\sim}$. This group is obviously isomorphic to $(A, \cdot)$ and so to $\mathrm{gal}(L/K)$ as well. Moreover, it is interpretable in C_K with parameters $\bar{a} \in L$ and the parameters involved in the definition of g. By Lemma 3.2.12, $B/\sim$ is interpretable in C_K (with parameters inside C_K).

As the derivation is trivial on C_K, $B/\sim$ is interpretable in the language $\{+, -, \cdot, ^{-1}, 0, 1\}$. By elimination of imaginaries in RCF (see Theorem 1.3.11), $B/\sim$ is definable in C_K.

Let G be the definable group $B/\sim$ with the group law given by the relation $R^{\sim}$.

The group $\mathrm{gal}(L/K)$ is isomorphic to G. □

Proof of Claim 3. Suppose $\bar{b} \in \psi(\mathscr{U}(i))$. By Lemma 3.2.11, there exists $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$ such that $\sigma(\bar{a}) = \bar{b}$ and since L/K is strongly normal, $\bar{b} \in \langle L, C_{\mathscr{U}(i)} \rangle$. So there is a tuple $\bar{h}$ of differential rational functions $h_1(\bar{X}, \bar{Y}), \dots, h_m(\bar{X}, \bar{Y}) \in K\langle \bar{X}, \bar{Y} \rangle$ and $\bar{c}_1 \in C_{\mathscr{U}(i)}$ such that $\bar{b} = \bar{h}(\bar{a}, \bar{c}_1)$. A priori $\bar{h}$ and $\bar{c}_1$ may depend on $\bar{b}$, so we will now work on getting a definable function that does not depend on $\bar{b}$.

We have some quantifier-free definable functions $(g_j(-,-))_{j \in J}$ such that for all $\bar{b} \in \psi(\mathscr{U}(i))$, there exists $j \in J$, $\mathscr{U}(i) \models \exists \bar{C} \quad (\bar{C}' = 0 \wedge \bar{b} = g_j(\bar{a}, \bar{C}))$. Therefore,

$$\mathrm{DCF}_0 \cup \mathrm{Diag}(L) \cup \{\psi(\bar{X})\} \cup \{\forall \bar{C} \quad (\bar{C}' = 0 \Rightarrow \bar{X} \neq g_j(\bar{a}, \bar{C})) : j \in J\}$$

is not consistent. By compactness, for some $j_1, \dots, j_n \in J$,

$$\mathrm{DCF}_0 \cup \mathrm{Diag}(L) \cup \{\psi(\bar{X})\} \cup \{\forall \bar{C} \quad (\bar{C}' = 0 \Rightarrow \bar{X} \neq g_j(\bar{a}, \bar{C})) : j \in \{j_1, \dots, j_n\}\}$$

is not consistant. So

$$\mathrm{DCF}_0 \cup \mathrm{Diag}(L) \cup \{\psi(\bar{X})\} \vdash \bigvee_{j=j_1,\dots,j_n} \exists \bar{C} \ \ (\bar{C}' = 0 \wedge \bar{X} = g_j(\bar{a}, \bar{C})).$$

So we have finitely many quantifier-free definable functions $g_1, \dots, g_n$ satisfying the required property. To get one function f, we let

$$\bigwedge_{j=1}^{n} f(\bar{a}, \bar{c}, j) = g_j(\bar{a}, \bar{c}) \wedge \big((d \neq 1 \wedge \cdots \wedge d \neq n) \to f(\bar{a}, \bar{c}, d) = g_n(\bar{a}, \bar{c})\big).$$

The function f is quantifier-free definable and for all $\bar{b} \in \psi(\mathscr{U}(i))$, there exists $\bar{c}_1 \in C_{\mathscr{U}(i)}$ such that

$$\bar{b} = f(\bar{a}, \bar{c}_1).$$

Suppose now that $\bar{b} \in \psi(L)$ and let $\hat{L}$ be a differential closure of L. By model completeness of DCF_0, we get $\bar{c}_1 \in C_{\hat{L}} = C_{L(i)} = C_{K(i)}$ such that $\bar{b} = f(\bar{a}, \bar{c}_1)$. Moreover, as $C_{K(i)} = C_K(i)$, there exists a quantifier-free definable function g and $\bar{c} \in C_K$ such that $\bar{b} = g(\bar{a}, \bar{c})$. (The tuple $\bar{c}$ contains imaginary and real parts of the components of the tuple $\bar{c}_1$.) □

We will now study another automorphism group associated with the extension L/K. It is called the full differential Galois group of L/K and denoted $\mathrm{Gal}(L/K)$:

Definition 3.2.14. $\mathrm{Gal}(L/K) := \mathrm{gal}(\langle L, C_{\mathscr{U}} \rangle / \langle K, C_{\mathscr{U}} \rangle)$.

We will now work on showing that one may embed $\mathrm{gal}(L/K)$ in $\mathrm{Gal}(L/K)$.

Lemma 3.2.15. *Let F be a differential field contained in $\mathscr{U}$. If $\bar{a}, \bar{b} \in \mathscr{U}$, $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F)$ iff $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F(i)) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F(i))$.*

Proof. The implication from right to left is evidently true.

As DCF_0 admits quantifier elimination, the type $\mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F(i))$ is determined by atomic formulas satisfied by $\bar{b}$ and so by formulas of the form $p(\bar{X}) = 0$ where $p(\bar{X}) \in F(i)\{\bar{X}\}$. This fact is also true for $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F(i))$ as well as for $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F)$ and $\mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F)$ but in the two last cases we only need differential polynomials with coefficients in F. Moreover, $p(\bar{X})$ can be written $q_1(\bar{X}) + iq_2(\bar{X})$ for some $q_1(\bar{X}), q_2(\bar{X}) \in F\{\bar{X}\}$. Clearly,

$$q_1(\bar{b}) + iq_2(\bar{b}) = 0 \text{ iff } q_1(\bar{b}) = 0 \text{ et } q_2(\bar{b}) = 0.$$

Hence, if $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F)$ then for all $p(\bar{X}) \in F(i)\{\bar{X}\}$, $p(\bar{a}) = 0$ iff $p(\bar{b}) = 0$. Therefore $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/F(i)) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/F(i))$. □

Lemma 3.2.16. *Let $\bar{a}$, $\bar{b} \in \mathscr{U}(i)$ such that $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ is isolated. Then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K(i)) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K(i))$ iff*

$$\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}}(i)\rangle) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/\langle K, C_{\mathscr{U}}(i)\rangle).$$

The argument of the proof of Lemma 3.2.16, as well as the one of Lemma 3.2.18, is essentially made in [45], Remark 2.6, even though the statement of that result does not directly imply our lemmas. For the ease of the reader we give the proofs below.

Proof. As $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ is isolated, by Lemma 3.2.15, $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K(i))$ is isolated as well.

Let $\phi(\bar{X})$ be a formula that isolates $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K(i))$, we will show that $\phi(\bar{X})$ isolates $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}}(i)\rangle)$ as well. Suppose not, i.e., there exist $\bar{a}_1, \bar{a}_2 \in \mathscr{U}(i)$, a formula $\psi(\bar{X}, \bar{Y})$ with parameters in K and $\bar{c} \in C_{\mathscr{U}}(i)$ such that

$$\mathscr{U}(i) \models \phi(\bar{a}_1) \wedge \phi(\bar{a}_2) \wedge \psi(\bar{a}_1, \bar{c}) \wedge \neg\psi(\bar{a}_2, \bar{c}).$$

Let $\hat{K} \subseteq \mathscr{U}(i)$ be a differential closure of K. As $\hat{K}$ is an elementary substructure of $\mathscr{U}(i)$, one can find $\bar{a}_3, \bar{a}_4 \in \hat{K}$ and $\bar{d} \in C_{\hat{K}} = C_K(i)$ such that $\hat{K} \models \phi(\bar{a}_3) \wedge \phi(\bar{a}_4) \wedge \psi(\bar{a}_3, \bar{d}) \wedge \neg\psi(\bar{a}_4, \bar{d})$. It contradicts the fact that ϕ isolates $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, i\rangle)$.

So it follows that if $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K(i)) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K(i))$ then

$$\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}}(i)\rangle) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/\langle K, C_{\mathscr{U}}(i)\rangle).$$

The converse is obvious. □

Lemma 3.2.17. *Let $\bar{a}, \bar{b} \in \mathscr{U}$ such that $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$ is isolated, then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$ iff $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}}\rangle) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/\langle K, C_{\mathscr{U}}\rangle)$.*

Proof. Follows from Lemmas 3.2.15 and 3.2.16. □

Lemma 3.2.18. *Let L/K be a strongly normal extension and σ be a differential field embedding from L into $\mathscr{U}$ fixing K. There exists a unique element of* $\mathrm{Gal}(L/K)$ *extending σ.*

Proof. By Remark 3.2.8 and Lemma 3.2.17,

$$\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}}\rangle) = \mathrm{tp}^{\mathscr{U}(i)}(\sigma(\bar{a})/\langle K, C_{\mathscr{U}}\rangle).$$

Thus there exists a $\langle K, C_{\mathscr{U}}\rangle$-isomorphism τ: $\langle L, C_{\mathscr{U}}\rangle \to \langle \sigma(L), C_{\mathscr{U}}\rangle$ such that $\tau(\bar{a}) = \sigma(\bar{a})$. By Lemma 3.2.9, $\langle L, C_{\mathscr{U}}\rangle = \langle \sigma(L), C_{\mathscr{U}}\rangle$.

Uniqueness is immediate. □

Any element of $\mathrm{Gal}(L/K)$ restricts to an embedding from L into $\mathscr{U}$. So Lemma 3.2.18 establishes a bijection $\mathrm{Hom}_K(L, \mathscr{U}) \to \mathrm{Gal}(L/K)$ sending any element of $\mathrm{Hom}_K(L, \mathscr{U})$ to its unique extension in $\mathrm{Gal}(L/K)$. So one may identify the subset $\mathrm{gal}(L/K)$ of $\mathrm{Hom}_K(L, \mathscr{U})$ with a subgroup of $\mathrm{Gal}(L/K)$.

Lemma 3.2.19. *Let $L = K\langle \bar{a} \rangle$ be a strongly normal extension of K. Let ψ be a formula isolating $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$, $\psi(\mathscr{U}) = \psi(\langle L, C_{\mathscr{U}} \rangle)$.*

Proof. We will show that $\psi(\mathscr{U}) \subseteq \psi(\langle L, C_{\mathscr{U}} \rangle)$, the other inclusion is straightforward. By Lemma 3.2.11, if $\bar{b} \in \psi(\mathscr{U})$ then there is an embedding $\sigma : L \to \mathscr{U}$ such that $\sigma(\bar{a}) = \bar{b}$. So, by Lemma 3.2.9, $\bar{b} \in \langle L, C_{\mathscr{U}} \rangle$. □

Note that if L/K is strongly normal then $\langle L, C_{\mathscr{U}} \rangle / \langle K, C_{\mathscr{U}} \rangle$ is also strongly normal. Actually, let $\tilde{\mathscr{U}}$ be a $|\langle L, C_{\mathscr{U}} \rangle|^+$-saturated model of CODF containing $\langle L, C_{\mathscr{U}} \rangle$. By Remark 3.2.4, as L/K is strongly normal, for all $\sigma \in \mathrm{gal}(\tilde{\mathscr{U}}(i)/K)$, $\langle L, C_{\tilde{\mathscr{U}}(i)} \rangle = \langle \sigma(L), C_{\tilde{\mathscr{U}}(i)} \rangle$. In particular, this is true for $\sigma \in \mathrm{gal}(\tilde{\mathscr{U}}(i)/\langle K, C_{\mathscr{U}} \rangle)$.

Therefore we may apply Theorem 3.2.13 to the (strongly normal) extension $\langle L, C_{\mathscr{U}} \rangle / \langle K, C_{\mathscr{U}} \rangle$. So we get that $\mathrm{Gal}(L/K)$ is isomorphic to a definable group of $C_{\mathscr{U}}$. We will make an explicit argument (see the proof of Theorem 3.2.20) laying emphasis on the following uniformity. The group $\mathrm{gal}(L/K)$, respectively $\mathrm{Gal}(L/K)$, can be identified with $\psi(L)$, respectively $\psi(\langle L, C_{\mathscr{U}} \rangle)$, where ψ is a formula isolating $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$, equivalently isolating $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/\langle K, C_{\mathscr{U}} \rangle)$, by Lemma 3.2.17. We will see in Lemma 3.2.21, that the definability of the corresponding groups in the constant fields C_K and $C_{\mathscr{U}}$ has the same uniformity.

Theorem 3.2.20. *If L/K is strongly normal then* $\mathrm{Gal}(L/K)$ *is isomorphic to a group G which is definable in $C_{\mathscr{U}}$.*

Proof. The proof is similar to the one of Theorem 3.2.13. Using the same notations, we indicate the main steps.

We first prove the following Claim:

Claim 4. There exists a quantifier-free definable function g (with parameters in K) such that for any $\bar{b} \in \psi(\mathscr{U})$, there exists $\bar{c} \in C_{\mathscr{U}}$ such that $\bar{b} = g(\bar{a}, \bar{c})$.

In the proof of Claim 3 we showed that one may find $\bar{c} \in C_{\mathscr{U}}(i)$. It suffices to take the real and imaginary parts $\bar{c}$ to find constants in $C_{\mathscr{U}}$ and so get Claim 4.

Let $\bar{b} \in \mathscr{U}$. If there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\bar{a}) = \bar{b}$, then σ restricts to an element of $\mathrm{Hom}_K(L, \mathscr{U})$ and by Remark 3.2.8, $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$.

Conversely, if $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$, then there is a K-embedding from L into $\mathscr{U}$ that maps $\bar{a}$ onto $\bar{b}$. By Lemma 3.2.18, there exists $\tau \in \mathrm{Gal}(L/K)$ such that $\tau(\bar{a}) = \bar{b}$.

Then we take $A = \psi(\mathscr{U})$. By Lemma 3.2.19, $\psi(\mathscr{U}) = \psi(\langle L, C_{\mathscr{U}} \rangle)$ and so one may reproduce the construction of the definable group G made in Theorem 3.2.13. □

Lemma 3.2.21. *Let ϕ be a formula such that the domain of the group G given by Theorem 3.2.20 is $\phi(C_{\mathscr{U}})$ then* $\mathrm{gal}(L/K)$ *is isomorphic to $\phi(C_K)$.*

Proof. From a formula ψ isolating the type of $\bar{a}$, the construction of the formula ϕ and the group law is made of elementary operations, compactness and the elimination of imaginaries in the theory RCF which is uniform (see Theorem 1.3.11). By uniformity of the elimination of imaginaries, a formula defining a set of representatives of a given definable equivalence relation may be chosen independently of the model of RCF we work in. □

3.2.3 Examples

We will see two classes of formally real differential field extensions that are strongly normal. The first one is the class of formally real Picard-Vessiot extensions with real closed fields of constants. Given any homogeneous linear differential equation with parameters in K, then assuming that C_K is real closed, K has a formally real Picard-Vessiot extension relative to that equation (see Theorem 3.1.20).

Lemma 3.2.22. *If L/K is a Picard-Vessiot extension and C_K is real closed, then L/K is strongly normal.*

Proof. Clearly, the conditions (1) and (2) of definition 3.2.3 follow from the definition of Picard-Vessiot extension (see Definition 3.1.4 above).

For (3), the argument is similar to the classical case, see [37], § 9. □

Theorem 3.2.23 (See [26], Theorem 5.5)**.** *If L/K is a Picard-Vessiot extension, then* $\mathrm{gal}(L/K)$ *is isomorphic to a linear group over C_K.*

Our second example is the class of Weierstrass extensions. Kolchin studied Weierstrass extensions with algebraically closed fields of constants in [29]. We follow the presentation of [37] and get analogous results assuming that the field of constants is real closed.

Definition 3.2.24. Let L/K be a differential field extension. An element $\alpha \in L$ is a Weierstrassian over K iff α is a zero of a differential polynomial of the form

$$(Y')^2 - k^2(4Y^3 - g_2 Y - g_3)$$

where $g_2, g_3 \in C_K$, $27g_3^2 - g_2^3 \neq 0$ and $k \in K$.

We will say that L/K is a Weierstrass extension iff $L = K\langle\alpha\rangle$ where α is a Weierstrassian (over K).

Remark 3.2.25. We make use of Singer's axioms of CODF (see [65]) to show the existence of Weierstrassian elements.

Let $k \in K$ and $g_2, g_3 \in C_K$, $27g_3^2 - g_2^3 \neq 0$. If M is a model of CODF containing K, and $\beta \in M$ is such that $k^2(4\beta^3 - g_2\beta - g_3) \geq 0$, then we can find $\gamma \in M$ such that $(\gamma)^2 - k^2(4\beta^3 - g_2\beta - g_3) = 0$. If moreover $\gamma \neq 0$, by Singer's axiomatisation of CODF, for any neighbourhood N of (β, γ) in M^2 (endowed with the product topology), there is $\alpha \in M$ such that

$$(\alpha')^2 - k^2(4\alpha^3 - g_2\alpha - g_3) = 0,$$

and $(\alpha, \alpha') \in N$.

Let E be the projective curve of equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

It is an elliptic curve defined over C_K. This curve is endowed with a rational group law denoted $\oplus$ (see Silverman [62], Chapter III, § 2) and which makes of E an algebraic group defined over $\mathbb{Q}(g_2, g_3)$ (see [62], Chapter III, Group Law Algorithm 2.3). The inverse of an element a will be denoted $\ominus a$ and we will write $a \ominus b$ to mean $a \oplus (\ominus b)$. Let F a be differential field containing K, $E(F)$ denotes the set of points of E in F.

Lemma 3.2.26. *If $L = K\langle\alpha\rangle$ is a Weierstrass extension of K and $C_K = C_L$ is real closed, then L/K is strongly normal.*

Proof. Conditions (1) and (2) from definition 3.2.3 are straightforward. As in [37], page 92, we use Lemma 2, Chapter 3, p. 805 from [29] to show that L/K satisfies (3). □

We denote (x, y) the affine coordinates of the projective point $(x : y : 1)$.

Let $L = K\langle\alpha\rangle$ be a Weierstrass extension with $(\alpha')^2 - k^2(4\alpha^3 - g_2\alpha - g_3)$ and $C_K = C_L$ is real closed. We denote $P(\sigma) := (\sigma(\alpha), \frac{\sigma(\alpha)'}{k}) \ominus (\alpha, \frac{\alpha'}{k})$, for $\sigma \in \text{gal}(L/K)$.

Lemma 3.2.27. *The map $P : \text{gal}(L/K) \to E(C_K) : \sigma \mapsto P(\sigma)$ is a group monomorphism.*

Proof. See for instance [37], page 92 or [29], Chapter III, § 6. □

Theorem 3.2.28. *The group* $\mathrm{gal}(L/K)$ *is isomorphic to a definable subgroup of* $E(C_K)$*, which is either finite or equal to* $E(C_K)$ *or of index* 2 *in* $E(C_K)$*.*

Proof. As in [37], take ψ be a quantifier-free formula that isolates $\mathrm{tp}^{\mathscr{U}(i)}(\alpha/K)$. We denote by $\pi_1 : L \times L \to L$ the projection onto the first coordinate. The subset $D = \{(c_1, c_2) \in E(C_K) : L \models \psi(\pi_1((c_1, c_2) \oplus (\alpha, \frac{\alpha'}{k})))\}$ of C_K^2 is definable with parameters from L and the map $P : \mathrm{gal}(L/K) \to D$ is a group isomorphism. By Lemma 3.2.12, D is definable in C_K.

We will first show that any infinite definable subgroup of $E(\mathbb{R})$ has either index 1 or index 2. It will follow from two facts. First, the group $\mathbb{R}/\mathbb{Z}$ has no proper subgroup of finite index. Second, if G is a definable group of dimension 1 in the field $\mathbb{R}$ then any infinite definable subgroup of G has finite index in G (see [43], Proposition 1.8 and Lemma 2.11).

By Corollary 2.3.1 of [63], $E(\mathbb{R})$ is either isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or to $\mathbb{R}/\mathbb{Z}$. Let H be an infinite proper definable subgroup of $E(\mathbb{R})$. As $E(\mathbb{R})$ has dimension 1, H has finite index in $E(\mathbb{R})$. Since $\mathbb{R}/\mathbb{Z}$ has no proper subgroup of finite index, $E(\mathbb{R})$ is isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{R}/\mathbb{Z} \times \{0\}$. This establishes the theorem when $C_K = \mathbb{R}$.

For any real closed field R_1, the same dichotomy holds for infinite definable subgroups of $E(R_1)$. Let $\phi(\bar{X})$ be an $\mathcal{L}_{df}$-formula. Then $\phi(R_1)$ is a subgroup of $E(R_1)$ iff $\phi(\mathbb{R})$ is a subgroup of $E(\mathbb{R})$. Moreover, $\phi(R_1)$ is of index n in $E(R_1)$ iff $\phi(\mathbb{R})$ is of index n in $E(\mathbb{R})$. All of that is due to the fact that these properties are first order and that RCF is a complete theory. So any infinite definable subgroup of $E(R_1)$ is either equal to $E(R_1)$ or of index 2 in $E(R_1)$. □

Remark 3.2.29. It would be interesting to get a result on the existence of Weierstrass extensions in the style of the results known on the existence of Picard-Vessiot extensions (see Theorem 3.1.7 and Theorem 3.1.20). More explicitely the question is: for a formally real field K where C_K is real closed, does K have a Weierstrass extension L which is formally real? As far as we know, it is not even clear when K is not formally real and assuming that C_K is algebraically closed that a Weierstrass extension does exist?

3.2.4 Intermediate extensions

Proposition 3.2.30. *Let* L/K *be a strongly normal extension and* $E \subseteq L$ *be a strongly normal extension of* K*. Let* $\mathcal{G} := \mathrm{Gal}(L/K)$ *and*

$$\mathcal{G}_E := \{\sigma \in \mathcal{G} : \text{ for all } x \in E, \sigma(x) = x\}.$$

1. *The group* $\mathcal{G}_E$ *is a normal subgroup of* $\mathcal{G}$*. Moreover,* $\mathcal{G}/\mathcal{G}_E$ *is isomorphic to a subgroup of* $\mathrm{Gal}(E/K)$*.*

2. *The groups* $\mathrm{Gal}(E/K)$ *and* $\mathcal{G}/\mathcal{G}_E$ *are definable in* $C_{\mathscr{U}}$.

Proof. 1. We have that $\mathcal{G}_E$ is normal in $\mathcal{G}$ iff
for all $\sigma \in \mathcal{G}_E$ and $\tau \in \mathcal{G}$, $\tau^{-1}\sigma\tau \in \mathcal{G}_E$ iff
for all $\sigma \in \mathcal{G}_E$, for all $\tau \in \mathcal{G}$ and for all $x \in E$, $\tau^{-1}\sigma\tau(x) = x$ i.e. $\sigma\tau(x) = \tau(x)$.

Let $\sigma \in \mathcal{G}_E$, $\tau \in \mathcal{G}$ and $x \in E$, by Lemma 3.2.9, $\langle E, C_{\mathscr{U}}\rangle = \tau(\langle E, C_{\mathscr{U}}\rangle)$, so $\tau(x) \in \langle E, C_{\mathscr{U}}\rangle$. Since σ fixes E and $C_{\mathscr{U}}$, $\sigma\tau(x) = \tau(x)$.

Let us show that $\mathcal{G}/\mathcal{G}_E$ is isomorphic to a subgroup of $\mathrm{Gal}(E/K)$.

The map $m : \mathcal{G} \to \mathrm{Gal}(E/K) : \sigma \mapsto \sigma\restriction_{\langle E, C_{\mathscr{U}}\rangle}$ is well defined (by Lemma 3.2.9) and is a group homomorphism. Moreover, $\ker m = \mathcal{G}_E$. Hence $\mathcal{G}/\mathcal{G}_E$ is isomorphic to a subgroup of $\mathrm{Gal}(E/K)$.

2. By Theorem 3.2.20, $\mathrm{Gal}(E/K)$, $\mathcal{G}$ and $\mathcal{G}_E$ are definable in $C_{\mathscr{U}}$. By elimination of imaginaries in RCF, the quotient group $\mathcal{G}/\mathcal{G}_E$ is definable in $C_{\mathscr{U}}$.

□

Example 3.2.31. We provide here an example of extensions $K \subset E \subset L$ satisfying the hypotheses of Proposition 3.2.30 and such that $\mathcal{G}/\mathcal{G}_E$ is isomorphic to a proper subgroup of $\mathrm{Gal}(E/K)$.

Let $K := \mathbb{R}$ endowed with the trivial derivation, $E := \mathbb{R}\langle t\rangle$ where $t' = t$ and t is transcendental over $\mathbb{R}$, $L := \mathbb{R}\langle u\rangle$ where $u^2 = t$, L/K and E/K are Picard-Vessiot (and so by Lemma 3.2.22 are strongly normal).

Take $\sigma \in \mathrm{Gal}(E/K)$ such that $\sigma(t) = -t$. Since t is a square in L and $-t$ is not, σ may clearly not be in the image of the map m from the proof of Proposition 3.2.30.

Actually $\mathrm{Gal}(E/K)$ is isomorphic to $\mathbb{G}_m(C_{\mathscr{U}})$, $\mathcal{G}$ is isomorphic to $\mathbb{G}_m(C_{\mathscr{U}})$ and $\mathcal{G}_E$ is isomorphic to the subgroup $\{-1, 1\}$ of $\mathbb{G}_m(C_{\mathscr{U}})$. Because $C_{\mathscr{U}}$ is real closed, any element of the group $\mathbb{G}_m(C_{\mathscr{U}})/\{-1, 1\}$ is a square, this is no longer true in the group $\mathbb{G}_m(C_{\mathscr{U}})$.

3.3 Regular strongly normal extensions

3.3.1 Normality

Definition 3.3.1. A differential field extension L/K is regular strongly normal iff $\mathrm{dcl}^{\mathscr{U}}(K) \cap L = K$ and L/K is strongly normal (definition 3.2.3).

Remark 3.3.2. Recall that (in characteristic 0), a field extension L/K is called regular iff no element of $L \setminus K$ is algebraic over K. We explain below why it is equivalent to our definition.

Since $\mathscr{U}$ is a model of CODF, $\mathrm{dcl}^{\mathscr{U}}(K)$ is the set of elements of $\mathscr{U}$ that are algebraic over K (see section 1.4.2). Hence, L/K is regular strongly normal iff it is strongly normal and K is relatively algebraically closed in L.

Note that $\mathrm{dcl}^{\mathscr{U}}(K)$ is the real closure of K inside $\mathscr{U}$.

Example 3.3.3. Let $K := \mathbb{R}(t)$ where t is transcendental, $t' = 1$ and for any $r \in \mathbb{R}$, $r' = 0$. Let $L := \mathbb{R}(u)$ where $u^3 = t$. The extension L/K is strongly normal, is not regular and fails to be $\mathrm{Hom}_K(L, \mathscr{U})$-normal. This example is mentioned in [30] (see Chapter VI, page 402-403, Exercise 1).

Now we show a property of normality for regular strongly normal extensions.

Lemma 3.3.4. *Suppose that L is a regular strongly normal extension of K. Then L/K is $\mathrm{Hom}_K(L, \mathscr{U})$-normal.*

Proof. Let $L := K\langle \bar{a} \rangle$ and ψ be a quantifier-free formula isolating the type $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$. Let $u \in L \setminus K$, we may write $u = \frac{p_1(\bar{a})}{p_2(\bar{a})}$ for some $p_1, p_2 \in K\{\bar{X}\}$. So $\mathrm{tp}^{\mathscr{U}}(u/K)$ contains the formula $\xi(X) := \exists \bar{Y} \big(\psi(\bar{Y}) \wedge X = \frac{p_1(\bar{Y})}{p_2(\bar{Y})}\big)$.

As L/K is regular then $u \notin \mathrm{dcl}^{\mathscr{U}}(K)$, so there is $d \in \mathscr{U}$ such that $d \neq u$ and $\mathscr{U} \models \xi(d)$. Thus for some $\bar{b} \in \psi(\mathscr{U})$, we have $d = \frac{p_1(\bar{b})}{p_2(\bar{b})}$. Since ψ isolates $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K)$, then $\mathrm{tp}^{\mathscr{U}(i)}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}(i)}(\bar{b}/K)$.

Therefore there is an isomorphism $\sigma : L \to K\langle \bar{b} \rangle$ such that $\sigma(\bar{a}) = \bar{b}$ and so $\sigma(u) = d \neq u$. ☐

Corollary 3.3.5. *Suppose that L is a regular strongly normal extension of K. Then for any $u \in L \setminus K$, there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(u) \neq u$.*

Proof. Because of Lemma 3.2.18, any element of $\mathrm{Hom}_K(L, \mathscr{U})$ extends to an element of $\mathrm{Gal}(L/K)$. So this corollary follows from Lemma 3.3.4. ☐

3.3.2 Partial correspondence

We summarize in the following theorem what we have proved before:

Theorem 3.3.6 (Partial correspondence)**.** *Let L/K be a regular strongly normal extension and G be a definable group in $C_{\mathscr{U}}$ which is isomorphic to $\mathrm{Gal}(L/K)$ (we will denote by $\eta : G \to \mathrm{Gal}(L/K)$ the isomorphism). Let G_E denote $\{h \in G :$ for all $x \in E, \eta(h)(x) = x\}$.*

1. *Let E be a differential subfield of L containing K, then L is a strongly normal extension of E, G_E is a definable subgroup of G and is isomorphic to* $\mathrm{Gal}(L/E)$.

2. *$E \mapsto G_E$ is an injective map from the set of intermediate extensions E of L/K such that L/E is regular (i.e.* $\mathrm{dcl}^{\mathscr{U}}(E) \cap L = E$*) into the set of definable subgroups of G.*

Proof. 1. It follows directly from the definition of strongly normal extensions that L/E is strongly normal and moreover by Theorem 3.2.13, G_E is definable in $C_{\mathscr{U}}$.

2. By Corollary 3.3.5.

□

Example 3.3.7. We present here an example of a regular strongly normal extension L/K (with G a definable group s.t. $\mathrm{Gal}(L/K) \simeq G$) and a definable subgroup H of G such that for any intermediate extension E, $H \neq G_E$. Take K and L as in Example 3.2.31, $G = \mathbb{G}_m(C_{\mathscr{U}})$. Take $H := \mathbb{G}_m(C_{\mathscr{U}})^2$ the set of squares of $\mathbb{G}_m(C_{\mathscr{U}})$.

Firstly, as $C_{\mathscr{U}}$ is real closed, H is an infinite subgroup of index 2 of $\mathbb{G}_m(C_{\mathscr{U}})$.

Secondly, L/K is an extension of transcendence degree 1. So let E be an intermediate extension, since $K := \mathbb{R}$ is real closed, E/K is transcendental and L/E is algebraic. Moreover, $L = E(t)$ is a finite algebraic extension. Therefore $\mathrm{Gal}(L/E)$ is finite and so G_E is finite.

It follows that G_E cannot be equal to H.

3.4 Increasing automorphisms

3.4.1 Motivations and notations

In this section since $\mathscr{U}$ has a unique order, any differential subfield of $\mathscr{U}$ is thought as an $\mathcal{L}_{odf}$-structure (with the order induced from $\mathscr{U}$).

In answer to the fact that the correspondence given in Theorem 3.3.6 is not 1 to 1, we will show below that some particular subgroups of $\mathrm{Gal}(L/K)$ correspond to the group of $\mathcal{L}_{odf}$-automorphisms of an ordered differential field extension.

We now explain the contrast between $\mathcal{L}_{odf}$-embeddings and $\mathcal{L}_{df}$-embeddings. Since $\mathscr{U}$ is $\mathcal{L}_{df}$-saturated, if $\bar{a}$, $\bar{b}$ are in $\mathscr{U}$ and $\mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{df}}(\bar{a}/K) = \mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{df}}(\bar{b}/K)$, then there is an $\mathcal{L}_{df}$-automorphism of $\mathscr{U}$ fixing K and sending $\bar{a}$ to $\bar{b}$. Note that since the order is definable in $\mathscr{U}$, any $\mathcal{L}_{df}$-automorphism of $\mathscr{U}$ is an $\mathcal{L}_{odf}$-automorphism. One has to be carefull that CODF eliminates quantifiers in $\mathcal{L}_{odf}$

and does not in $\mathcal{L}_{df}$. So elements of $\mathrm{Isom}_K(K\langle\bar{a}\rangle, K\langle\bar{b}\rangle)$, which are not assumed to be $\mathcal{L}_{odf}$-isomorphisms, do not necessarily respect types in $\mathscr{U}$ (i.e. are not elementary in $\mathscr{U}$).

We still work with L/K a strongly normal extension, assuming $L \subseteq \mathscr{U}$. We investigate the group of $\mathcal{L}_{odf}$-automorphisms of $\langle L, C_{\mathscr{U}}\rangle$ fixing $\langle K, C_{\mathscr{U}}\rangle$ and denote it $\mathrm{Aut}(L/K)$. Thus $\mathrm{Aut}(L/K)$ is just the set of increasing elements of $\mathrm{Gal}(L/K)$ and so it is a subgroup of $\mathrm{Gal}(L/K)$.

Let $L := K\langle\bar{a}\rangle$. Since the $\mathcal{L}_{odf}$-type of $\bar{a}$ over K in $\mathscr{U}$ is in general not isolated, the proof of Theorem 3.2.20 cannot be adapted to $\mathrm{Aut}(L/K)$.

3.4.2 Alternative correspondence

Let $\mathcal{G} := \mathrm{Gal}(L/K)$. By Theorem 3.2.20, there is a group isomorphism $\eta : G \to \mathcal{G}$ for some definable group G in $C_{\mathscr{U}}$.

Below $\langle L, C_{\mathscr{U}}\rangle^{rc}$ denotes the unique real closure of $\langle L, C_{\mathscr{U}}\rangle$ in $\mathscr{U}$.

Proposition 3.4.1. *Let G_0 be a definable subgroup of G (possibly over some parameters in $C_{\mathscr{U}}$). There is a finite tuple $\bar{d} \in \langle L, C_{\mathscr{U}}\rangle^{rc}$ such that the group $\eta(G_0) \cap \mathrm{Aut}(L/K)$ is isomorphic to* $\mathrm{Aut}(L(\bar{d})/K(\bar{d}))$.

Proof. Let W be $\{\tau(\bar{a}) : \tau \in \eta(G_0)\} = \{g(\bar{a}, \bar{c}) : \bar{c} \in G_0\}$ for the $\langle K, C_{\mathscr{U}}\rangle$-definable map g given by Theorem 3.2.20 (see Claim 4 in the proof). The set W is $\mathcal{L}_{odf}$-definable in $\mathscr{U}$ with parameters in $\langle L, C_{\mathscr{U}}\rangle$ and $W \subseteq \langle L, C_{\mathscr{U}}\rangle$. By [46], Theorem 0.3, CODF admits elimination of imaginaries in the language $\mathcal{L}_{odf}$. So (by Theorem 1.1.22) W has a canonical parameter $\bar{d} \in \mathrm{dcl}^{\mathscr{U}}(\langle L, C_{\mathscr{U}}\rangle) = \langle L, C_{\mathscr{U}}\rangle^{rc}$.

Let σ be any element of $\eta(G_0) \cap \mathrm{Aut}(L/K)$, we will show that σ lifts to an element of $\mathrm{Aut}(L(\bar{d})/K(\bar{d}))$. As $\langle L, C_{\mathscr{U}}\rangle^{rc}$ is the definable closure of $\langle L, C_{\mathscr{U}}\rangle$, there is a unique extension $\tilde{\sigma} : \langle L, C_{\mathscr{U}}\rangle^{rc} \to \langle L, C_{\mathscr{U}}\rangle^{rc}$ of σ (still fixing $\langle K, C_{\mathscr{U}}\rangle$).

Claim 5. $\tilde{\sigma}(\bar{d}) = \bar{d}$.

Hence, $\tilde{\sigma}\restriction_{\langle L(\bar{d}), C_{\mathscr{U}}\rangle} \in \mathrm{Aut}(L(\bar{d})/K(\bar{d}))$.

We set

$$\begin{aligned} f :& \eta(G_0) \cap \mathrm{Aut}(L/K) \to \mathrm{Aut}(L(\bar{d})/K(\bar{d})) \\ & \sigma \mapsto \tilde{\sigma}\restriction_{\langle L(\bar{d}), C_{\mathscr{U}}\rangle} . \end{aligned}$$

As it is clear that f is an injective group homomorphism, it remains to show that f is surjective.

Let $\sigma \in \mathrm{Aut}(L(\bar{d})/K(\bar{d}))$. Note that by Lemma 3.2.9, $\sigma(\langle L, C_{\mathscr{U}}\rangle) = \langle L, C_{\mathscr{U}}\rangle$. Thus $\sigma\restriction_{\langle L, C_{\mathscr{U}}\rangle} \in \mathrm{Aut}(L/K)$. Since $\mathscr{U}$ is saturated, $\sigma\restriction_{L(\bar{d})}$ extends to an automorphism λ of $\mathscr{U}$, leaving W invariant (because it fixes $\bar{d}$). Let us show that

$\sigma\restriction_{\langle L,C_{\mathscr{U}}\rangle}\in \eta(G_0)$. Because $\lambda(W) = W$, $\lambda(\bar{a}) = \tau(\bar{a})$ for some $\tau \in \eta(G_0)$. So $\sigma(\bar{a}) = \tau(\bar{a})$. Therefore $\tau = \sigma\restriction_{\langle L,C_{\mathscr{U}}\rangle}$. Thus $\sigma\restriction_{\langle L,C_{\mathscr{U}}\rangle}\in \eta(G_0)$. □

Proof of Claim 5. Let $\bar{p} \in \langle L, C_{\mathscr{U}}\rangle$ be such that W is $\bar{p}$-definable (i.e. the elements of $\bar{p}$ are the parameters involved in the $\mathcal{L}_{odf}$-definition of W).

We will show that $\tilde{\sigma}\restriction_{L\langle \bar{d},\bar{p}\rangle}$ can be extended to an element $\gamma \in \mathrm{Aut}(\mathscr{U}/K)$ such that $\gamma(W) = W$.

Since $\tilde{\sigma}$ fixes $\langle K, C_{\mathscr{U}}\rangle$, using the elimination of quantifiers and the $|K|^+$-saturation of $\mathscr{U}$, the restriction $\tilde{\sigma}\restriction_{L\langle \bar{d},\bar{p}\rangle}$ can be extended to an element $\gamma \in \mathrm{Aut}(\mathscr{U}/K)$. Since W is $\bar{p}$-definable in $\mathscr{U}$ and $\sigma(\bar{p}) = \gamma(\bar{p})$, $\sigma(W) = \gamma(W)$. As $\sigma \in \eta(G_0)$, $W = \{\tau(\bar{a}) : \tau \in \eta(G_0)\}$ is invariant under σ, i.e. $\sigma(W) = W$. So $\gamma(W) = W$.

Because $\bar{d}$ is a canonical parameter, $\gamma(\bar{d}) = \bar{d}$. So $\tilde{\sigma}(\bar{d}) = \bar{d}$. □

Note that $\mathrm{Gal}(L(\bar{d})/K(\bar{d}))$ is isomorphic to a subgroup of $\mathrm{Gal}(L/K)$. For any $\sigma \in \mathrm{Gal}(L(\bar{d})/K(\bar{d}))$, $\sigma\restriction_{\langle L,C_{\mathscr{U}}\rangle}$ belongs to $\mathrm{Gal}(L/K)$. But in general an element σ of $\mathrm{Gal}(L/K)$ cannot be extended to an element of $\mathrm{Gal}(L(\bar{d})/K(\bar{d}))$.

3.5 Non-finitely generated strongly normal extensions

In [32], Chapter II, Kolchin's notion of strongly normal extension is extended by J.J. Kovacic to the context of non-finitely generated extensions. We consider that extended notion in the case of formally real differential fields. Note that in [35], Chapter 3, A. Magid presents the construction of the full Picard-Vessiot compositum (in the case of an algebraically closed field of constants).

In this section, when a strongly normal extension is assumed to be finitely generated, we specify it explicitely:

Definition 3.5.1. By finitely generated strongly normal (f.g.s.n.) extension, we mean a strongly normal extension according to Definition 3.2.3.

We still work with the fixed differential fields $K \subset \mathscr{U}$.

3.5.1 The extended notion of strongly normal extension

We take the following definition of strongly normal extension from [32].

Definition 3.5.2. Let $L \subseteq \mathscr{U}$ be a differential field extension of K, L is strongly normal iff L is a union of f.g.s.n. extensions of K (inside $\mathscr{U}$).

The following lemma is a direct consequence of Definition 3.5.2.

Lemma 3.5.3. *If L/K be strongly normal then*

- $C_K = C_L$
- *For all* $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$, $\langle \sigma(L), C_{\mathscr{U}}(i)\rangle = \langle L, C_{\mathscr{U}}(i)\rangle$.

It is not clear whether the converse of this lemma is true.

Let $M \subseteq \mathscr{U}$ be a maximal extension of K such that $C_K = C_M$ (it does exist by Zorn Lemma). Let $\mathfrak{N}$ be the set of f.g.s.n. extensions of K such that $L \subseteq M$ and $\bar{K} := \bigcup_{L\in\mathfrak{N}} L$.

Remark 3.5.4. $\bar{K}$ is the maximal strongly normal extension of K inside M.

We now prove a short lemma, before showing that $\bar{K}$ may have a proper strongly normal extension inside M.

Lemma 3.5.5. *Let $L_1, L_2 \subset \mathscr{U}$ be f.g.s.n. extensions of K and $L := \langle L_1, L_2\rangle$. If $C_L = C_K$ then L is a f.g.s.n. extension of K.*

Proof. The fact that L is finitely generated is obvious.

Let $\sigma \in \mathrm{gal}(\mathscr{U}(i)/K)$,

$$\begin{aligned}\langle \sigma(L), C_{\mathscr{U}}(i)\rangle &= \langle \sigma(L_1), \sigma(L_2), C_{\mathscr{U}}(i)\rangle\\ &= \langle L_1, L_2, C_{\mathscr{U}}(i)\rangle.\end{aligned}$$

□

Lemma 3.5.6. *$\bar{K}$ may have a proper strongly normal extension in M.*

Proof. We will show that there exists an example of a formally real differential field K where $\bar{K}$ has a proper Picard-Vessiot extension L_2. Let K be a real closed field with trivial derivation. Let $L_1 \subseteq \mathscr{U}$ be a Picard-Vessiot extension of K and $L_2 \subseteq \mathscr{U}$ be a Picard-Vessiot extension of L_1 such that $L_2(i)$ is not contained in any Picard-Vessiot extension of $K(i)$ (it does exist by Lemma 3.1.19).

We consider $\bar{K}$ inside a maximal extension M of K containing L_2 and such that $C_K = C_M$.

Suppose $L_2 \subseteq \bar{K}$, then by Lemma 3.5.5 there is $L \in \mathfrak{N}$ such that $L_2 \subseteq L$. So $L_2(i) \subseteq L(i)$, then by Proposition 3.2.2, $L_2(i)$ is contained in some Picard-Vessiot extension of $K(i)$. Since we got a contradiction, $L_2 \nsubseteq \bar{K}$. So $\langle L_2, \bar{K}\rangle$ is a proper Picard-Vessiot extension of $\bar{K}$. □

Remark 3.5.7. One may iterate the construction of $\bar{K}$ in order to get a formally real differential field that does not have any strongly normal extension.

3.5.2 Differential Galois group

Theorem 3.5.8. *Let L/K be a strongly normal extension,* $\mathrm{Gal}(L/K)$ *is isomorphic to a subgroup of a projective limit of semialgebraic groups over $C_{\mathscr{U}}$. More precisely, let $\mathfrak{F} := \{F \subset L : F$ is a f.g.s.n. extensions of $K\}$,* $\mathrm{Gal}(L/K)$ *is isomorphic to a subgroup of* $\varprojlim_{F\in\mathfrak{F}} \mathrm{Gal}(F/K)$.

In [32], Chapter II, Theorem 1, a similar result is proved. The automorphism group considered in [32] is $\mathrm{gal}(\langle L, C_M\rangle/\langle K, C_M\rangle)$ where M is a model of DCF_0, so that result does not apply for our group $\mathrm{Gal}(L/K)$.

Proof. Note that $L = \bigcup_{F\in\mathfrak{F}} F$.

For any $F \in \mathfrak{F}$, $\mathcal{G}_F := \mathrm{Gal}(L/F)$ is a subgroup of $\mathcal{G} := \mathrm{Gal}(L/K)$. By the same argument as in the proof of Proposition 3.2.30, $\mathcal{G}_F$ is normal in $\mathcal{G}$ and there is an embedding $\mathcal{G}/\mathcal{G}_F \hookrightarrow \mathrm{Gal}(F/K)$. Hence the canonical projections $f_F : \mathcal{G} \to \mathcal{G}/\mathcal{G}_F$ induce homomorphisms $f_F : \mathcal{G} \to \mathrm{Gal}(F/K)$.

1. First note that if $F_1, F_2 \in \mathfrak{F}$ then $\mathcal{G}_{\langle F_1,F_2\rangle} = \mathcal{G}_{F_1} \cap \mathcal{G}_{F_2}$ and so $\mathcal{G}_{\langle F_1,F_2\rangle} \subseteq \mathcal{G}_{F_i}$ (for $i \in \{1, 2\}$). We will endow the family $(\mathcal{G}/\mathcal{G}_F)_{F\in\mathfrak{F}}$ of groups with a structure of projective system. If $F \subset F'$, we let $f_F^{F'} : \mathcal{G}/\mathcal{G}_{F'} \to \mathcal{G}/\mathcal{G}_F : g.G_{F'} \mapsto g.G_F$. Note that if $F_3 \subset F_2 \subset F_1 \in \mathfrak{F}$ then $f_{F_1}^{F_3} = f_{F_1}^{F_2} \circ f_{F_2}^{F_3}$ and $f_F^F = \mathrm{id}$.

   We also endow the family $(\mathrm{Gal}(F/K))_{F\in\mathfrak{F}}$ with a structure of projective system, taking the restrictions $r_F^{F'} : \mathrm{Gal}(F'/K) \to \mathrm{Gal}(F/K)$.

   Hence we may consider

   $$\varprojlim_{F\in\mathfrak{F}} \mathrm{Gal}(\mathcal{G}/\mathcal{G}_F) \text{ and } \varprojlim_{F\in\mathfrak{F}} \mathrm{Gal}(F/K).$$

   For short, we write those limits $\varprojlim \mathcal{G}/\mathcal{G}_F$ and $\varprojlim \mathrm{Gal}(F/K)$. As the following diagram commutes, $\varprojlim \mathcal{G}/\mathcal{G}_F$ is a subgroup of $\varprojlim \mathrm{Gal}(F/K)$.

   $$\begin{array}{ccc} \mathcal{G}/\mathcal{G}_{F'} & \hookrightarrow & \mathrm{Gal}(F'/K) \\ {\scriptstyle f_F^{F'}}\downarrow & & \downarrow{\scriptstyle r_F^{F'}} \\ \mathcal{G}/\mathcal{G}_F & \hookrightarrow & \mathrm{Gal}(F/K) \end{array}$$

2. Let

   $$\begin{aligned} f : \quad & \mathcal{G} \to \varprojlim \mathcal{G}/\mathcal{G}_F \\ & g \mapsto (g.\mathcal{G}_F)_{F\in\mathfrak{F}}. \end{aligned}$$

The map f is a group homomorphism and $\ker f = \bigcap_{f\in\mathfrak{F}} \mathcal{G}_F = \{g \in \mathcal{G} : \forall F \in \mathfrak{F}, g \in \mathcal{G}_F\} = \{\mathrm{id}_L\}$.

Hence f is an embedding and as $\varprojlim \mathcal{G}/\mathcal{G}_F \subseteq \varprojlim \mathrm{Gal}(F/K)$, we get the required embedding of $\mathcal{G}$ into $\varprojlim \mathrm{Gal}(F/K)$.

By Theorem 3.2.20, for all $F \in \mathfrak{F}$, $\mathrm{Gal}(F/K)$ is a semialgebraic group in $C_{\mathscr{U}}$.

□

Open Question

In [29], Theorem 2, Chapter III, E. Kolchin shows for a (finitely generated) strongly normal extension F/E where C_E is algebraically closed that $\mathrm{gal}(F/E)$ is isomorphic to an algebraic group over C_E. Moreover, he shows that the transcendence degree of F over E is the dimension of $\mathrm{gal}(F/E)$ as an algebraic group. It is a natural question to ask in our context of formally real differential fields whether there is a relation between the transcendence degree of a strongly normal extension L/K and the semialgebraic dimension of $\mathrm{gal}(L/K)$, denoted $\dim(\mathrm{gal}(L/K))$, where $\mathrm{gal}(L/K)$ is identified with the semialgebraic group whose it is isomorphic to. The best relation we hope is an equality.

Question 3.5.9. Let L/K be a strongly normal extension of transcendence degree d. Is that true that $\dim(\mathrm{gal}(L/K)) = d$?

Chapter 4

Definable types and VC-density

Introduction

In this chapter, we consider definable types. For a language $\mathcal{L}$, a subset A of an $\mathcal{L}$-structure M and $n \in \mathbb{N} \setminus \{0\}$, a type $p \in S_n^M(A)$ is said to be definable iff for any $\mathcal{L}$-formula $\phi(\bar{X}, \bar{Y})$, there is an $\mathcal{L}$-formula $d\phi(\bar{Y})$ with parameters in A such that for all $\bar{b} \in A$, $(\phi(\bar{X}, \bar{b}) \in p$ iff $M \models d\phi(\bar{b}))$. We call $d\phi$ a ϕ-definition of the type p.

Definable types were studied in various contexts, for instance in stable and o-minimal theories. A theory T is stable iff for any model M of T and any $n \in \mathbb{N} \setminus \{0\}$, it holds that any $p \in S_n^M(M)$ is definable.

A characterisation of definable types in o-minimal structures was given by D. Marker and C. Steinhorn in [38] and A. Pillay in [44]. That characterisation implies that even though an o-minimal theory is unstable, any type over a Dedekind complete model of an o-minimal theory is definable. It is a generalisation of a more particular statement proved by L. van den Dries in [15]: any type over the real field $(\mathbb{R}, +, \cdot, 0, 1)$ is definable.

Since then we know other examples of models M of unstable theories such that all types over M are definable: for instance the p-adic fields $\mathbb{Q}_p$ (see F. Delon [12]).

We show for the theory CODF of closed ordered differential fields that the type of a tuple $\bar{u} := (u_1, \ldots, u_n)$ over a real closed differential subfield A of a model of CODF is definable iff A is Dedekind complete in $A\langle\bar{u}\rangle^{rc}$. Then it follows that if $A = \mathbb{R}$, then any type over A is definable.

Let M be a model of CODF. By Remark 1.4.7, isolated types are in general not dense in $S_n^M(A)$. We show that definable types are dense in $S_n^M(A)$.

We then consider the dp-rank of types in CODF. We know that for any type p in CODF, dp-rk$(p) < \aleph_1$. We construct an element whose type has dp-rank $\aleph_0$. Therefore, CODF is not a strongly dependent theory (strong dependence is a generalisation of dp-minimality, see section 4.5.2 for the definitions).

We recall the definitions of the VC-dimension and the VC-density. We show how the VC-dimension and the VC-density of an $\mathcal{L}_{odf}$-formula ϕ in the theory CODF are related to the VC-dimension and the VC-density of ψ in RCF, where ψ is taken in such a way that

$$\text{CODF} \vdash \forall\bar{X}\forall\bar{Y}\big(\psi(\bar{X},\ldots,\bar{X}^{(k)};\bar{Y},\ldots,\bar{Y}^{(l)}) \leftrightarrow \phi(\bar{X};\bar{Y})\big),$$

for some $k, l \in \mathbb{N}$.

Conventions and notations

We fix a saturated model $\mathscr{U}$ of CODF and $A \subset \mathscr{U}$ such that $|A| < |\mathscr{U}|$. In particular, $\mathscr{U}$ is a model of the $\mathcal{L}_{of}$-theory RCF of real closed fields. So we will make use of $\mathscr{U}$ as a saturated model of RCF as well.

For any subfield K of $\mathscr{U}$ and $\bar{u} := (u_1, \ldots, u_n) \in \mathscr{U}$, $\bar{u} > K$ means $u_i > v$ (for all $i \in \{1, \ldots, n\}$, $v \in K$). The absolute value of an element u of an ordered field is denoted $|u|$. Moreover, K^{rc} denotes the real closure of K inside $\mathscr{U}$.

In this chapter, unless noted otherwise, $\mathcal{L}$-definable functions and $\mathcal{L}$-formulas are without parameters.

For a tuple $\bar{X} := (X_1, \ldots, X_n)$, we denote $|\bar{X}|$ the length n of $\bar{X}$. We will deal with formulas involving two tuples of variables playing different roles. Such formulas are called partitioned formulas, we emphasise this separation with a sumicolon between the variables. In a partitioned formula $\phi(\bar{X};\bar{Y})$, the variables of $\bar{X}$ are thought as object variables while the ones of $\bar{Y}$ are thought as parameters variables. We always assume that the variables of $\bar{X}$ and $\bar{Y}$ are pairwise distinct.

Let χ be a partitioned $\mathcal{L}_{odf}$-formula of the shape $\bigvee_i \bigwedge_j p_{ij}(\bar{X};\bar{Y})\square_{ij}0$ for some $p_{ij} \in \mathbb{Q}\{\bar{X};\bar{Y}\}$ and $\square_{ij} \in \{=,\neq,>,\geq,<,\leq\}$. We denote $\chi^*(\bar{W};\bar{Z})$ the following partitioned $\mathcal{L}_{of}$-formula:

$$\bigvee_i \bigwedge_j p^*_{ij}(\bar{W};\bar{Z})\square_{ij}0.$$

Then χ^* is a partitioned $\mathcal{L}_{of}$-formula such that

$$\text{ODF} \vdash \forall\bar{X}\forall\bar{Y}\big(\chi^*(\bar{X},\ldots,\bar{X}^{(k)};\bar{Y},\ldots,\bar{Y}^{(l)}) \leftrightarrow \chi(\bar{X};\bar{Y})\big)$$

for some $k, l \in \mathbb{N}$ which will be assumed to be minimal in the sequel.

Remark 4.0.1. For any partitioned $\mathcal{L}_{odf}$-formula $\phi(\bar{X};\bar{Y})$ there is a partitioned $\mathcal{L}_{of}$-formula $\psi(\bar{W};\bar{Z})$ such that

$$\mathrm{CODF} \vdash \forall \bar{X} \forall \bar{Y} \big(\psi(\bar{X},\dots,\bar{X}^{(k)};\bar{Y},\dots,\bar{Y}^{(l)}) \leftrightarrow \phi(\bar{X};\bar{Y})\big),$$

for some $k, l \in \mathbb{N}$.

Proof. Since CODF eliminates quantifiers, then ϕ is equivalent to a quantifier-free $\mathcal{L}_{odf}$-formula ξ. Since ξ is quantifier-free, there are finitely many differential polynomials $p_{ij} \in \mathbb{Q}\{\bar{X};\bar{Y}\}$ and $\square_{ij} \in \{=,\neq,>,\geq,<,\leq\}$ such that

$$\mathrm{ODF} \vdash \forall \bar{X} \forall \bar{Y} \Big(\xi(\bar{X};\bar{Y}) \leftrightarrow \bigvee_i \bigwedge_j p_{ij}(\bar{X};\bar{Y}) \square_{ij} 0\Big).$$

Letting $\chi :=$

$$\bigvee_i \bigwedge_j p_{ij}(\bar{X};\bar{Y}) \square_{ij} 0,$$

and $\psi := \chi^*$, we get the required equivalence in CODF:

$$\psi(\bar{X},\dots,\bar{X}^{(k)};\bar{Y},\dots,\bar{Y}^{(l)}) \leftrightarrow \phi(\bar{X};\bar{Y}).$$

$\square$

Note that

- the same thing may be done with formulas which are not partitioned;
- in the proof above, p_{ij} and $\square_{ij}$ are not unique, so neither is the formula ψ. Actually, ψ is not even unique up to equivalence in the theory RCF. For instance, if we take $X \neq 0 \vee X' = 0$ for the formula $\phi(X)$ then one may take $W_0 \neq 0 \vee W_1 = 0$ or $1 = 1$ for the formula $\psi(\bar{W})$.

Let $\bar{u} \in \mathscr{U}$ and $B \subset \mathscr{U}$. We write $\mathrm{tp}_<(\bar{u}/B)$ for $\mathrm{tp}_{\{<\}}(\bar{u}/B)$, i.e., the order type of $\bar{u}$ in $\mathscr{U}$ over B. Recall that since we fixed $\mathscr{U}$ which is saturated, if $|B| < |\mathscr{U}|$, then $S_n^{\mathscr{U}}(B) = \{\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/B) : \bar{u} \in \mathscr{U}$ and the length of $\bar{u}$ is $n\}$, so we use the notation $S_n^{\mathrm{CODF}}(B)$ for $S_n^{\mathscr{U}}(B)$. Moreover, we make use of $\mathscr{U}$ as a saturated model of RCF and we write $S_n^{\mathrm{RCF}}(B)$ for $\{\mathrm{tp}_{\mathcal{L}_{of}}(\bar{u}/B) : \bar{u} \in \mathscr{U}$ and the length of $\bar{u}$ is $n\}$.

We denote $\mathrm{dcl}^{\mathscr{U}}(B)$ the definable closure of B in $\mathscr{U}$ for the language $\mathcal{L}_{odf}$. Recall that $\mathrm{dcl}^{\mathscr{U}}(B) = \langle B \rangle^{rc}$ (see section 1.4.2).

Note that $\mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{of}}(\bar{u}/A) \vdash \mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{of}}(\bar{u}/\operatorname{dcl}(A))$. Therefore, if $\mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{of}}(\bar{u}/\operatorname{dcl}(A))$ is definable then $\mathrm{tp}^{\mathscr{U}}_{\mathcal{L}_{of}}(\bar{u}/A)$ is definable. So we make the assumption that $A = \operatorname{dcl}^{\mathscr{U}}(A)$, equivalently, A is a real closed subfield of $\mathscr{U}$ which is a differential subfield as well.

We will make use of the fact that any model of CODF is definably complete (see Proposition 2.2 of [46]). This property of our model $\mathscr{U}$ of CODF states as follows: every non-empty $\mathcal{L}_{odf}(\mathscr{U})$-definable subset D of $\mathscr{U}$ has a supremum in $\mathscr{U} \cup \{+\infty\}$. Moreover, since $A = \operatorname{dcl}^{\mathscr{U}}(A)$, if D is $\mathcal{L}_{odf}(A)$-definable then the supremum of D belongs to $A \cup \{+\infty\}$.

4.1 On cuts and definable types in real closed fields

In this section, we review a result on definable types in o-minimal structures presented in [38]. In particular, that result applies to types in the $\mathcal{L}_{of}$-theory RCF of real closed fields. We will state it in that context which is the one we are interested in. We also give the definitions and some basic properties of cuts and Dedekind completeness.

Throughout this section, we work inside $\mathscr{U}$ and use it as a saturated model of RCF. We borrow the following definition of cut from [38].

Definition 4.1.1. Let $K \subseteq L$ be ordered fields and $u \in L$. We say $\mathrm{tp}_<(u/K)$ is a cut of K iff there are nonempty subsets C_1, C_2 of K such that

- $C_1 \cup C_2 = K$;
- C_1 has no greatest element and C_2 has no least element;
- for any $c_1 \in C_1$ and $c_2 \in C_2$, $c_1 < u < c_2$.

We will say that u is a cut of K iff $\mathrm{tp}_<(u/K)$ is a cut of K.

Definition 4.1.2. Let $K \subseteq L$ be ordered fields. Then K is Dedekind complete in L iff no cut of K is realised in L.

Definition 4.1.3. Let $K \subseteq L$ be ordered fields and $\epsilon \in L$. We say that ϵ is K-infinitesimal if for all $v \in K$ it holds that if $0 < v$ then $|\epsilon| < v$.

Lemma-Definition 4.1.4. Let $K \subseteq L$ be ordered fields and $u \in L$. If u is not a cut of K and $|u| \not> K$, then there exist a unique $v \in K$ and a unique K-infinitesimal $\epsilon \in L$ such that $u = v + \epsilon$. We say that the standard part of u in K is v and we write $\mathrm{st}_K(u) := v$.

Proof. This follows from the definitions given above. □

When there is no possible confusion, we write $\mathrm{st}(u)$ for $\mathrm{st}_K(u)$. Note that $\mathrm{st}(u) = 0$ iff u is K-infinitesimal and $\mathrm{st}(u) = u$ iff $u \in K$. Moreover, if we denote $L_{st} := \{u \in L : |u| \not> K \text{ and } u \text{ is not a cut of } K\}$, then L_{st} is a subring of L and the map $L_{st} \to K : u \mapsto \mathrm{st}(u)$ is a ring homomorphism.

We will often say that u has a standard part in K instead of u belongs to L_{st}.

From now to the end of the section, we assume that K is a real closed subfield of $\mathscr{U}$ and $|K| < |\mathscr{U}|$.

D. Marker and C. Steinhorn's characterisation of definable types states as follows in the case of the theory RCF:

Theorem 4.1.5 (see [38], Theorem 2.1)**.** *Let $n \in \mathbb{N} \setminus \{0\}$ and $p \in S_n^{\mathrm{RCF}}(K)$. Then p is definable iff for any realisation $\bar{u} \in \mathscr{U}$ of p, it is the case that K is Dedekind complete in $K(\bar{u})^{rc}$.*

Remark 4.1.6. For an element $u \in \mathscr{U}$, K is Dedekind complete in $K(u)^{rc}$ iff u is not a cut of K. Moreover, for all $p \in S_1^{\mathrm{RCF}}(K)$, p is definable iff for any realisation $u \in \mathscr{U}$ of p, u is not a cut of K.

Lemma 4.1.7. *Let $K \subseteq L \subseteq M$ be real closed fields. If K is Dedekind complete in L and L is Dedekind complete in M then K is Dedekind complete in M.*

Corollary 4.1.8. *Let K be a real closed field, let t_i where $i \in \mathbb{N}$ be such that for all $j \in \mathbb{N}$, t_j is $K(t_i : i < j)^{rc}$-infinitesimal, then K is Dedekind complete in $K(t_i : i \in \mathbb{N})^{rc}$.*

Proof. Follows from Remark 4.1.6 and Lemma 4.1.7. □

4.2 Differential cuts

Definition 4.2.1. Let $u \in \mathscr{U}$. We say that $\mathrm{tp}_{\mathcal{L}_{odf}}(u/A)$ is a differential cut of A (shortly, we call it a d-cut of A) iff there is $k \in \mathbb{N}$ such that $\mathrm{tp}_{<}(u^{(k)}/A)$ is a cut of A.

The map $L_{st} \to K : u \mapsto \mathrm{st}(u)$ is generally not a differential ring homomorphism, i.e, for some $u \in L_{st}$, it may hold that $\mathrm{st}(u)' \neq \mathrm{st}(u')$. Actually, u' does not even need to belong to L_{st}. In other words, L_{st} is not a differential ring and so $\mathrm{st}(u')$ does not exist for all $u \in L_{st}$.

Remark 4.2.2. Suppose that for all $n \in \mathbb{N}$, there is $v \in A$ such that $|u^{(n)}| < v$. Then $\mathrm{tp}_{\mathcal{L}_{odf}}(u/A)$ is a d-cut of A if and only if there is $f \in A\{X\}$ such that $\mathrm{tp}_<(f(u)/A)$ is a cut of A.

Proof. [$\Rightarrow$] Follows from the definition of d-cut.

[$\Leftarrow$] Note that for $x \in \mathscr{U}$, $\mathrm{tp}_<(x/A)$ is not a cut iff x has a standard part $\mathrm{st}(x)$ in A or $|x| > A$.

So if $\mathrm{tp}_{\mathcal{L}_{odf}}(u/A)$ is not a d-cut of A, then for all $n \in \mathbb{N}$, $u^{(n)}$ has a standard part in A. Since $x \mapsto \mathrm{st}(x)$ is a ring homomorphism fixing A pointwise, it follows that for all $f \in A\{X\}$, $\mathrm{st}(f(u)) = f^*(\mathrm{st}(u), \mathrm{st}(u'), \ldots, \mathrm{st}(u^{(k)}))$ for some $k \in \mathbb{N}$. So $f(u)$ is not a cut of A since it has a standard part in A. □

Note that in Remark 4.2.2, the hypothesis that $|u^{(n)}| < v$ is necessary. Take $u > A$ such that $u' = u - c$ where c is a cut of A and $u^{(n)} = 0$ for $n \geq 2$. Clearly $\mathrm{tp}_{\mathcal{L}_{odf}}(u/A)$ is not a d-cut of A. Let $f(X) := X - X'$. Then $f(u) = c$ is a cut. Assuming that A is not Dedekind complete in $\mathscr{U}$, one shows the existence of such a u, using the axioms of CODF and the compactness theorem.

4.3 Definable types in CODF

Lemma 4.3.1. *Let $p \in S_1^{\mathrm{CODF}}(A)$. If p is definable then p is not a d-cut of A.*

Proof. Let $n \in \mathbb{N}$ and let the formula $\phi(X, b)$ be $b < X^{(n)}$. Then there is an $\mathcal{L}_{odf}(A)$-formula $d\phi$ such that $\{b \in A : \phi(X, b) \in p\} = d\phi(\mathscr{U}) \cap A$. By Proposition 2.2 from [46], $\mathscr{U}$ is definably complete. Either $d\phi(\mathscr{U})$ is empty or $d\phi(\mathscr{U})$ has a supremum in $\mathrm{dcl}^{\mathscr{U}}(A) = A$ or $d\phi(\mathscr{U})$ is not bounded. So let a be a realisation of p then $\mathrm{tp}_<(a^{(n)}/A)$ may not be a cut.

So there is no $n \in \mathbb{N}$ such that $\mathrm{tp}_<(a^{(n)}/A)$ is a cut of A, namely p is not a d-cut. □

Lemma 4.3.2. *Let $\bar{u} \in \mathscr{U}$ such that $\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$ is definable. Then A is Dedekind complete in the real closure of $A\langle\bar{u}\rangle$.*

Proof. This proof is similar to the one of the analogue statement for o-minimal structures (see [38], Corollary 2.4).

Suppose to the contrary that there is an element a in the real closure K of $A\langle\bar{u}\rangle$ which realises a cut, i.e. $C_1 < a < C_2$, where C_1, C_2 are as in definition 4.1.1. As the real closure is prime and so atomic over $A\langle\bar{u}\rangle$, $\mathrm{tp}^K_{\mathcal{L}_{of}}(a/A\langle\bar{u}\rangle)$ is isolated by an $\mathcal{L}_{of}$-formula $\psi(X, \bar{b}, \bar{\mathbf{u}})$, where $\bar{b} \in A$ and $\bar{\mathbf{u}} \in \bigcup_{n\in\mathbb{N}} \bar{u}^{(n)}$. In particular for all $c_1 \in C_1, c_2 \in C_2$,

$$\mathscr{U} \models (\psi(X, \bar{b}, \bar{\mathbf{u}}) \rightarrow c_1 < X < c_2).$$

Moreover, RCF has definable Skolem functions (see Theorem 1.3.12), and so there exists an $\mathcal{L}_{of}$-definable function f such that $K \models \psi(f(\bar{b}, \bar{\mathbf{u}}), \bar{b}, \bar{\mathbf{u}})$. Let $g(\bar{b}, \bar{u}) := f(\bar{b}, \bar{\mathbf{u}})$. Then g is an $\mathcal{L}_{odf}$-definable function and $g(\bar{b}, \bar{u})$ realises the cut $C_1 < X < C_2$, so $p := \mathrm{tp}_{\mathcal{L}_{odf}}(g(\bar{b}, \bar{u})/A)$ is a cut of A.

Let $\phi(X, \bar{Y})$ be an $\mathcal{L}_{odf}$-formula and $\bar{d} \in A$ then

$$\phi(X, \bar{d}) \in p(X) \text{ iff } \phi(g(\bar{b}, \bar{Y}), \bar{d}) \in \mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A).$$

As $\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$ is definable, there is a formula $\theta(\bar{X}, \bar{Z})$ such that for any $\bar{z}, \bar{v} \in A$, $\mathscr{U} \models \theta(\bar{z}, \bar{v})$ iff $\phi(g(\bar{z}, \bar{Y}), \bar{v}) \in \mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$. Therefore $\mathscr{U} \models \theta(\bar{b}, \bar{d})$ iff $\phi(X, \bar{d}) \in p(X)$, namely p is definable.

But by Lemma 4.3.1, as p is a cut, p may not be definable. A contradiction. □

Lemma 4.3.3. *Let $\bar{u} \in \mathscr{U}$. Suppose A is Dedekind complete in the real closure of $A\langle \bar{u} \rangle$. Then $\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$ is definable.*

Proof. In particular, for any $n \in \mathbb{N}$, A is Dedekind complete in the real closure of $A(\bar{u}, \ldots, \bar{u}^{(n)})$. One may apply Theorem 4.1.5, so $\mathrm{tp}_{\mathcal{L}_{of}}(\bar{u}, \ldots, \bar{u}^{(n)}/A)$ is definable in $\mathcal{L}_{of}$. More precisely, for any $\mathcal{L}_{of}$-formula $\psi(\bar{X}_0, \ldots, \bar{X}_n, \bar{Z})$ there is an $\mathcal{L}_{of}$-formula $d\psi(\bar{Z})$ such that it holds that for all $\bar{d} \in A$,

$$\psi(\bar{X}_0, \ldots, \bar{X}_n, \bar{d}) \in \mathrm{tp}_{\mathcal{L}_{of}}(\bar{u}, \ldots, \bar{u}^{(n)}/A) \text{ iff } \mathscr{U} \models d\psi(\bar{d}).$$

Let $\chi(\bar{X}, \bar{Y})$ be an $\mathcal{L}_{odf}$-formula and $\bar{b} \in A$. By Remark 4.0.1, for some $k, l \in \mathbb{N}$ and $\bar{d} := (\bar{b}, \ldots, \bar{b}^{(l)})$, there is an $\mathcal{L}_{of}$-formula ψ such that

$$\mathrm{CODF} \vdash \forall \bar{X} \big(\psi(\bar{X}, \ldots, \bar{X}^{(k)}, \bar{d}) \leftrightarrow \chi(\bar{X}, \bar{b}) \big).$$

Hence, the following equivalences hold

$$\begin{aligned} &\chi(\bar{X}, \bar{b}) \in \mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A) \\ \text{iff } &\psi(\bar{X}_0, \ldots, \bar{X}_k, \bar{d}) \in \mathrm{tp}_{\mathcal{L}_{of}}(\bar{u}, \ldots, \bar{u}^{(k)}/A) \\ \text{iff } &\mathscr{U} \models d\psi(\bar{d}) \\ \text{iff } &\mathscr{U} \models d\psi(\bar{b}, \ldots, \bar{b}^{(l)}). \end{aligned}$$

Namely, $d\psi(\bar{X}, \ldots, \bar{X}^{(l)})$ is a χ-definition of the type $\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$. □

Then the following statement follows from Lemma 4.3.2 and Lemma 4.3.3.

Corollary 4.3.4. *Let $\bar{u} \in \mathscr{U}$. Then $\mathrm{tp}_{\mathcal{L}_{odf}}(\bar{u}/A)$ is definable if and only if A is Dedekind complete in the real closure of $A\langle \bar{u} \rangle$.*

4.4 Density of definable types

We consider the topology of $S_n^{\mathrm{CODF}}(A)$ defined in section 1.1.

Lemma 4.4.1. *Definable* 1*-types are dense in* $S_1^{\mathrm{CODF}}(A)$.

Proof. Let ϕ be an $\mathcal{L}_{odf}$-formula with parameters from A such that $[\phi]$ is non-empty, equivalently, such that there is $u \in \mathscr{U}$ such that $\mathscr{U} \models \phi(u)$. We will show that there is $v \in L$ for some differential real closed extension L of A such that $\mathscr{U} \models \phi(v)$ and K is Dedekind complete in L. By Corollary 4.3.4, the type of v is definable.

By quantifier elimination in CODF, there are $p_{ik},\ q_{jk} \in A\{X\}$ such that $\phi(X)$ is equivalent (in $\mathscr{U}$) to $\bigvee_k(\bigwedge_i p_{ik}(X) = 0 \wedge \bigwedge_j q_{jk}(X) > 0)$. Without loss of generality, one may assume that $\phi(X)$ is equivalent to

$$\bigwedge_i p_i(X) = 0 \wedge \bigwedge_j q_j(X) > 0$$

for some $p_i, q_j \in A\{X\}$.

Below we write $\mathcal{I}(u)$ of $\mathcal{I}_A(u)$.

Suppose that $\mathcal{I}(u) \neq 0$. As $\mathcal{I}(u)$ is a non-zero prime differential ideal of $A\{X\}$, by Lemma 1.2.7, there is $f \in A\{X\}$ such that

$$\mathcal{I}(u) = I(f) := \{g \in A\{X\} : g \cdot s_f^l \in [f] \text{ for some } l \in \mathbb{N}\}.$$

Moreover, $s_f(u) \neq 0$ and $p_i \in \mathcal{I}(u)$. So

$$\mathscr{U} \models \forall X\Big(f(X) = 0 \wedge s_f(X) \neq 0 \wedge \bigwedge_j q_j(X) > 0 \rightarrow \phi(X)\Big).$$

Let d be the order of f. By Lemma 1.2.8, for all $z \in \mathscr{U}$, $f(z) = 0$ implies that for any $l \in \mathbb{N}$, $z^{(l)} \in A(z, z^{(1)}, \ldots, z^{(d)})$. Therefore there are differential polynomials $\tilde{q}_k$ of order $\leq d$ such that $f(X) = 0 \wedge s_f(X) \neq 0 \wedge \bigwedge_j q_j(X) > 0$ is equivalent to

$$f(X) = 0 \wedge s_f(X) \neq 0 \wedge \bigwedge_k \tilde{q}_k(X) > 0.$$

Let $\bar{u} := (u, u', u^{(2)}, \ldots, u^{(d)})$, then

$$\mathscr{U} \models f^*(\bar{u}) = 0 \wedge s_f^*(\bar{u}) \neq 0 \wedge \bigwedge_k \tilde{q}_k^*(\bar{u}) > 0.$$

As RCF is model complete there is $\bar{a} := (a_0, \ldots, a_d) \in A$ such that $A \models f^*(\bar{a}) = 0 \wedge s^*_f(\bar{a}) \neq 0 \wedge \bigwedge_k \tilde{q}^*_k(\bar{a}) > 0$.

Let $t_0, \ldots, t_{d-1}$ be algebraically independent over A and such that for all $j \in \{0, \ldots, d-1\}$, t_j is $A(t_0, \ldots, t_{j-1})^{rc}$-infinitesimal. By Lemma 1.4.2, there is a derivation on $L := A(t_0, \ldots, t_{d-1})^{rc}$ such that for some $v \in L$, $L \models f(v) = 0 \wedge s_f(v) \neq 0 \wedge \bigwedge_k \tilde{q}_k(v) > 0$. Since $\mathscr{U}$ is saturated, by Lemma 1.1.10, we may identify L with a subfield of $\mathscr{U}$. Then

$$\mathscr{U} \models f(v) = 0 \wedge s_f(v) \neq 0 \wedge \bigwedge_k \tilde{q}_k(v) > 0$$

which implies that $\mathscr{U} \models \phi(v)$.

Since for all j, t_j is $A(t_0, \ldots, t_{j-1})^{rc}$-infinitesimal, by Corollary 4.1.8, A is Dedekind complete in L.

Suppose $\mathcal{I}(u) = 0$. Then $\phi(X)$ is equivalent to $\bigwedge_j q_j(X) > 0$. By model completeness of RCF, there is some $\bar{a} := (a_0, \ldots, a_n) \in A$ such that $A \models \bigwedge_j q^*_j(\bar{a}) > 0$. Now we use the axioms of CODF to show that the system $(X^{(n)} = a_n) \bigwedge_j q_j(X) > 0$, has a solution w in $\mathscr{U}$. Since $\mathcal{I}(w) \neq 0$, it works now exactly as in the first case.

□

Lemma 4.4.2. *Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory. Let L be a model of T, $E \subseteq L$, $\bar{u}, v \in L$ and $D := \mathrm{dcl}^L(E, \bar{u})$. If $\mathrm{tp}(v/D)$ is definable and $\mathrm{tp}(\bar{u}/E)$ is definable then $\mathrm{tp}(v\bar{u}/E)$ is definable.*

Proof. For any $\mathcal{L}$-formula $\phi(X\bar{Y}, \bar{Z})$, we will show that there is a ϕ-definition of $\mathrm{tp}(v\bar{u}/E)$.

Since $\mathrm{tp}(v/D)$ is definable, there is an $\mathcal{L}(D)$-formula $\phi_1(\bar{Z})$ such that for all $\bar{a} \in D$,

$$L \models \phi_1(\bar{a}) \text{ iff } \phi(X\bar{u}, \bar{a}) \in \mathrm{tp}(v/D) \text{ iff } \phi(X\bar{Y}, \bar{a}) \in \mathrm{tp}(v\bar{u}/D).$$

We rewrite $\phi_1(\bar{Z})$ as $\phi_2(\bar{Z}, \bar{d})$ where ϕ_2 is an $\mathcal{L}$-formula and $\bar{d} \in D$. Because $D = \mathrm{dcl}^L(E, \bar{u})$ there is a tuple of $\mathcal{L}$-definable functions $\bar{f}$ such that $\bar{d} = \bar{f}(\bar{e}, \bar{u})$ for some $\bar{e} \in E$. Then

$$L \models \phi_2(\bar{a}, \bar{f}(\bar{e}, \bar{u})) \text{ iff } \phi(X\bar{u}, \bar{a}) \in \mathrm{tp}(v/D).$$

Suppose now that $\bar{a} \in E$, then

$$L \models \phi_2(\bar{a}, \bar{f}(\bar{e}, \bar{u})) \text{ iff } \phi_2(\bar{a}, \bar{f}(\bar{e}, \bar{W})) \in \mathrm{tp}(\bar{u}/E).$$

Since $\mathrm{tp}(\bar{u}/E)$ is definable, then there is an $\mathcal{L}(E)$-formula ϕ_3 such that for all $\bar{a} \in E$,

$$\phi_2(\bar{a}, \bar{f}(\bar{e}, \bar{W})) \in \mathrm{tp}(\bar{u}/E) \text{ iff } L \models \phi_3(\bar{a}, \bar{e}).$$

Finally, we get for all $\bar{a} \in E$,

$$L \models \phi_3(\bar{a}, \bar{e}) \text{ iff } \phi(X\bar{Y}, \bar{a}) \in \mathrm{tp}(v\bar{u}/D),$$

namely $\phi_3(\bar{Z}, \bar{e})$ is a ϕ-definition of $\mathrm{tp}(v\bar{u}/E)$. □

Lemma 4.4.3. *Let T be an $\mathcal{L}$-theory, $\mathcal{M}$ be a saturated model of T. If for all $A \subset \mathcal{M}$ such that $|A| < |\mathcal{M}|$ it holds that definable 1-types are dense in $S_1^{\mathcal{M}}(A)$ then for all $A \subset \mathcal{M}$ such that $|A| < |\mathcal{M}|$ and all $n \in \mathbb{N}$, definable n-types are dense in $S_n^{\mathcal{M}}(A)$.*

Proof. We proceed by induction on n. Suppose that the lemma is true when $n = k$ for some $k \in \mathbb{N} \setminus \{0\}$.

Let the length of $\bar{X}$ be k and $\phi(\bar{X}, Y)$ be a formula with parameters in A which is realised in $\mathcal{M}$ (i.e. the open set $[\phi]$ is nonempty).

The formula $\exists Y \phi(\bar{X}, Y)$ is realised in $\mathcal{M}$ and so by induction hypothesis, there is $\bar{u} \in \mathcal{M}$ such that

- $\mathscr{U} \models \exists Y \phi(\bar{u}, Y)$;
- $\mathrm{tp}(\bar{u}/A)$ is definable.

The formula $\phi(\bar{u}, Y)$ is consistent and with parameters in $\mathrm{dcl}^{\mathcal{M}}(A, \bar{u})$.

By the hypothesis of density in $S_1^{\mathcal{M}}(A)$, there is v in $\mathcal{M}$ such that

- $\mathcal{M} \models \phi(\bar{u}, v)$;
- $\mathrm{tp}(v/\,\mathrm{dcl}^{\mathcal{M}}(A, \bar{u}))$ is definable.

By Lemma 4.4.2, $\mathrm{tp}(\bar{u}, v/A)$ is definable. Since $\phi(\bar{u}, v)$ holds in $\mathcal{M}$, $\mathrm{tp}(\bar{u}, v/A)$ belongs to the open subset $[\phi]$ of $S_{k+1}^{\mathcal{M}}(A)$. □

Corollary 4.4.4. *Let $n \in \mathbb{N} \setminus \{0\}$. Definable n-types are dense in $S_n^{\mathrm{CODF}}(A)$.*

Proof. This follows immediately from Lemma 4.4.1 and Lemma 4.4.3. □

4.5 Non Independence Property

The non independence property (NIP) of a theory first appeared in S. Shelah's works on stability theory.

Many well-studied theories have NIP. For instance o-minimal and stable theories which were believed to be fundamentally different have NIP. So the study of NIP theories provides a common approach to the study of both o-minimal and stable theories. Other examples of NIP theories are known, for instance weakly o-minimal and C-minimal theories.

We refer the reader to P. Simon's monograph [64] for a detailed text on NIP theories.

Among many others, one of the interests of NIP theories is that the complexity of a formula is measured by the VC-dimension and the VC-density (definitions below).

Let T be a complete $\mathcal{L}$-theory with no finite model and $\mathcal{M}$ be a saturated model of T.

4.5.1 VC-dimension

Definition 4.5.1. Let $\phi(\bar{X};\bar{Y})$ be a partitioned $\mathcal{L}$-formula and $E \subseteq \mathcal{M}^{|\bar{X}|}$. We say that E is shattered by ϕ iff for all $E_0 \subseteq E$, there is $\bar{b} \in \mathcal{M}$ such that for all $\bar{a} \in E$,

$$\mathcal{M} \models \phi(\bar{a};\bar{b}) \text{ iff } \bar{a} \in E_0.$$

Definition 4.5.2. Let $\phi(\bar{X};\bar{Y})$ be a partitioned $\mathcal{L}$-formula and $n \in \mathbb{N}$. We say that the VC-dimension of $\phi(\bar{X};\bar{Y})$ in T is n if ϕ shatters a set $E \subseteq \mathcal{M}^{|\bar{X}|}$ of cardinality $|E| = n$ and no set $E \subseteq \mathcal{M}^{|\bar{X}|}$ of cardinality $|E| > n$ is shattered by ϕ. If ϕ shatters arbitrary large sets then we say that the VC-dimension of ϕ is infinite.

Note that the definition above does not depend on the saturated model $\mathcal{M}$ we work with. When there is no possible confusion, we do not always make explicit mention of which theory we consider the VC-dimension in. The VC-dimension of ϕ is denoted VC-dim ϕ.

Definition 4.5.3. The theory T does not have the independence property (for short, we say T has NIP) iff any partitioned $\mathcal{L}$-formula $\phi(\bar{X};\bar{Y})$ has finite VC-dimension.

Theorem 4.5.4 (See [42], Théorème 2.2)**.** *The theory CODF has NIP.*

Let ϕ be a partitioned $\mathcal{L}_{odf}$-formula. There is, by Remark 4.0.1, a partitioned $\mathcal{L}_{of}$-formula ψ such that

$$\text{CODF} \vdash \forall \bar{X} \forall \bar{Y} \big(\psi(\bar{X}, \dots, \bar{X}^{(k)}; \bar{Y}, \dots, \bar{Y}^{(l)}) \leftrightarrow \phi(\bar{X}; \bar{Y}) \big),$$

for some $k, l \in \mathbb{N}$. We will consider VC-dim ϕ in CODF and VC-dim ψ in RCF. Recall that we will make use of $\mathscr{U}$ as a saturated model of CODF and RCF at the same time.

Lemma 4.5.5. VC-dim $\phi \leq$ VC-dim ψ.

Proof. Let $E \subseteq \mathcal{M}^{|\bar{X}|}$. Suppose E is shattered by ϕ, i.e., for all $E_0 \subseteq E$, there is $\bar{b} \in \mathscr{U}$ such that for all $\bar{a} \in E$, $\mathscr{U} \models \phi(\bar{a}; \bar{b})$ iff $\bar{a} \in E_0$.

Let $\tilde{E} := \{(\bar{a}, \bar{a}', \dots, \bar{a}^{(k)}) : \bar{a} \in E\}$, then $|\tilde{E}| = |E|$. Let us show that $\tilde{E}$ is shattered by ψ.

Let $\tilde{E}_0$ be a subset of $\tilde{E}$, there is $E_0 \subseteq E$ such that

$$\tilde{E}_0 = \{(\bar{a}, \bar{a}', \dots, \bar{a}^{(k)}) : \bar{a} \in E_0\}.$$

For all $\bar{a} \in E$,

$$\begin{aligned} \mathscr{U} \models \psi(\bar{a}, \bar{a}', \dots, \bar{a}^{(k)}; \bar{b}, \bar{b}', \dots, \bar{b}^{(l)}) & \text{ iff } \mathscr{U} \models \phi(\bar{a}; \bar{b}) \\ & \text{ iff } \bar{a} \in E_0 \\ & \text{ iff } (\bar{a}, \bar{a}', \dots, \bar{a}^{(k)}) \in \tilde{E}. \end{aligned}$$

□

4.5.2 dp-rank

We recall the definitions of indiscernible sequence and dp-rank. For more details, we refer the reader to [64], chapter 4.

Definition 4.5.6. Let S be a totally ordered set and let B be a subset of $\mathcal{M}$. A sequence $I := (a_n : n \in S)$ of elements of $\mathcal{M}$ is indiscernible over B iff for any $k \in \mathbb{N} \setminus \{0\}$, for any $m_1, \dots, m_k, n_1, \dots, n_k \in S$ such that $m_1 < \dots < m_k$ and $n_1 < \dots < n_k$, any $\mathcal{L}_{odf}$-formula $\phi(\bar{X}, \bar{Y})$ and any $\bar{b} \in B$ it holds that

$$\mathcal{M} \models \phi(a_{m_1}, \dots, a_{m_k}, \bar{b}) \text{ iff } \mathcal{M} \models \phi(a_{n_1}, \dots, a_{n_k}, \bar{b}).$$

Definition 4.5.7. Let κ be a cardinal. Sequences $I_i, i < \kappa$ of elements of $\mathcal{M}$ are mutually indiscernible over B iff for all $l < \kappa$, I_l is indiscernible over $B \cup \bigcup_{i<\kappa, i \neq l} I_i$.

By a partial type p over B, we mean a subset of a type $q \in S_n^{\mathcal{M}}(B)$.

Definition 4.5.8 (dp-rank)**.** Let p be a partial type over B and κ be a cardinal then dp-rk$(p/B) < \kappa$ iff for any sequences $I_i, i < \kappa$ which are mutually indiscernible over B and any realisation $\bar{u}$ of p, there is $i < \kappa$ such that I_i is indiscernible over $B \cup \bar{u}$.

Let $\bar{u} \in \mathcal{M}$, we write dp-rk$(\bar{u}/B)$ for dp-rk$(\mathrm{tp}_{\mathcal{L}}^{\mathcal{M}}(\bar{u}/B))$. Note that if $p \subseteq q$ then dp-rk$(p/B) \geq$ dp-rk(q/B).

The theory CODF has NIP (Theorem 4.5.4) and has a countable axiomatisation (section 1.4). Then by Observation 4.13 from [64], for any partial type $p \subseteq q \in S_n^{\mathscr{U}}(B)$, dp-rk$(p/B) < \aleph_1$. Moreover, there is no better bound on dp-rk(p/B) in CODF as shown by the following lemma:

Lemma 4.5.9. *There is* $u \in \mathscr{U}$ *such that* dp-rk$(u/\varnothing) \geq \aleph_0$.

Proof. Since $\mathrm{dcl}^{\mathscr{U}}(\varnothing) = \mathbb{Q}^{rc}$, indiscernible sequences over $\varnothing$ are exactly indiscernible sequences over $\mathbb{Q}^{rc}$ and so for any $u \in \mathscr{U}$, dp-rk$(u/\varnothing) =$ dp-rk$(u/\mathbb{Q}^{rc})$.

Let S be a countable ordered set which is supposed to contain 0 and 1 and such that $0 < 1$. Let R be the real closure of $\mathbb{Q}^{rc}(t_{ij}, u_i : i \in \aleph_0, j \in S)$ where t_{ij}, u_j are algebraically independent over $\mathbb{Q}^{rc}$. We let $t'_{ij} = t_{ij}$, $u'_i = u_{i+1}$ and $r' = 0$ when $r \in \mathbb{Q}^{rc}$. We also let

- $\mathbb{Q}^{rc} < t_{ij}$;
- if $(i, j) < (k, l)$ with respect to lexicographical order then $t_{ij} < t_{kl}$;
- $t_{i0} < u_i < t_{i1}$.

It determines a structure of ordered differential field on

$$\mathbb{Q}^{rc}(t_{ij}, u_i : i \in \aleph_0, j \in S).$$

The derivation extends to R in a unique way (Lemma 1.2.11). Since $\mathscr{U}$ is saturated and R is countable then there is a copy of R in $\mathscr{U}$. We identify this copy with R.

For $i \in \aleph_0$, let I_i be the sequence $(t_{ij} : j \in S)$.

Let $l \in \aleph_0$ and $F_l := \mathrm{dcl}^{\mathscr{U}}(\mathbb{Q}^{rc} \cup \bigcup_{i \in \aleph_0, i \neq l} I_i)$.
For any $m_1, \ldots, m_k, n_1, \ldots, n_k \in S$ such that $m_1 < \cdots < m_k$ and $n_1 < \cdots < n_k$, since t_{ij} are $\mathbb{Q}^{rc}$-algebraically independent then for all j, t_{lj} are F_l-algebraically independent. Since moreover $t'_{lj} = t_{lj}$, there is a unique $\mathcal{L}_{odf}$-automorphism

$$\sigma : F_l(t_{lm_1}, \ldots, t_{lm_k}) \to F_l(t_{ln_1}, \ldots, t_{ln_k})$$

fixing F_l and given by $(\sigma(t_{lm_1}), \ldots, \sigma(t_{lm_k})) = (t_{ln_1}, \ldots, t_{ln_k})$. Since $\mathscr{U}$ is a model of CODF and CODF eliminates quantifiers in $\mathcal{L}_{odf}$, σ is a partial $\mathcal{L}_{odf}$-elementary map of $\mathscr{U}$ (see Remark 1.1.14). Then I_l is indiscernible over F_l, namely $(I_i, i \in \aleph_0)$ are mutually indiscernible over $\mathbb{Q}^{rc}$.

But letting $u := u_0$, for any $i \in \aleph_0$, I_i is not indiscernible over $\mathbb{Q}^{rc} \cup \{u\}$ because $u^{(i)} = u_i$ and $t_{i0} < u_i < t_{i1}$. So dp-rk$(u/\mathbb{Q}^{rc}) \geq \aleph_0$. □

Definition 4.5.10. The theory T is called strongly dependent iff for all $u \in \mathcal{M}$, dp-rk$(u/\varnothing) < \aleph_0$.

Definition 4.5.11. The theory T is called dp-minimal iff for all $u \in \mathcal{M}$, dp-rk$(u/\varnothing) \leq 1$.

Note that in the definitions of dp-minimal and strongly dependent, u is a single element.

Examples of dp-minimal theories are o-minimal theories, C-minimal theories and for any prime number p, the theory of the p-adic field. Superstable theories are strongly dependent. See [64], section 4.3 for more examples.

By Lemma 4.5.9, CODF is not strongly dependent (and so it is not dp-minimal).

4.5.3 VC-density

Definition and motivations

The VC-density was intensely studied in NIP theories for instance in [1] and [18].

We are still working in a saturated model $\mathcal{M}$ of an $\mathcal{L}$-theory T (not supposed to have NIP). For a partitioned $\mathcal{L}$-formula $\phi(\bar{X}; \bar{Y})$, we associate a function

$$\pi_\phi : \mathbb{N} \to \mathbb{N}$$

called the shatter function of ϕ (in T) such that $\pi_\phi(n)$ is the least natural number such that

$$\pi_\phi(n) \geq \left|\left\{\phi(\mathcal{M}^{|\bar{X}|}; \bar{b}) \cap E : \bar{b} \in \mathcal{M}\right\}\right|$$

for all $E \subset \mathcal{M}^{|\bar{Y}|}$ of cardinality $|E| = n$.

Moreover, we may require $\bar{b}$ to satisfy a given (partial) type p in the variable $\bar{Y}$, so we define $\pi_{\phi,p}(n)$ to be the least natural number such that

$$\pi_{\phi,p}(n) \geq \left|\left\{\phi(\mathcal{M}^{|\bar{X}|}; \bar{b}) \cap E : \bar{b} \in \mathcal{M} \text{ and } \mathcal{M} \models p(\bar{b})\right\}\right|$$

for all $E \subset \mathcal{M}^{|\bar{Y}|}$ of cardinality $|E| = n$.

When T is NIP, since ϕ has finite VC-dimension, $\pi_\phi(n) < 2^n$ for some n. Then by Sauer-Shelah Lemma (see Lemma 2.1 of [1]), there is a polynomial bound on $\pi_\phi(n)$. So we naturally define the VC-density of ϕ as follows:

Definition 4.5.12. Let $d \in \mathbb{R}$, we say that a partitioned $\mathcal{L}$-formula $\phi(\bar{X};\bar{Y})$ has VC-density $\leq d$ with respect to p if there exists $K \in \mathbb{R}$ such that the following holds for all $n \in \mathbb{N}$:

$$\pi_{\phi,p}(n) \leq K \cdot n^d.$$

We denote $\text{VC-dens}_p\,\phi$ the VC-density of ϕ with respect to p.

When T is NIP, for any formula ϕ, $\text{VC-dens}_p\,\phi$ does exist in $\mathbb{R}$.

Note that letting $p := \{\bar{Y} = \bar{Y}\}$, we get $\pi_{\phi,p} = \pi_\phi$. Then in that case we write $\text{VC-dens}\,\phi$ for $\text{VC-dens}_p\,\phi$.

Remark 4.5.13. VC-density does not respect the property that if $\phi \to \psi$ then $\text{VC-dens}_p\,\phi \leq \text{VC-dens}_p\,\psi$.

Proof. To see this, we observe that for all partitioned formulas $\phi(\bar{X};\bar{Y})$ and $\psi(\bar{X};\bar{Y})$, it holds that $\phi(\bar{X};\bar{Y}) \to \psi(\bar{X};\bar{Y})$ iff $\neg\psi(\bar{X};\bar{Y}) \to \neg\phi(\bar{X};\bar{Y})$ and $\text{VC-dens}_p\,\phi = \text{VC-dens}_p\,\neg\phi$ (because $\pi_{\phi,p}(n) = \pi_{\neg\phi,p}(n)$).

Another convincing argument is that

$$(\bar{X} \neq \bar{X}) \to \phi(\bar{X};\bar{Y}) \to \psi(\bar{X};\bar{Y}) \to (\bar{X} = \bar{X})$$

and $\text{VC-dens}_p(\bar{X} \neq \bar{X}) = \text{VC-dens}_p(\bar{X} = \bar{X}) = 0$. □

Remark 4.5.14. Suppose $p \subseteq q$. Then $\text{VC-dens}_p(\phi) \geq \text{VC-dens}_q(\phi)$.

Proof. Since for any $\bar{b} \in \mathcal{M}$, $\mathcal{M} \models q(\bar{b})$ implies that $\mathcal{M} \models p(\bar{b})$, for any $n \in \mathbb{N}$, $\pi_{\phi,p}(n) \geq \pi_{\phi,q}(n)$. So $\text{VC-dens}_p(\phi) \geq \text{VC-dens}_q(\phi)$. □

We will establish in the next section a relation between VC-density in RCF and CODF. There is a bound for the VC-density in RCF:

Theorem 4.5.15. *Let $\phi(\bar{X};Y_1,\ldots,Y_n)$ be an $\mathcal{L}_{of}$-formula and p be a partial type. Then* $\text{VC-dens}_p\,\phi \leq n$ *in RCF.*

That result is proved in [1] in the more general context of weakly o-minimal theories. Note that polynomial bounds on the VC-dimension in real closed fields were already studied by Karpinski and Macintyre (see [27] and [28]).

Firstly, note that there is no better uniform bound on the VC-density, since (for any theory T), the partitioned formula $\phi(X;\bar{Y}) :=$

$$(X = Y_1) \vee \cdots \vee (X = Y_n)$$

has VC-dimension n (see [1], section 1.4). Secondly, it is known by a result of A. Dolich, J. Goodrick and D. Lippel that if T is an $\mathcal{L}$-theory such that for any partitioned formula $\phi(\bar{X};Y)$ and partial type p, VC-dens$_p\,\phi \leq 1$ then T is dp-minimal (see [13], Proposition 3.2). So one may not expect that bound for the theory CODF, since it is not dp-minimal.

A stronger relationship between the VC-density and the dp-rank (stated for any theory T) is the object of the following open question (see [18], Problem 1.3).

Question 4.5.16. Let $n \in \mathbb{N}$, p be a partial type. Is it true that dp-rk$(p) \leq n$ iff for all partitioned formulas $\phi(\bar{X};\bar{Y})$, it holds that VC-dens$_p\,\phi \leq n$?

In [1], they show for some NIP $\mathcal{L}$-theories that for every $m \in \mathbb{N}$,

$$\sup\{\text{VC-dens}\,\phi : \phi(\bar{X};\bar{Y}) \text{ is a part. } \mathcal{L}\text{-form. and } |\bar{Y}| = m\} < +\infty.$$

We do not know whether the theory CODF does have that property. If it does then the answer to Question 4.5.16 is "no" since dp-rk$(\{\bar{Y} = \bar{Y}\}/\varnothing) = \aleph_0$.

Relationship between VC-density in CODF and RCF

We now consider VC-density in CODF and RCF (in their respective languages) and make a comparison.

Let $\phi(\bar{X};\bar{Y})$ be a partitioned $\mathcal{L}_{odf}$-formula, by Remark 4.0.1, there is some partitioned $\mathcal{L}_{of}$-formula ψ such that

$$\text{CODF} \vdash \forall\bar{X}\forall\bar{Y}\big(\psi(\bar{X},\ldots,\bar{X}^{(k)};\bar{Y},\ldots,\bar{Y}^{(l)}) \leftrightarrow \phi(\bar{X};\bar{Y})\big).$$

We denote for a partial $\mathcal{L}_{odf}$-type $p(\bar{Y})$,

$$p^* := \bigcap_{\bar{v}\in\mathscr{U},\,\mathscr{U}\models p(\bar{v})} \text{tp}_{\mathcal{L}_{of}}(\bar{v},\bar{v}',\ldots,\bar{v}^{(l)}).$$

Lemma 4.5.17. VC-dens$_p\,\phi \leq$ VC-dens$_{p^*}\,\psi$.

Proof. Let $E \subset \mathscr{U}^{|\bar{X}|}$ be finite and let

$$E_{\phi,p} := \big\{\phi(\mathscr{U}^{|\bar{X}|};\bar{b}) \cap E : \bar{b} \in \mathscr{U} \text{ and } \mathscr{U} \models p(\bar{b})\big\}.$$

Let us denote $\tilde{E} := \{(\bar{u},\bar{u}',\ldots,\bar{u}^{(k)}) : \bar{u} \in E\}$ and similarly for $F \in E_{\phi,p}$, $\tilde{F} := \{(\bar{u},\bar{u}',\ldots,\bar{u}^{(k)}) : \bar{u} \in F\}$.

Note that if $\mathscr{U} \models p(\bar{b})$ then $\mathscr{U} \models p^*(\bar{b},\bar{b}',\ldots,\bar{b}^{(l)})$.

Clearly $|E| = |\tilde{E}|$. Note that

$$\tilde{F} := \psi(\mathscr{U}^{|\bar{X}|\cdot(k+1)}; \bar{b}, \bar{b}', \dots, \bar{b}^{(l)}) \cap \tilde{E},$$

where $\bar{b}$ is such that $F = \phi(\mathscr{U}^{|\bar{X}|}; \bar{b}) \cap E$, so $\tilde{F} \in \tilde{E}_{\psi,p^*}$.

Then the map $E_{\phi,p} \to \tilde{E}_{\psi,p^*} : F \mapsto \tilde{F}$ is injective.

So $\pi_{\phi,p}(n) \leq \pi_{\psi,p^*}(n)$ and $\text{VC-dens}_p\, \phi \leq \text{VC-dens}_{p^*}\, \psi$. □

Combining this lemma with Theorem 4.5.15, we obtain the following bound on the VC-density in CODF:

$$\text{VC-dens}_p\, \phi \leq (l+1) \cdot |\bar{Y}|.$$

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